On class numbers of positive definite binary quaternion hermitian forms (II)

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To the memory of Takuro Shintani

In the previous paper [3], we obtain an explicit formula for the class number of the principal genus of the positive definite binary hermitian space over a definite quaternion algebra B over the rational number field. A similar formula has been obtained also in some ternary case [2]. In this paper, we shall give an explicit formula for the class number of *any genus* of maximal lattices of the binary hermitian space over B. In [4], one of the authors has given some numerical examples and proposed conjectures on a generalization of Eichler's correspondence between automorphic forms belonging to $SL_2(R)$ and $SU(2)$ to genus two case. There, it is essential to consider all parahoric subgroups of the division quaternion hermitian groups over Q_p. There are three parahoric subgroups up to conjugation, one is minimal parahoric, and the other two are maximal compact, each of which fixes a maximal lattice in the principal, or nonprincipal genus, respectively. So, we should not content ourselves only to the principal genus, and this is our motivation to this paper. When the discriminant of B is a prime number p, 'the main term' of the class number formula for the nonprincipal genus is $(p^2-1)/2880$, while $(p-1)(p^2+1)/2880$ for the principal genus. In this paper, we only announce the results, and local data (some local masses and number of optimal embeddings) which are needed to obtain an explicit formula will be written in the forthcoming paper.

§1. Let (V, f) be the two dimensional positive definite quaternion hermitian space over B, and O be a maximal order of B. Denote by $G(V, f)$ the similitude group of (V, f). Put $V_p=V\otimes Q_p$, and define B_p, O_p, and $G(V_p, f_p)$ in the similar way. We can take a basis e_1, e_2 of V_p such that

$$f_p(e_1, e_2)=1\,, \qquad f_p(e_1, e_1)=f_p(e_2, e_2)=0\,.$$

Identify V_p with B_p^2 by this basis. When $B_p=M_2(Q_p)$, all maximal left O_p-lattices form a single class which is represented by (O_p, O_p). When B_p is division, there exist exactly two classes of maximal left O_p-lattices, each of which is represented by (O_p, O_p) or $(O_p, \pi O_p)$, where π is a prime element of O_p.

Let D be the discriminant of B. Put $D=D_1D_2$, where D_1, D_2 are natural integers. Denote by $\mathcal{L}(D_1, D_2)$ the set of maximal left O-lattices L in V such that for every prime p, $Lg_p=(O_p, O_p)$ if $p\nmid D_2$, and $Lg_p=(O_p, \pi O_p)$ if $p\mid D_2$, for some $g_p\in G(V_p, f_p)$, taking basis of V_p as above. Then, $\mathcal{L}(D_1, D_2)$ form a genus of V, and any genus in V can be obtained in this way by choosing D_1, D_2 suitably. Denote by $H(D_1, D_2)$ the number of classes in $\mathcal{L}(D_1, D_2)$. An explicit formula for $H(D, 1)$ has been given in [3]. The purpose of this paper is to give an explicit formula for $H(D_1, D_2)$ for any D_2.

§2. We denote by H_i the total contribution to the class number formula for $H(D_1, D_2)$, of those conjugacy classes whose principal polynomials are of the form $f_i(\pm x)$, where $f_i(x)$ are defined as in [3] p. 590.

THEOREM. *The class number $H(D_1, D_2)$ of $\mathcal{L}(D_1, D_2)$ is given by*

$$H=\sum_{i=1}^{12} H_i\,,$$

where H_i are as follows:

$$H_1=\frac{1}{2^6 3^2 5}\prod_{p\mid D_1}(p-1)(p^2+1)\prod_{p\mid D_2}(p^2-1)\,,$$

$$H_2=\frac{1}{2^6 3^2}\prod_{p\mid D_1}(p-1)^2\times\begin{cases}7\cdots \textit{if } 2\nmid D_1,\ D_2=1\,,\\ 13\cdots \textit{if } 2\mid D_1,\ D_2=1\,,\\ 3\cdots \textit{if } 2\nmid D_1,\ D_2=2\,,\\ 0\cdots \textit{otherwise,}\end{cases}$$

$$H_3=\frac{1}{2^4 3}\prod_{p\mid D_1}(p-1)\Big(1-\Big(\frac{-1}{p}\Big)\Big)\times\begin{cases}1\cdots \textit{if } D_2=1\,,\\ 3\cdots \textit{if } D_2=2\,,\\ 0\cdots \textit{otherwise,}\end{cases}$$

$$H_4=\frac{1}{2^3 3^2}\prod_{p\mid D_1}(p-1)\Big(1-\Big(\frac{-3}{p}\Big)\Big)\times\begin{cases}1\cdots \textit{if } D_2=1\,,\\ 8\cdots \textit{if } D_2=3\,,\\ 0\cdots \textit{otherwise,}\end{cases}$$

$$H_5=\frac{1}{2^3 3^2}\prod_{p\mid D_1}(p-1)\Big(1-\Big(\frac{-3}{p}\Big)\Big)\times\begin{cases}1\cdots \textit{if } D_2=1\,,\\ 0\cdots \textit{otherwise,}\end{cases}$$

$$H_6=\frac{1}{2^5 3}\sideset{}{^*}\sum_{D^*\mid 2D}\prod_{p\mid D^*}(p-1)\prod_{\substack{p\mid 2D/D^*\\ p\mid D_1}}\Big(1-\Big(\frac{-1}{p}\Big)\Big)\prod_{\substack{p\mid 2D/D^*\\ p\mid D_2,\ p\neq 2}}\left(\frac{p+1}{2}\Big(1-\Big(\frac{-1}{p}\Big)\Big)\right)\times a\,,$$

where D^* *runs through the set of divisors of* $2D$ *which are products of odd number of primes, and* $a=0$ *if there exists a prime such that* $p\mid D_2$, $p\mid D^*$, $p\equiv 3 \bmod 4$, *and if there exists no such prime,* $a=3$ *if* $2\nmid D$, $2\mid D^*$, $=5$ *if* $2\mid D_1$, $2\mid D^*$, *or* $2\nmid D$, $2\nmid D^*$, $=11$ *if* $2\mid D_1$, $2\nmid D^*$, $=7$ *if* $2\mid D_2$, $2\mid D^*$, $=9$ *if* $2\mid D_2$, $2\nmid D^*$,

$$H_7=\frac{1}{2^3 3^2}\sum_{D^*\mid 3D}^{*}\prod_{p\mid D^*}(p-1)\prod_{\substack{p\mid 3D/D^*\\ p\mid D_1}}\Big(1-\Big(\frac{-3}{p}\Big)\Big)\prod_{\substack{p\mid 3D/D^*\\ p\mid D_2,\, p\neq 3}}\left(\frac{p+1}{2}\Big(1-\Big(\frac{-3}{p}\Big)\Big)\right)\times a\,,$$

where D^* *is a divisor of* $3D$ *as in* H_6, *and* $a=0$ *if there exists a prime* p *such that* $p\mid D_2$, $p\mid D^*$, $p\equiv 2 \bmod 3$, *and if there exists no such prime,* $a=1$ *if* $3\nmid D$, $3\mid D^*$, *or if* $3\mid D_1$, $3\mid D^*$, $=4$ *if* $3\mid D_2$, $3\mid D^*$, *or* $3\nmid D$, $3\nmid D^*$, $=16$ *if* $3\mid D_1$, $3\nmid D^*$, $=10$ *if* $3\mid D_2$, $3\nmid D^*$,

$$H_8=\frac{1}{2^2 3}\prod_{p\mid D_1}\Big(1-\Big(\frac{-1}{p}\Big)\Big)\Big(1-\Big(\frac{-3}{p}\Big)\Big)\times\begin{cases}1\cdots \textit{if } D_2=1\,,\\ 0\cdots \textit{if } D_2\neq 1\,,\end{cases}$$

$$H_9=\frac{1}{2^3 3^2}\prod_{p\mid D_1}\Big(1-\Big(\frac{-3}{p}\Big)\Big)^2\times\begin{cases}8\cdots \textit{if } 2\nmid D_1,\ D_2=1\,,\\ 5\cdots \textit{if } 2\mid D_1,\ D_2=1\,,\\ 12\cdots \textit{if } 2\nmid D_1,\ D_2=2\,,\\ 0\cdots \textit{otherwise},\end{cases}$$

$$H_{10}=\frac{1}{10}\prod_{p\mid D}2\prod_{p\in D_1(-1;\,5)}2\times\begin{cases}0\cdots \textit{if } \bigcup_{i=1}^{3}D_1(i\,;5)\neq\varnothing \textit{ or } \bigcup_{i=1,-1}D_2(i\,;5)\neq\varnothing\,,\\ 1\cdots \textit{if } \bigcup_{i=1}^{3}D_1(i\,;5)=\bigcup_{i=1,-1}D_2(i\,;5)=\varnothing \textit{ and } 5\mid D\,,\\ 2\cdots \textit{otherwise},\end{cases}$$

where we put $D_k(i\,;j)=\{p\mid D_k\,;\ p\equiv i \bmod j\}$,

$$H_{11}=\frac{1}{2^3}\prod_{\substack{p\mid D\\ p\neq 2}}2\prod_{p\in D_1(-1;\,8)}2\times\begin{cases}0\cdots \textit{if } D(1,8)\neq\varnothing \textit{ or } D_2(-1\,;8)\neq\varnothing\,,\\ 1\cdots \textit{if } D(1,8)=\varnothing,\ D_2(-1\,;8)=\varnothing\,,\end{cases}$$

where $D(i\,;j)=\{p\mid D\,;\ p\equiv i \bmod j\}$,

$$H_{12}=0 \textit{ if } D(1\,;12)\neq\varnothing \textit{ or } D_2(-1\,;12)\neq\varnothing,$$

and in other cases, it is as follows:

$$H_{12}=\frac{1}{2^4 3}\prod_{p\mid D}2\prod_{p\in D_1(-1;12)}2\times a\,,$$

where a *is given by the following table*:

	I	II	III
$6 \nmid D$	0	2	4
$3 \mid D_1,\ 2 \nmid D$	2	3	4
$3 \mid D_2,\ 2 \nmid D$	0	1	2
$2 \mid D_1,\ 3 \nmid D$	4	3	2
$2 \mid D_2,\ 3 \nmid D$	2	1	0
$6 \mid D_1$	5	9/2	4
$3 \mid D_1,\ 2 \mid D_2$	2	3/2	1
$2 \mid D_1,\ 3 \mid D_2$	2	3/2	1
$6 \mid D_2$	1	1/2	0

Here, I, II, III *means the following cases:*

I $\cdots D_1(-1\,;12)=\varnothing$ *and* $\sharp D(5\,;12)=even$

II $\cdots D_1(-1\,;12)\neq\varnothing$,

III $\cdots D_1(-1\,;12)=\varnothing$ *and* $\sharp D(5\,;12)=odd$.

Numerical examples

(i) $D_1=1$ and $D_2=$prime,

D_2	2	3	5	7	11	13	17	19	23	29	31	37	41
$H(1, D_2)$	1	1	1	1	1	2	2	2	2	3	3	5	4

(ii) $D=2\cdot3\cdot5$,

(D_1, D_2)	$(2\cdot3\cdot5, 1)$	$(2\cdot3, 5)$	$(2\cdot5, 3)$	$(3\cdot5, 2)$
$H(D_1, D_2)$	12	5	8	8

(D_1, D_2)	$(3, 2\cdot5)$	$(5, 2\cdot3)$	$(2, 3\cdot5)$	$(1, 2\cdot3\cdot5)$
$H(D_1, D_2)$	3	6	5	5

Examples of lattices

For $(D_1, D_2)=(1, 2)$ and $(1, 3)$, we have $H(D_1, D_2)=1$, and the following lattices are representatives of $\mathcal{L}(D_1, D_2)$:

(i) $D=D_2=2,\ D_1=1,$

$$L=(O,\ O)g\,,\qquad \text{where}$$

$$O=\boldsymbol{Z}+\boldsymbol{Z}i+\boldsymbol{Z}j+\boldsymbol{Z}\frac{1+i+j+k}{2},\ i^2=j^2=-1,\ ij=-ji,$$

$$\text{and}\ \ g=\begin{pmatrix}1 & -1\\ 0 & i-k\end{pmatrix},$$

(ii) $D=D_2=3,\ D_1=1,$

$$L=(O,\ O)g\,,\qquad \text{where}$$

$$O=\boldsymbol{Z}+\boldsymbol{Z}\frac{1+\alpha}{2}+\boldsymbol{Z}\beta+\boldsymbol{Z}\frac{(1+\alpha)\beta}{2},\ \alpha^2=-3,\ \beta^2=-1,\ \alpha\beta=-\beta\alpha,$$

$$\text{and}\ \ g=\begin{pmatrix}1 & -1-\beta\\ 0 & \alpha\end{pmatrix}.$$

References

[1] Hashimoto, K., On Brandt matrices associated with the positive definite quaternion hermitian forms, J. Fac. Sci. Univ. Tokyo Sect. IA, **27** (1980), 227-245.

[2] Hashimoto, K., On class numbers of positive definite ternary quaternion hermitian forms, in preparation.

[3] Hashimoto, K. and T. Ibukiyama, On class numbers of positive definite binary quaternion hermitian forms, J. Fac. Sci. Univ. Tokyo Sect. IA, **27** (1980), 549-601.

[4] Ibukiyama, T., On symplectic Euler factors of genus two, Doctorial thesis, Univ. Tokyo, 1980.

[5] Ihara, Y., On certain arithmetical Dirichlet series, J. Math. Soc. Japan, **16** (1964), 214-235.

[6] Satake, I., Theory of spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic fields, Inst. Hautes Études Sci. Publ. Math. **18** (1963), 5-69.

[7] Shimura, G., Arithmetic of alternating forms and quaternion hermitian forms, J. Math. Soc. Japan **15** (1963), 33-65.

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