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On Differential Operators on Automorphic Forms and Invariant Pluri-harmonic Polynomials

by

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Several mathematicians have given holomorphic differential operators $\mathscr{D}$ on automorphic forms F on bounded symmetric domains D such that the restriction of $\mathscr{D}F$ on some bounded domain Δ of lower dimension in D is also automorphic (cf. [5], [7], [11], [12], [2], [3]). The aim of this paper is to give a characterization of such holomorphic linear differential operators with constant coefficients in symplectic cases by using harmonic polynomials on $\mathbb{R}^n$, or pluri-harmonic polynomials on matrix arguments with some invariance property.

For any natural number m, we denote by H_m the Siegel upper half space of degree m. We shall treat the following two kinds of pairs (D, Δ) of domains:

(I) $D=H_n$ and $\Delta=H_{n_1}\times\cdots\times H_{n_r}$ with $n=n_1+\cdots+n_r$,

(II) $D=(H_n)^r$ and $\Delta=H_n$ (diagonally embedded in D).

The case (I) contains the cases treated by Eichler-Zagier [7] ($n=r=2$), Böcherer-Satoh-Yamazaki [2] ($n\geq 2, r=2$ with certain vector valued weights), Ibukiyama-Zagier [8] ($n=r\geq 2$), and the case (II) contains the Rankin-Cohen operators in Cohen [5], and Choie-Eholzer [4], Eholzer-Ibukiyama [6] ($r=2, n=1, 2, \geq 3$, respectively), and Satoh's operators in [12]. These two cases are in a sense "dual" to each other. Our theorems assert that, in the above two cases, everything is reduced to the representation theory of the orthogonal groups. Our differential operators $\mathscr{D}$ are expressed as $Q\left(\dfrac{\partial}{\partial Z}\right)$, where Q are obtained from pluri-harmonic polynomial maps which are characterized by some invariance property (cf. §1, Theorems 1 and 2). The polynomial maps Q will be given explicitly in several cases in section 3. It is plausible that whole theory might be generalized for any dual reductive pairs or any tube domains.

This paper is organized as follows: In section 1, after reviewing some standard notions, we shall state our main results Theorems 1 and 2. In section 2, we shall give proofs of these theorems. In section 3, we shall give several explicit examples of the differential operators we are concerned with.

This work is motivated by Zagier's interest in differential operators and triple L function. During his stay in Kyushu University in fall in 1990, he convinced me earnestly that this problem is very important. Without his strong interest in

this problem, this work would not have been done. The author would like to thank him for this point and also for several discussions. He also thanks the Max Planck Institute for Mathematics for the kind hospitality while he was revising the paper.

1. Main results

1.1. Notation

First, we shall fix some notation. For any natural integer n, let $Sp(n, \mathbb{R})$ be the usual split symplectic group of size $2n$. This group acts on H_n in the usual way by

$$gZ=(AZ+B)(CZ+D)^{-1},$$

where $g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp(n, \mathbb{R})$. Now, let (τ, V_τ) be an irreducible finite dimensional rational representation of $GL(n, \mathbb{C})$. For any $Z\in H_n$ and any $g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we write $j_\tau(g, Z)=\tau(CZ+D)$. For any V_τ-valued function F on H_n, and any element $g\in Sp(n, \mathbb{R})$, we write

$$(F|_\tau[g])(Z)=j_\tau(g, Z)^{-1}F(gZ).$$

In particular, when τ is a power of the determinant, that is, when $\tau(CZ+D)=\det(CZ+D)^k$ for some natural number k, we write $F|_k[g]=F|_\tau[g]$ as usual. More generally, for a fixed natural number r, let $n_1, \cdots, n_r$ be natural numbers and for each j with $1\le j\le r$, let (τ_i, V_{τ_i}) be an irreducible representation of $GL(n_i, \mathbb{C})$. Take an irreducible representation (τ, V_τ) of $GL(n_1, \mathbb{C})\times\cdots\times GL(n_r, \mathbb{C})$ defined by $\tau=\tau_1\otimes\cdots\otimes\tau_r$ and $V_\tau=V_{\tau_1}\otimes\cdots\otimes V_{\tau_r}$. For any $g=(g_1, \cdots, g_r)\in Sp(n_1, \mathbb{R})\times\cdots\times Sp(n_r, \mathbb{R})$, and $Z=(Z_1, \cdots, Z_r)\in H_{n_1}\times\cdots\times H_{n_r}$, we write

$$j_\tau(g, Z)=\tau_1(C_1Z_1+D_1)\otimes\cdots\otimes\tau_r(C_rZ_r+D_r),$$

where we write $g_i=\begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ for each i with $1\le i\le r$. For any V_τ-valued function $F(Z)$ on $Z\in H_{n_1}\times\cdots\times H_{n_r}$ and any $g\in Sp(n_1, \mathbb{R})\times\cdots\times Sp(n_r, \mathbb{R})$, we write

$$(F|_\tau[g])(Z)=j_\tau(g, Z)^{-1}F(gZ).$$

When $\tau_i=\det^{k_i}$ for each i with $1\le i\le r$, we write $F|_{k_1,\cdots,k_r}[g]=F|_\tau[g]$. We say that a function on $H_{n_1}\times\cdots\times H_{n_r}$ is C^∞, if it is a C^∞ function as a function of $\sum_{j=1}^r n_j(n_j+1)$ components of real symmetric matrices X_j and Y_j, where $Z_j=X_j+iY_j\in H_{n_j}$ $(1\le j\le r)$. For any discrete subgroup Γ of $Sp(n_1, \mathbb{R})\times\cdots\times Sp(n_r, \mathbb{R})$ with covolume finite, we say that a C^∞ function F transforms like a modular form of weight τ, if $F|_\tau[\gamma]=F$ for any $\gamma\in\Gamma$. Now, we fix a natural number n and r, and choose natural numbers $n_1, \cdots, n_r$ so that $n_1+\cdots+n_r=n$. For any matrix $M=(m_{kl})_{1\le k,l\le n}\in M_n(\mathbb{C})$, and any pair of integers (i, j) with $1\le i, j\le r$, we denote by M_{ij} the submatrix (m_{kl}) of M with $n+\cdots+n_{i-1}+1\le k\le n_1+\cdots+n_i$ and $n_1+\cdots+n_{j-1}+1\le l\le n_1+\cdots+n_j$. That is, we write $M=(M_{ij})_{1\le i,j\le r}$ by submatrices. For the sake of simplicity, we sometimes write $M_i = M_{ii}$ for each i with $1 \le i \le r$. Now, we put

$$G(n_1, \cdots, n_r) = Sp(n_1, \mathbb{R}) \times Sp(n_2, \mathbb{R}) \times \cdots \times Sp(n_r, \mathbb{R})$$

and we regard it as a subgroup of $Sp(n, \mathbb{R})$ by the "diagonal embedding" ι, where the group homomorphism ι of $G(n_1, \cdots, n_r)$ into $Sp(n, \mathbb{R})$ is defined by

$$\iota(g_1, \cdots, g_r) = \begin{pmatrix} (A_{ij}) & (B_{ij}) \\ (C_{ij}) & (D_{ij}) \end{pmatrix} \in Sp(n, \mathbb{R})$$

with $g_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ and $A_{ij} = B_{ij} = C_{ij} = D_{ij} = 0$ for pairs (i, j) with $i \ne j$.

1.2. Pluri-harmonic polynomials

We review pluri-harmonic polynomials of matrix arguments. Let m and d be natural integers and let P be a $\mathbb{C}$-valued polynomial of md variables x_{ij} ($1 \le i \le m$, $1 \le j \le d$) regarded as a function of $X = (x_{i\nu}) \in M_{m,d} = M_{m,d}(\mathbb{C})$. For each i, j with $1 \le i, j \le m$, denote by $\Delta_{i,j}$ the following differential operator:

$$\Delta_{i,j} = \sum_{\nu=1}^{d} \frac{\partial^2}{\partial x_{i\nu} \partial x_{j\nu}}.$$

A polynomial $P(X)$ on $M_{m,d}$ is called *harmonic* if $\sum_{i=1}^{m} \Delta_{i,i} P = 0$, and it is said to be *pluri-harmonic* if $\Delta_{i,j} P = 0$ for each pair (i, j) with $1 \le i, j \le m$. This is equivalent to say that $P(AX)$ is harmonic for any $A \in GL(m, \mathbb{C})$ (cf. Kashiwara and Vergne [9]). We denote by $\mathscr{H}_{m,d}$ the space of all pluri-harmonic polynomials on $M_{m,d}$. The group $GL(m, \mathbb{C}) \times O(d)$ acts on $M_{m,d}$ by: $P(X) \to P({}^t AXB)$ for $A \in GL(m, \mathbb{C})$ and $B \in O(d)$. The irreducible decomposition of $\mathscr{H}_{m,d}$ has been given in Kashiwara and Vergne [9]. They have shown there for example that each irreducible component has multiplicity one, and if an irreducible representation $\tau \otimes \lambda$ of $GL(m, \mathbb{C}) \times O(d)$ appears in $\mathscr{H}_{m,d}$, then the representation τ of $GL(m, \mathbb{C})$ is uniquely determined by the representation λ of $O(d)$ and vice versa. (cf. Kashiwara and Vergne loc. cit. (5.7), (6.14).) We denote this τ by $\tau(\lambda)$. It is also clear that pluri-harmonic polynomials in each irreducible representation space in $\mathscr{H}_{m,d}$ are homogeneous. Now, assume that an irreducible representation $\tau \otimes \lambda$ appears as an irreducible component of $\mathscr{H}_{m,d}$. We denote by V_τ the representation space of τ. Then, pluri-harmonic polynomials maps P of $M_{m,d}$ to V_τ such that $P(aX) = \tau(a)P(X)$ form a vector space of dimension $\dim \lambda$. We denote this vector space by $\mathscr{H}_{m,d}(\tau, \lambda) = \mathscr{H}(\tau, \lambda)$.

1.3. Invariant pluri-harmonic polynomials and Theorems

We first treat the case (I) in the introduction. Fix $n_1, \cdots, n_r$ with $n = n_1 + \cdots + n_r$ as in the introduction. Also put $D = H_n$ and $\Delta = H_{n_1} \times \cdots \times H_{n_r}$. We fix a natural number d. Let λ_i be an irreducible representation of $O(d)$ which appears in $\mathscr{H}_{n_i,d}$. We take the tensor representation $\lambda = \lambda_1 \otimes \cdots \otimes \lambda_r$ of the direct product $O(d)^r$ of r copies of $O(d)$. We can take the representation space of λ as the tensor product $V(\lambda) = \mathscr{H}(\tau(\lambda_1), \lambda_1) \otimes \cdots \otimes \mathscr{H}(\tau(\lambda_r), \lambda_r)$. We fix a basis of $V(\lambda)$. Then, each element of $V(\lambda)$ is regarded as a vector each of whose component is a polynomial $P(X_1, \cdots, X_r)$ of $X_i \in M_{n_i,d}$ $(1 \leq i \leq r)$ such that P is pluri-harmonic for each X_i $(1 \leq i \leq r)$. Besides, the above polynomial P is a linear combination of products $P_{\nu_1}(x_1) \cdots P_{\nu_r}(X_r)$ of polynomials P_{ν_i} such that each P_{ν_i} is homogeneous and pluri-harmonic. We denote by $\Delta(O(d))$ the image of the diagonal embedding of $O(d)$ into $O(d)^r$. Now, assume that the restriction of λ to $\Delta(O(d))$ contains the trivial representation of $\Delta(O(d))$. (The multiplicity of the trivial representation is not necessarily one, even if it is not zero.) We denote by $\mathscr{H}^{inv} = \mathscr{H}^{inv}_{n_1,\cdots,n_r}(\lambda)$ the isotypic component of the trivial representation of $\lambda \mid \Delta(O(d))$. This space consists of polynomial maps $P(X_1, \cdots, X_r)$ of $(X_1, \cdots, X_r) \in M_{n_1,d} \times \cdots \times M_{n_r,d}$ to $V = V(\tau(\lambda_1)) \otimes \cdots \otimes V(\tau(\lambda_r))$ such that the following three conditions are satisfied.

(1) $P(X_1, \cdots, X_r)$ is pluri-harmonic for each X_i $(1 \leq i \leq r)$.

(2) $P(X_1 g, \cdots, X_r g) = P(X_1, \cdots, X_r)$ for each $g \in O(d)$.

(3) $P(a_1 X_1, \cdots, a_r X_r) = (\tau_1(a_1) \otimes \cdots \otimes \tau_r(a_r)) P(X_1, \cdots, X_r)$ for each $a_i \in GL(n_i, \mathbb{C})$ $(1 \leq i \leq r)$.

Now, denote by X the $n \times d$-matrix defined by:

$$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_r \end{pmatrix},$$

where $X_i \in M_{n_i,d}$ as before. From now on, we assume that $d \geq n$. Then, by virtue of Weyl [13], any $O(d)$-invariant polynomial of the components of X is a polynomial of the components of the $n \times n$ symmetric matrix $X^t X$, and $n(n+1)/2$ different components of $X^t X$ are algebraically independent. Hence, under the assumption that $d \geq n$, for each element $P \in \mathscr{H}^{inv}$, there exists a unique polynomial map Q of the set of $n \times n$ symmetric matrices to V such that $P(X_1, \cdots, X_r) = Q(X^t X)$. We call Q the *associated polynomial* of P. For $Z = (z_{kl})_{1 \leq k,l \leq n} \in H_n$, we put

$$\frac{\partial}{\partial Z} = \left(\frac{1+\delta_{kl}}{2} \frac{\partial}{\partial z_{kl}} \right).$$

For the case (I) in the introduction, we consider the commutation relation below. For each $i = 1, \cdots, r$, we take a finite dimensional irreducible representation (τ_i, V_{τ_i}) of $GL(n_i, \mathbb{C})$ and put $V = \bigotimes_{i=1}^r V_{\tau_i}$. We take an even natural number $d = 2k$ and put $\tau_i' = \tau_i \otimes \det^k$. We put $n = \sum_{i=1}^r n_j$ as before. We take a polynomial map Q of the set of $n \times n$ symmetric matrices to V, and put $\mathscr{D} = Q\left(\frac{\partial}{\partial Z}\right)$.

CONDITION 1. *For any C^∞ function F on H_n, and any element $g = (g_1, \cdots, g_r) \in G(n_1, \cdots, n_r)$, we get*

$$(Res_\Delta(\mathscr{D}F))\big|_{\tau_1' \otimes \cdots \otimes \tau_r'}[g] = Res_\Delta(\mathscr{D}(F|_k[g])).$$

(Here we denote by Res_Δ the restriction of maps to Δ.)

THEOREM 1. *Suppose that $d \geq n$. Let Q be a polynomial map from the set of $n \times n$ symmetric matrices to V. Then the associated differential operators $\mathscr{D} = Q\left(\frac{\partial}{\partial Z}\right)$ satisfies the commutation relation, Condition 1, if and only if the polynomial P on $M_{n,d}$ defined by $P(X) = Q(X^t X)$ belongs to $\mathscr{H}^{inv} = \mathscr{H}^{inv}_{n_1, \cdots, n_r}(\lambda_1 \otimes \cdots \otimes \lambda_r)$, where λ_i is such that $\tau_i = \tau(\lambda_i)$ for each i.*

COROLLARY 1. *Assumptions and notation being as above, assume that a V-valued C^∞ function F on H_n transforms like a modular form of weight k with respect to a discrete subgroup Γ of $Sp(n, \mathbb{R})$. Then, the function $Res_\Delta(\mathscr{D}F)$ transforms like a modular form on Δ of weight $\det^k \tau_1 \otimes \cdots \otimes \det^k \tau_r$ with respect to $\Gamma \cap G(n_1, \cdots, n_r)$. In particular, if F is a holomorphic modular form, that $Res_\Delta(\mathscr{D}F)$ is also a holomorphic modular form.*

Next, we shall treat the case (II) in the introduction. Put $D = H_n^r$ and $\Delta = H_n$, and regard Δ as a subset of D through the diagonal embedding. For each i with $1 \leq i \leq r$, we fix natural numbers d_i such that $d_i \geq n$. Put $d = d_1 + \cdots + d_r$. We regard the group $K = O(d_1) \times \cdots \times O(d_r)$ as a subgroup of $O(d)$ through the "diagonal" embedding:

$$O(d_1) \times \cdots \times O(d_r) \in (k_1, \cdots, k_r) \longrightarrow \begin{pmatrix} k_1 & 0 & \cdots & \cdots & 0 \\ 0 & k_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & k_r \end{pmatrix} \in O(d).$$

Take a representation $\tau \otimes \lambda$ of $GL(n, \mathbb{C}) \times O(d)$ which appears in $\mathscr{H}_{n,d}$. We denote by $\mathscr{H}^{inv} = \mathscr{H}^{inv}(d_1, \cdots, d_r; \lambda)$ the subspace of $\mathscr{H}(\tau, \lambda)$ consisting of pluri-harmonic polynomial maps P in $\mathscr{H}(\tau, \lambda)$ which is invariant by the action of K, namely $P(X_1 k_1, \cdots, X_r k_r) = P(X_1, \cdots, X_r)$ for all $k_i \in O(d_i)$) $(1 \leq i \leq r)$. If the restriction of λ to K has the trivial representation as an irreducible component of multiplicity l, then the dimension of $\mathscr{H}^{inv}$ is l. For each i with $1 \leq i \leq r$, denote the $n \times d_i$ matrix argument by X_i. By virtue of H. Weyl [13], for each element $P \in \mathscr{H}^{inv}$, there exists a polynomial map Q such that $Q(X_1^t X_1, \cdots, X_r^t X_r) = P(X_1, \cdots, X_r)$. We call Q the associated map of P.

Now for the case (II), we consider the commutation relation below. Let (τ, V) be a finite dimensional irreducible representation of $GL(n, \mathbb{C})$. Let Q be a polynomial map of r product of the space of $n \times n$ symmetric matrices to V. Define a differential operator $\mathscr{D}$ by $\mathscr{D} = Q\left(\frac{\partial}{\partial Z_1}, \cdots, \frac{\partial}{\partial Z_r}\right)$. We assume that each d_i $(1 \leq i \leq r)$ is even and put $d_i = 2k_i$ and $d = 2k$. We regard $Sp(n, \mathbb{R})$ as a subgroup of $Sp(n, \mathbb{R})^r$ through the diagonal embedding.

CONDITION 2. *For any C^∞ function $F(Z_1, \cdots, Z_r)$ on H_n^r and any element $g \in Sp(n, \mathbb{R})$, we get the following commutation relation.*

$$Res_\Delta(\mathscr{D}(F|_{k_1,\cdots,k_r}[g]))=(Res_\Delta(\mathscr{D}F))|_{\det^k\otimes\tau}[g]\ ,$$

where Res_Δ *is the restriction of the maps to* Δ.

THEOREM 2. *Notation being as above, we assume that* $d_i\geq n$ *for each* i. *We take* $X_i\in M_{n,d_i}$ $(1\leq i\leq r)$ *and define a map* P *by* $P(X_1,\cdots,X_r)=Q(X_1^tX_1,\cdots,X_r^tX_r)$. *Then* $\mathscr{D}$ *satisfies the above commutation relation, Condition 2, if and only if* P *is a pluri-harmonic polynomial map belonging to* $\mathscr{H}^{inv}=\mathscr{H}^{inv}(d_1,\cdots,d_r;\lambda)$, *where* $\tau=\tau(\lambda)$.

Corollary 2. *Notation and assumptions being the same as above, we get following results*:

(1) *If a function* F *on* H_n^r *transforms like a modular forms of weight* $(k_1,\cdots,k_r)$ *with respect to some discrete subgroup* Γ *of* $Sp(n,R)^r$, *then* $Res_\Delta(\mathscr{D}F)$ *transforms like a modular form of weight* $\det^k\otimes\tau$ *with respect to* $\Gamma\cap\Delta(Sp(n,\mathbb{R}))$, *where* $\Delta(Sp(n,\mathbb{R}))$ *is the image of* $Sp(n,\mathbb{R})$ *in* $Sp(n,\mathbb{R})$. *under the diagonal embedding* ι.

(2) *Let* Γ *be a discrete sybgfoup of* $Sp(n,\mathbb{R})$. *For each* i *with* $1\leq i\leq r$, *let* $F_i(Z_i)$ *be a* $\mathbb{C}$-*valued modular form on* H_n *of weight* k_i *with respect to* Γ. *Take the product* $F(Z_1,\cdots,Z_r)=F_1(Z_1)\times\cdots\times F_r(Z_r)$. *Then,* $Res_\Delta(\mathscr{D}F)$ *is a modular form on* H_n *of weight* $\det^k\otimes\tau$ *with respect to* Γ.

REMARK. Of course, these theorems can be applied also for modular embeddings, if we change the coordinates according to the embedding. Also we can get differential operators on Jacobi forms, using the Fourier-Jacobi expansion.

2. Proofs

2.1.

In this section, we prove Theorems 1 and 2. The idea of the proofs of both theorems is based on the following two facts. Firstly, for any natural number n and d with $d\geq n$, let Y be a $n\times d$ matrix whose components are algebraically independent nd variables over $\mathbb{C}$. For each i with $1\leq i\leq n$, denote by y_i the i-th row vector of Y. Then, by virtue of the invariant theory on the orthogonal groups (H. Weyl, loc.cit.), the $n(n+1)/2$ inner products (in the usual sense) (y_i,y_j) $(1\leq i\leq j\leq n)$ are algebraically independent over $\mathbb{C}$. Secondly, we have the following transfomation formula for harmonic polynomials and besides we can show a lemma which asserts the converse.

LEMMA 1 (Kashiwara-Vergne [9]). *For any pluri-harmonic polynomial* $P(X)$ *on* $M_{n,d}$ *and* $Z\in H_n$ *and any* $Y\in M_{n,d}$ *we get*

$$\int_{M_{n,d}} e^{iTr({}^tXY)}e^{\frac{i}{2}Tr({}^tXZX)}P(X)dX=(2\pi)^{nd/2}(\det(Z/i))^{-\frac{d}{2}}e^{\frac{i}{2}Tr({}^tY(-Z^{-1})Y)}P(-Z^{-1}Y)\ ,$$

where dX *is the usual Lebesque measure on* $M_{n,d}(\mathbb{R})$, *and by* $\det(Z/i)^{\frac{d}{2}}$ *we mean the branch which is* 1 *for* $Z=i1_n$.

LEMMA 2. *We fix any homogeneous polynomials* $P_1(X)$ *and* $P_2(X)$ *of variable* $X\in\mathbb{R}^N$. *We assume that the following equality*

$$\int_{\mathbb{R}^N} e^{i^tXY} e^{-\frac{{}^tXX}{2}} P_1(X)dX = (2\pi)^{N/2} e^{-\frac{{}^tYY}{2}} P_2(iY)$$

is satisfied for any vector $Y=(y_1, \cdots, y_N) \in \mathbb{R}^N$. *Then we get* $P_1(X)=P_2(X)$ *and besides this is a harmonic polynomial.*

Proof of Lemma 2. This lemma seems more or less known (cf. Kashiwara and Vergne loc. cit. p. 4), but we give here a proof for the reader's convenience. We write the total degree of $P_i(X)$ by $\deg(P_i)$. For $X={}^t(x_1, \cdots, x_N) \in \mathbb{R}^N$, write $P_1(X)$ as

$$P_1(X) = \sum_{\alpha} c(\alpha) x^{\alpha} ,$$

where α runs over multi-indices $(\alpha_1, \cdots, \alpha_N)$ such that $\alpha_1 + \cdots + \alpha_N = \deg P_1$ with $\alpha_i \geq 0$, and we write $x^{\alpha} = \prod_{i=1}^N x_i^{\alpha_i}$. We calculate the left hand side. First we get

$$-\frac{{}^tXX}{2} + i^tXY = -\frac{1}{2}\sum_{j=1}^N (x_j - iy_j)^2 - \frac{{}^tYY}{2} .$$

Hence, for a fixed Y, by changing the variable X into $X+iY$, we get

$$\int_{\mathbb{R}^N + iY} e^{-\frac{{}^tXX}{2}} P_1(X+iY)dX = (2\pi)^{N/2} P_2(iY)$$

Here the integrand is holomorphic with respect to X and $e(-{}^tXX/2)$ tends rapidly to zero when X tends to $\pm\infty$. Hence the integral remains unchanged if we replace the path $\mathbb{R}^N + iY$ of the integral by $\mathbb{R}^N$. Now we can calculate the integral term by term. First of all, we have

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} x^l dx = \begin{cases} 2^{\frac{l+1}{2}} \Gamma(\frac{l+1}{2}) & \text{if } l \text{ is even} , \\ 0 & \text{if } l \text{ is odd} . \end{cases}$$

Hence, we get

$$\sum_{\alpha} c(\alpha) \int_{\mathbb{R}^N} \prod_{j=1}^N e^{-\frac{1}{2}x_j^2} (x_j + iy_j)^{\alpha_j} \prod_{j=1}^N dx_j$$

$$= \sum_{\alpha} c(\alpha) \prod_{j=1}^N \sum_{\beta_j=0,\, \beta_j:\text{even}}^{\alpha_j} \binom{\alpha_j}{\beta_j} (iy_j)^{\alpha_j - \beta_j} 2^{(\beta_j+1)/2} \Gamma(\tfrac{\beta_j+1}{2}) .$$

By our assumption, this is equal to $(2\pi)^{N/2} P_2(iY)$. Since we assumed this equality for all $Y \in \mathbb{R}^N$, this is a polynomial identity. By definition, the right hand side is homogeneous with respect to y_j. On the left hand side, the terms of highest degree with respect to Y come from those with $\beta_j = 0$ for all j. We get $2^{1/2}\Gamma(1/2) = (2\pi)^{1/2}$. Hence, comparing the terms of highest degree with respect to Y of both sides, we get $P_1(iY) = P_2(iY)$. Besides, the terms of $\deg P - 2$ of the left hand side with respect to Y, that is, those terms with $\beta_j = 2$ for one j and $\beta_l = 0$ for all the other l $(1 \leq j, l \leq N)$ vanishes identically. Also we have $2^{3/2}\Gamma(3/2) = 2^{1/2}\Gamma(1/2) = (2\pi)^{1/2}$. Hence we get

$$\sum_{\alpha}\sum_{j=1}^{N}\alpha_j(\alpha_j-1)c(\alpha)(iy_j)^{\alpha_j-2}\prod_{l=1,l\neq j}^{N}(iy_l)^{\alpha_l}=0\,,$$

where the sum is over indices such that $\sum_{j=1}^{N}\alpha_j=\deg P$. This means that $P_1(X)$ is harmonic. q.e.d.

2.2. Proof of Theorem 1

First for a polynomial map Q of $n\times n$ symmetric matrices to V associated to an invariant pluri-harmonic polynomial P, we put $\mathscr{D}=Q\left(\frac{\partial}{\partial Z}\right)$. We show that $\mathscr{D}$ satisfies the commutation relation in Theorem 1. Let F be as in Theorem 1. Then, by definition, $F|_k[g]=\deg(CZ+D)^{-k}F(gZ)$. We must calculate $Res_{\Delta}(\mathscr{D}(\det(CZ+\ D)^{-k}F(gZ)))$ for any $g\in G=G(n_1, n_2, \cdots, n_r)$. Obivously, it is sufficient to show the commutation relation for generators of G. It can be easily shown that the group G is generated by the following three kinds of elements.

(1) $u(B)=\begin{pmatrix}1_n & B\\ 0 & 1_n\end{pmatrix}$, where $B\in M_n(\mathbb{R})$, ${}^tB=B$, and $B_{ij}=0$ if $i\neq j$.

(2) $d(A)=\begin{pmatrix}A & 0\\ 0 & {}^tA^{-1}\end{pmatrix}$, where $A\in GL(n,\mathbb{R})$ and $A_{ij}=0$ if $i\neq j$.

(3) $\rho=\begin{pmatrix}0 & 1_n\\ -1_n & 0\end{pmatrix}$.

Among the above elements, the commutation relation for elements of type (1) is trivial. As for elements of type (2), if we put $W=AZ^tA$, then we get $\frac{\partial}{\partial Z}={}^tA\frac{\partial}{\partial W}A$, and by our choice of Q, we get $Q\left(\frac{\partial}{\partial Z}\right)=\tau_1({}^tA_1)\otimes\cdots\otimes\tau_r({}^tA_r)Q\left(\frac{\partial}{\partial W}\right)$. Hence we get

$$Q\left(\frac{\partial}{\partial Z}\right)(F(AZ^tA)(\det A)^k)=\left(\bigotimes_{j=1}^{r}\det(A_j)^{d/2}\tau_j({}^tA_j^{-1})^{-1}\right)\left(Q\left(\frac{\partial}{\partial Z}\right)F\right)(AZ^tA)\,.$$

To show the relation for ρ, we need some tricks. We denote by $\boldsymbol{\mu}$ multi-index $(\boldsymbol{\mu}_{ij})_{1\le i\le j\le n}$. We put $|\boldsymbol{\mu}|=\sum_{1\le i\le j\le n}\mu_{ij}$ and denote by $D_{\boldsymbol{\mu}}$ the following differential operator

$$D_{\boldsymbol{\mu}}=\prod_{1\le i\le j\le n}\frac{\partial^{\mu_{ij}}}{\partial z_{ij}^{\mu_{ij}}}\,.$$

Then, it is clear that there exist V-valued functions $Q_{\boldsymbol{\mu}}(Z)$ which does not depend on F such that

$$\mathscr{D}(\det(-Z)^{-k}F(-Z^{-1}))=\sum_{\boldsymbol{\mu}}Q_{\boldsymbol{\mu}}(Z)(\mathscr{D}_{\boldsymbol{\mu}}F)(-Z^{-1})\,,$$

where $\boldsymbol{\mu}$ runs over multi-indices such that $|\boldsymbol{\mu}|\le\deg(Q)$. So, the rough idea to prove the commutation relation is to find a function F such that $D_{\boldsymbol{\mu}}F$ are linearly independent at any point Z (or over the ring of holomorphic functions) and satisfies the commutation relation. If we put $F_0(Z)=\exp(\frac{i}{2}Tr({}^tYZY))$ for a matrix Y of size $n\times d$ of independent variables, then

$$(\mathscr{D}_{\boldsymbol{\mu}}F_0)(Z)=\left(\frac{i}{2}\right)^{|\boldsymbol{\mu}|}\prod_{1\le i\le j\le n}(y_i, y_j)^{\mu_{ij}}F_0(Z),$$

and the polynomials $y_{\boldsymbol{\mu}}=\prod_{1\le i\le j\le n}(y_i, y_j)^{\mu_{ij}}$ ($\boldsymbol{\mu}$: multi-indices) are linearly independent over $\mathbb{C}$. So, it is enough to show the relation only for F_0. We calculate the action of $\mathscr{D}$ on this F_0. By Lemma 1, we get

$$\begin{aligned}\mathscr{D}(F_0|_k[\rho])&=i^{nd/2}(2\pi)^{-nd/2}\int_{M_{n,d}}e^{iTr({}^tXY)}\mathscr{D}(e^{\frac{i}{2}Tr({}^tXZX)})dX\\ &=i^{nd/2}\left(\frac{i}{2}\right)^{\deg Q}(2\pi)^{-nd/2}\int_{M_{n,d}}e^{iTr({}^tXY)}e^{\frac{i}{2}Tr({}^tXZX)}Q(X^tX)dX.\end{aligned}$$

We write $Z_0=(Z_{ij})$ where $Z_{ij}=0$ for all $i\neq j$ and $Z_{ii}=Z_i$. Then, by the Fubini Theorem and iterated use of Lemma 1, we get

$$\begin{aligned}&Res_\Delta(\mathscr{D}(F_0|_k[\rho]))\\ &=i^{nd/2}(2\pi)^{-nd/2}\left(\frac{i}{2}\right)^{\deg Q}\int_{M_{n,d}}e^{i\sum_{i=1}^r Tr({}^tX_iY_i)}e^{\frac{i}{2}\sum_{i=1}^r Tr({}^tX_iZ_iX_i)}Q(X^tX)dX_1dX_2\cdots dX_r\\ &=\left(\frac{i}{2}\right)^{\deg Q}\prod_{i=1}^r(det(-Z_i)^{-d/2}e^{\frac{i}{2}Tr({}^tY_i(-Z_i^{-1})Y_i)})P(-Z_0^{-1}Y))\\ &=\left(\frac{i}{2}\right)^{\deg Q}\prod_{i=1}^r(det(-Z_i)^{-\frac{d}{2}}e^{\frac{i}{2}Tr({}^tY(-Z_i^{-1})Y)})\times(\tau_1(-Z_1^{-1})\otimes\cdots\otimes\tau_r(-Z_r^{-1})P(Y))\\ &=\prod_{i=1}^r det(-Z_i)^{-\frac{d}{2}}\times(\tau_1(-Z_1^{-1})\otimes\cdots\otimes\tau_r(-Z_r^{-1}))\times(\mathscr{D}F_0)(-Z_0^{-1}).\end{aligned}$$

Now we prove the converse. Let Q be a polynomial map of $n\times n$ symmetric matrices Y to V_τ, and put $\mathscr{D}=Q\left(\frac{\partial}{\partial Z}\right)$. Assuming that $\mathscr{D}$ satisfies the commutation relation, we shall show that Q is an associated polynomial of invariant pluri-harmonic polynomial P. First, taking the element of type (1) above and $F=F_0$, we see that

$$Q({}^tAYA)=(\tau_1({}^tA_1)\otimes\cdots\otimes\tau_r({}^tA_r))Q(Y).$$

A fortiori, if we put $P(X_1,\cdots,X_r)=Q(X^tX)$, then this P is homogeneous for each X_i. We shall show that P is pluri-harmonic for each X_i. Taking the above relation into account, the commutation relation for this Q and $\rho\in G$ and F_0 is written as

$$\int_{M_{n,d}(\mathbb{R})} \prod_{j=1}^{r} e^{iTr({}^tX_jY_j)} e^{\frac{i}{2}Tr({}^tX_jZ_jX_j)} P(X)dX$$

$$= \prod_{j=1}^{r} (2\pi)^{n_jd/2} \det(Z_j/i)^{-d/2} e^{\frac{i}{2}Tr({}^tY_j(-Z_j^{-1})Y_j)} P(-Z_1^{-1}Y_1, \cdots, -Z_r^{-1}Y_r) .$$

We put $Z_j = i\alpha_j^2$ for some positive definite real symmetric $n \times n$ matrices α_j $(1 \le j \le r)$. So changing the variable X_j into $\alpha_j^{-1}X_j$, we get

$$\int_{M_{n,d}(\mathbb{R})} \prod_{j=1}^{r} e^{i\,Tr({}^tX_j(\alpha_j^{-1}Y_j))} e^{-\frac{1}{2}Tr({}^tX_jX_j)} P(\alpha_1^{-1}X_1, \cdots, \alpha_r^{-1}X_r)dX$$

$$= P(i\alpha_1^{-1}(\alpha_1^{-1}Y_1), \cdots, i\alpha_r^{-1}(\alpha_r^{-1}Y_r)) \prod_{j=1}^{r} ((2\pi)^{n_jd/2} \det(\alpha_j)^{-d} e^{-\frac{1}{2}Tr({}^t(\alpha_j^{-1}Y_j)(\alpha_j^{-1}Y_j))}) .$$

Hence, by iterated use of Lemma 2, we can show that $P(\alpha_1^{-1}Y_1, \cdots, \alpha_r^{-1}Y_r)$ is harmonic for each Y_j for any α_j. Hence $P(X_1, \cdots, X_r)$ is pluri-harmonic for each X_j $(1 \le j \le r)$. Thus, Theorem 1 is proved.

2.3. Proof of Theorem 2

Define a function $F_1(Z_1, \cdots, Z_r)$ on H_n^r by

$$F_1(Z_1, \cdots, Z_r) = e^{\frac{i}{2}\sum_{j=1}^{r} Tr({}^tY_jZ_jY_j)} ,$$

where $Y_j \in M_{n,d_j}$ $(1 \le j \le r)$ are matrices of independent variables. Let P and Q be the polynomial maps in Theorem 2 and assume that $P \in \mathscr{H}^{inv}$. The differential operator $\mathscr{D}$ is defined by $\mathscr{D} = Q\left(\frac{\partial}{\partial Z_1}, \cdots, \frac{\partial}{\partial Z_r}\right)$. Then, the commutation relation for the elements of the form $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in Sp(n, \mathbb{R})$ is obvious as before. As for the inversion $\rho(Z) = -Z^{-1}$ $(Z \in H_n)$, we get by Lemma 1 that

$$\mathscr{D}(F_1|_{k_1,\cdots,k_r}[\rho]) = i^{nd/2}(2\pi)^{-nd/2} \int_{M_{n,d}} e^{iTr({}^tXY)} (\mathscr{D}F_1)(Z_1, \cdots, Z_r)dX$$

$$= i^{nd/2}(2\pi)^{-nd/2} \left(\frac{i}{2}\right)^{\deg Q}$$

$$\times \int_{M_{n,d}} e^{iTr({}^tXY)} Q(X_1^tX_1, \cdots, X_r^tX_r) F_1(Z_1, \cdots, Z_r)dX .$$

Denothing the variable on H_n by Z, we get $F_1(Z, \cdots, Z) = \exp(\frac{i}{2}Tr({}^tYZY))$, where we put $Y = (Y_1, \cdots, Y_r)$. Applying Lemma 1 to $P(X) = Q(X_1^tX_1, \cdots, X_r^tX_r)$, we get

$$Res_{\Delta}(\mathscr{D}(F_1|_{k_1,\cdots,k_r}[\rho]))$$
$$=i^{nd/2}\left(\frac{i}{2}\right)^{\deg Q} P(-Z^{-1}Y_1,\cdots,-Z^{-1}Y_r)(\det Z/i)^{-\frac{d}{2}}F_1(-Z^{-1},\cdots,-Z^{-1})$$
$$=\left(\frac{i}{2}\right)^{\deg P}\det(-Z)^{-\frac{d}{2}}(\tau(-Z^{-1}))P(Y_1,\cdots,Y_r)F_1(-Z^{-1},\cdots,-Z^{-1})$$
$$=\det(-Z)^{-\frac{d}{2}}(\tau(-Z^{-1}))(\mathscr{D}F_1)(-Z^{-1},\cdots,-Z^{-1}).$$

On the other hand, $\mathscr{D}_{\boldsymbol{\mu}}F_1$ ($\boldsymbol{\mu}$: multi-indices) are linearly independent over $\mathbb{C}$. Hence, for any C^{∞} function F on H_n, we get

$$Res_{\Delta}(\mathscr{D}(F|_{k_1,\cdots,k_r}[\rho]))=\det(-Z)^{-\frac{d}{2}}\tau(-Z^{-1})\mathscr{D}F(-Z^{-1},\cdots,-Z^{-1}).$$

The proof of the converse is almost similar to that of Theorem 1, or even easier, if we use F_1. So, we omit the details here. Hence, Theorem 2 is proved.

3. Examples of invariant pluri-harmonic polynomials

In sections 1 and 2, we have shown that "invariant" pluri-harmonic polynomials give the differential operators which satisfy the commutation relation and vice versa. Here remains two problems.

(1) When do there exist "invariant" pluri-harmonic polynomials?

(2) How can one describe the "invariant" pluri-harmonic polynomials explicitly?

In this section, we give partial answers to these questions.

3.1. Case I

3.1.1. The case $r=2$

Assume that representation λ_1 or λ_2 of $O(d)$ appear in $\mathscr{H}_{n_1,d}$ or $\mathscr{H}_{n_2,d}$, respectively. We get $\mathscr{H}_{n_1,n_2}^{inv}(\lambda_1\otimes\lambda_2)\neq\varnothing$, if and only if λ_1 is equivalent to λ_2, and besides, $\dim\mathscr{H}_{n_1,n_2}^{inv}(\lambda\otimes\lambda)=1$ in this case. This fact is easily shown by Schur's Lemma.

The case $n=2$, $n_1=n_2=1$ In this case, if $d\geq 3$, then $\mathscr{H}^{inv}\neq 0$ if and only if $\lambda_1=\lambda_2=$ the ν-th spherical representation ρ_ν, that is, the representation of $O(d)$ whose representation space is the space of harmonic polynomials of d variables of homogeneous degree ν. Now, for vectors $x, y\in\mathbb{R}^d$, denote by (x, y) the usual inner product and put $n(x)=(x, x)$, $n(y)=(y, y)$. Define the homogeneous polynomials $G_d^\nu(s, m)$ of degree 2ν of 2 variables as follows:

$$\frac{1}{(1-2st+mt^2)^{\frac{(d-2)}{2}}}=\sum_{\nu=0}^{\infty}G_d^\nu(s, m)t^\nu.$$

The polynomials $G_d^\nu(s, 1)$ are called the Gegenbauer polynomials. In particular, when $d=3$, those polynomials are the Legendre polynomials $P_\nu(s)$. For each ν and each d, the polynomial $G_d^\nu((x, y), n(x)n(y))$ of total degree 2ν of $2d$ variables gives the basis of one dimensional space $\mathscr{H}_{1,1}(\rho_\nu\otimes\rho_\nu)$. So, if F is a Siegel modular form of degree 2 of weight $k=d/2$, where we assume d is even, then $\left(G_{2k}^{\nu}\left(\frac{1}{2}\frac{\partial}{\partial z_{12}}, \frac{\partial^2}{\partial z_{11}\partial z_{22}}\right)F(Z)\right)_{z_{12}=0}$ is in the space of tensor products of modular forms of one variables of weight $k+\nu$. This fact was essentially known by Eichler and Zagier (cf. [7] p. 28.)

Spherical case When n, n_1 and n_2 are general, then $\mathcal{H}^{inv}\neq 0$ for the representation λ which is not necessarily spherical. But, in the case that $\lambda=\rho_{2l}$ for some l, the description of $\mathcal{H}_{n_1,n_2}^{inv}(\rho_{2l}\otimes\rho_{2l})$ is easy. In fact, let u_i $(1\leq i\leq n)$ be independent variables. For each i with $1\leq i\leq n$, denote by x_i a d-dimensional variable vector and put $X={}^t(x_1, \cdots, x_n)$. For each multi-index $m=(m_1, \cdots, m_n)$, denote by $P_m(X)$ the coefficient of $\prod_{i=1}^n u_i^{m_i}$ of the polynomial

$$G_d^{\nu}\left(\left(\sum_{i=1}^{n_1} u_i x_i, \sum_{j=n_1+1}^{n} u_j x_j\right), \quad n\left(\sum_{i=1}^{n_1} u_i x_i\right) n\left(\sum_{j=n_1+1}^{n} u_j x_j\right)\right).$$

Then, $P(X)=(P_m(X))_m$ is a pluri-harmonic polynomial map which gives the basis of $\mathcal{H}_{n_1,n_2}^{inv}(\rho_{2l}\otimes\rho_{2l})$. In this case, for Siegel modular forms on H_n of weight k, $\mathcal{D}F$ gives an automorphic form on $H_{n_1}\times H_{n_2}$ of weight $(\det^k Sym_{n_1}^{2l})\otimes(\det^k Sym_{n_2}^{2l})$, where $Sym_{n_i}^{2l}$ is the symmetric tensor representation of GL_{n_i} for each $i=1$, or 2. This operator is the one which appears in Böcherer-Satoh-Yamazaki [2].

The case $n=4$, $n_1=n_2=2$ We assume that $d\geq 4$. We treat the case $\tau(\lambda)=\det^{\nu}$. Then we get operators to map Siegel modular form F on H_4 of weight k to a modular form $Res_{\Delta}(\mathcal{D}F)$ on $H_2\times H_2$ of homogeneous weight $k+\nu$. This corresponds to the case $\lambda=(\nu, \nu, 0, \cdots, 0)_+$ in Kashiwara-Vergne's notation ([9] p. 27), where the notation $(\nu, \nu, 0, \cdots, 0)$ is regarded as the Young diagram of $SO(d)$ which corresponds to the restriction of the representation λ of $O(d)$ to $SO(d)$. For the sake of simplicity, we write $\mathcal{H}_{2,2}^{inv}(\lambda\otimes\lambda)=\mathcal{H}^{inv}(\nu)$. We have already explained that $\dim\mathcal{H}^{inv}(\nu)=1$. For each non-negative integer ν, the space $\mathcal{H}^{inv}(\nu)$ consists of ($\mathbb{C}$-valued) pluri-harmonic polynomials $P(X, Y)$ of $4d$-variables $(X, Y\in M_{2,d})$ such that $P(AX, BY)=\det(A)^{\nu}\det(B)^{\nu}P(X, Y)$ and that $P(Xg, Yg)=P(X, Y)$ for each $g\in O(d)$. Now, we describe the basis of $\mathcal{H}^{inv}(\nu)$. Put

$$f_1=f_1(X, Y)=\det(X^tY),$$
$$f_2=f_2(X, Y)=\det(X^tX)\det(Y^tY),$$
$$f_3=f_3(X, Y)=\det\begin{pmatrix} X^tX & X^tY \\ Y^tX & Y^tY\end{pmatrix}.$$

Let t be a variable and put

$$\Delta_0(X, Y, t)=1-2f_1t+f_2t^2, \quad \text{and}$$

$$R(X, Y, t)=(\Delta_0(X, Y, t)+\sqrt{\Delta_0(X, Y, t)^2-4f_3t^2})/2.$$

Then, the each coefficient $G_\nu(X, Y)$ of t^ν of the following formal power series

$$G(X, Y, t)=\sum_{\nu=0}^{\infty} G_\nu(X, Y)t^\nu=\frac{1}{R(X, Y, t)^{\frac{d-5}{2}}\sqrt{\Delta_0(X, Y, t)^2-4f_3t^2}}$$

gives the basis of $\mathscr{H}^{inv}(v)$. For example,

$$G_1(X, Y)=(d-3)\det(X^t Y)$$

$$G_2(X, Y)=\frac{(d-1)(d-3)}{2}f_1^2-\frac{d-3}{2}f_2+\frac{d-1}{2}f_3$$

Since the calculation is slightly complicated, we sketch the proof. We must show that $G(X, Y, t)$ is pluri-harmonic for each X and Y. If we write $X^tX=R=(r_{ij})$, $Y^tY=S=(s_{ij})$ and $X^tY=T=(t_{ij})$, then as for X, we get

$$\begin{aligned}\Delta_{11}=&2d\frac{\partial}{\partial r_{11}}+4r_{11}\frac{\partial^2}{\partial r_{11}^2}+4r_{12}\frac{\partial^2}{\partial r_{11}\partial r_{12}}+r_{22}\frac{\partial^2}{\partial r_{12}^2}\\&+s_{11}\frac{\partial^2}{\partial t_{11}^2}+2s_{12}\frac{\partial^2}{\partial t_{11}\partial t_{12}}+s_{22}\frac{\partial^2}{\partial t_{12}^2}\\&+4t_{11}\frac{\partial^2}{\partial r_{11}\partial t_{11}}+4r_{12}\frac{\partial^2}{\partial r_{11}\partial t_{12}}+2t_{21}\frac{\partial^2}{\partial r_{12}\partial t_{11}}+2t_{22}\frac{\partial^2}{\partial r_{12}\partial t_{12}}.\end{aligned}$$

Since it is complicated to show the harmonicity of $G(X, Y, t)$ directly, we use the following formula of power series expansion.

$$\frac{1}{\sqrt{1-4t}\,((1+\sqrt{1-4t})/2)^s}=\sum_{n=1}^{\infty}\binom{2n+s}{n}t^n$$

To apply this in our case, we can show the following identity.

$$\begin{aligned}&\Delta_{11}\left(\frac{f_3^n}{\Delta_0(X, Y, t)^{\frac{d-3}{2}+2n}}\right)\\&=(t_{22}^2s_{11}-2t_{22}t_{21}s_{12}+t_{12}^2s_{22}-r_{22}\det(S))\\&\times\left((4n+d-3)(4n+d-1)\frac{f_3^nX^2}{\Delta_0(X, Y, t)^{\frac{d-3}{2}+2n+2}}-n(4n+2d-10)\frac{f_3^{n-1}}{\Delta_0(X, Y, t)^{\frac{d-3}{2}+2n}}\right),\end{aligned}$$

Taking both formula above into account, we see $\Delta_{11}(G(X, Y, t))=0$. In a similar way, we can show $\Delta_{ij}(G(X, Y, t))=0$ and the pluri-harmonicity of $G(X, Y, t)$ for Y follows from the symmetry.

More generally, in the case $r=2$ and $n=2n_1=2n_2$, the polynomial $\det(X^tY)$ gives always an invariant harmonic polynomial corresponding to $\tau(g)=\det(g)$. As for this case, some related and a little more complicated operators were also introduced by Böcherer [1] in fairly different formulation.

Next, we treat the case where $n=4$, $n_1=n_2=2$ and $\lambda=(2, 1, 0, \cdots, 0)$. In this case, the base of $\mathscr{H}^{inv}$ is given by the vector ${}^t(v_1, v_2)$, where

$$v_1 = d(x_1, y_1)\det(X^t Y) + \begin{vmatrix} n(x_1) & (x_1, y_1) & (x_1, y_2) \\ (x_1, x_2) & (x_2, y_1) & (x_2, y_2) \\ (y_1, x_1) & n(y_1) & (y_1, y_2) \end{vmatrix}$$

$$v_2 = d(x_2, y_2)\det(X^t Y) + \begin{vmatrix} n(x_2) & (x_2, y_2) & (x_2, y_1) \\ (x_1, x_2) & (x_1, y_2) & (x_1, y_1) \\ (y_2, x_2) & n(y_2) & (y_1, y_2) \end{vmatrix}$$

The weight of $Res_\Delta(\mathscr{D}F)$ is $(\det \cdot Sym^1) \otimes (\det \cdot Sym^1)$.

3.1.2. The case $r=3$

$n=3$, $n_1=n_2=n_3=1$ This case is treated in [8] in detail. Here, λ_1, λ_2, λ_3 are automatically spherical, if we assume $\mathscr{H}^{inv} \neq 0$. Hence, denote them by ρ_{a_1}, ρ_{a_2} and ρ_{a_3}. Then $\mathscr{H}(\rho_{a_1}, \rho_{a_2}, \rho_{q_3}) \neq \emptyset$, if and only if $a_1 = v_2 + v_3$, $a_2 = v_3 + v_1$, $a_3 = v_1 + v_2$ for some non-negative integers v_1, v_2, v_3. Besides, under this condition, $\dim \mathscr{H}(\rho_{a_1}, \rho_{a_2}, \rho_{a_3}) = 1$. As for the generating function of the bases and many other properties, see [8].

3.2. The case II

Spherical representation Take a spherical representation ρ_v of $SO(d)$. When $d_1 + \cdots + d_r = d$, the restriction $\mathrm{Res}\,\rho_v$ of ρ_v to $SO(d_1) \times \cdots \times S(d_r)$ contains the trivial representation as an irreducible component, if and only if v is even. If $v = 2l$ for some integer l, then the multiplicity of the trivial representation of $\mathrm{Res}\,\rho_v$ is equal to $\binom{r+l-2}{l}$. In particular, if $r=2$, then the multiplicity is one. If $n=1$ and $r=2$, then for modular forms f of weight k_1 and g of weight k_2 on H_1, our operators $Res_\Delta \mathscr{D}(f(z_1)f(z_2))$ are nothing but Cohen's operators in [5]. When $n=2$ and $l=2$, our operators are Satoh's operators in [12]. The case where n, l general and $r=2$ was written in Eholzer and Ibukiyama [6].

We assume that $n=1$ now. Each element of $\mathscr{H}^{inv}(d_1, d_2; \rho_v)$ is a harmonic polynomial P of $x \in \mathbb{R}^d$, and a function of $m_1 = n(x_1), \cdots, m_r = n(x_r)$, where $x_i \in \mathbb{R}^{d_i}$ $(1 \le i \le r)$ and $x = {}^t(x_1, \cdots, x_r)$. Hence, writing the condition $\Delta P = 0$ by the variables m_i $(1 \le i \le r)$, we get the following differential equations:

$$\frac{1}{2}\Delta f = \sum_{i=1}^{r} d_i \frac{\partial f}{\partial m_i} + 2\sum_{i=1}^{r} m_i \frac{\partial^2 f}{\partial m_i^2} = 0$$

Take a solution $f(m_1, \cdots, m_r)$ of the homogeneous degree l with respect to $m_1, \cdots, m_r$ and write it by the coefficients as follows:

$$f(m_1, \cdots, m_r) = \sum_{\mu_1 + \cdots + \mu_r = l} C(\mu_1, \cdots, \mu_r) m_1^{\mu_1} \cdots m_r^{\mu_r}.$$

Then, we have the following difference equation:

$$\begin{aligned}(1+\mu_r)(d_r + 2\mu_r)&C(\mu_1, \cdots, \mu_{r-1}, \mu_r + 1) \\ &= -\sum_{i=1}^{r-1} (d_i + 2\mu_i) C(\mu_1, \cdots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \cdots, \mu_r)\end{aligned}$$

When $r=2$, by the above relation, we get Cohen's operator explicitly.

The case $r=2$ and n general After this work had been finished, an explicit formula for generalized Cohen's operator was considered by Choie and Eholzer [4] in case $n=2$ for $\mathbb{C}$-valued Siegel modular forms of degree 2. This was generalized to the case of general n by Eholzer and Ibukiyama [6]. So, we would like to add some explanation on this. We take the representation (τ, λ) of $GL(n)\times O(2k)$ on the pluriharmonic polynomials such that $\tau(g)=\det(g)^v$. In our setting, this case corresponds to the differential operators on the product of two $\mathbb{C}$-valued Siegel modular forms of weight k_1 and k_2 of degree n to $\mathbb{C}$-valued Siegel modular forms of weight k_1+k_2+v. We consider the restriction of this representation (τ, λ) to $\{1\}\times(O(d_1)\times O(d_2))$. Under the assumption that $d_1\geq n$ and $d_2\geq n$, the multiplicity of the trivial representation in this restriction is one. Hence using this invariant pluri-harmonic polynomial, we can construct the desired differential operators. If $d_1, d_2\geq 2n$, the assertion for multiplicity is almost a direct consequence of Koike and Terada [10] p. 115 Corollary 2.6. The details was given in [6], as well as the method to get explicit operators in this case.

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