LIFTING CONJECTURES FROM VECTOR VALUED SIEGEL MODULAR FORMS OF DEGREE TWO

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Abstract. We give two kinds of conjecture on liftings from vector valued Siegel modular forms of degree two to Siegel modular forms of higher degrees with respect to the full Siegel modular groups. The degree four case of our conjectures answers to the *lifting puzzle* posed by Ryan, Poor, and Yuen on the liftings to degree 4, and our conjectures are based on the numerical examples which fit their concrete examples of Euler 2 factors, as well as a coincidence of the gamma factors and existence of consistent homomorphisms between L groups. The other reasons for the conjectures are that the first one contains the Ikeda lift as a special case, and the second one contains so called Ikeda-Miyawaki lift as a special case.

1. Introduction

Ryan, Poor and Yuen calculated in their paper [16] all Euler 2 factors of the standard L functions of Siegel cusp eigenforms of degree 4 of weight 16 of level 1. There are 7 eigenforms denoted by h_1 to h_7 and they claimed all seem to be a kind of liftings judging from the Euler 2 factors. Among them, they explained four forms by known liftings and left the remaining three forms as a problem. These three forms are divided into two groups $\{h_3, h_4\}$ and $\{h_7\}$ from their shapes of Euler factors, and they suggested that both groups seem to be unknown different types of liftings from somewhere. They call this problem a lifting puzzle. In Luminy conference in May 2011, Anton Mellit told the author an idea to explain the second group by taking a vector valued Siegel modular form of degree two of some weight. After the conference, the author checked the idea of Mellit by giving an explicit numerical example, and made a further guess and gave an example to explain not only the second but also the first group, then based on these examples, state two different conjectures on liftings from vector valued Siegel modular forms of degree two to general degree. By exchanging this information by emails with some attendants of Luminy conference, the author learned that Jonas Bergström together with Martin Raum had already guessed equality (2.6) in the case where F equals h_3 or h_4

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by considering the corresponding conjectural motives and their Hodge structures, and they had checked this equality numerically for $p = 2$. The numerical calculation was based on work by Faber and van der Geer on the cohomology of local system of the moduli space of abelian surfaces, and this calculation is not the same as the one given here by explicit constructions of vector valued Siegel modular forms.

In this paper, we propose two kinds of general conjecture, one is from vector valued Siegel modular forms of weight $\det^{n+2} Sym(2m-3n-2)$ of degree 2 to Siegel modular forms of weight m of degree $2n$ for even natural numbers n and m, and the other is from a pair of weight $2m-2n$ of degree one and weight $\det^{m-2n+2} Sym(2n-2)$ of degree two, to weight m of degree $2n$ for even m. In both cases, conjectural relations of L functions are explicitly given. Each of which explains each group of the lifting puzzles of Ryan, Poor and Yuen as a special case. Evidence of these conjectures are

(1) Each conjecture fits each numerical example in [16] for degree 4.
(2) The gamma factors coincide.
(3) The former conjecture includes the Ikeda lift of degree $2n$ with even n and the other includes a part of the Ikeda-Miyawaki lift.
(4) There exist consistent homomorphisms between L groups.

It seems that the above conjectures can be regarded as a part of Arthur conjectures in [3]. Panchishkin also gave some conjectures on general lifting in his paper [15], but he treated only scalar valued cases and also the shapes of the L functions seem different from ours. After reviewing notation and definitions in section 2.1, we state our conjectures for degree 4 in section 2.2, give numerical examples in section 2.3, state our conjectures for general degree in section 3.1 with explanation of the above evidence (2) and (3) in section 3.2, 3.3. In section 3.4, we explain L-group morphisms in the Langlands conjecture compatible to our conjectural liftings. In the Appendix, we shortly explain explicit relations between the standard L functions and the spinor L functions for small degree.

2. Conjectures for degree four and numerical examples

2.1. **Notation and definitions.** We denote by $\mathrm{Sp}(n,\mathbb{R})$ the real symplectic group of size $2n$ and by Γ_n the full Siegel modular group of degree n defined by $\Gamma_n = \mathrm{Sp}(n,\mathbb{R}) \cap M_{2n}(\mathbb{Z})$. We denote by $\mathfrak{H}_n$ the Siegel upper half space of degree n. For any natural number k, we denote by $A_k(\Gamma_n)$ or $S_k(\Gamma_n)$ the space of Siegel modular forms, or Siegel cusp forms of Γ_n of weight k, respectively. We denote by $A_{k,j}(\Gamma_2)$ or $S_{k,j}(\Gamma_2)$ the space of vector valued Siegel modular forms, or Siegel cusp forms, of weight $\det^k \mathrm{Sym}(j)$ of Γ_2. More concretely, an element of $A_{k,j}(\Gamma_2)$ is a homogeneous polynomial $F(Z,u) = \sum_{i=0}^{j} f_i(Z) u_1^{j-i} u_2^j$ in variables u_1, u_2 with coefficients $f_i(Z)$ in holomorphic functions of $Z \in \mathfrak{H}_2$ such

that
$$F(\gamma Z, u) = \det(cZ+d)^k F(Z, u(cZ+d))$$
for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_2$. We note that $A_{k,j}(\Gamma_2) = \{0\}$ if j is odd and $A_{k,j}(\Gamma_2) = S_{k,j}(\Gamma_2)$ if k is odd. For a Hecke eigenform $f = \sum_{n=0}^{\infty} a(n)q^n \in A_k(\Gamma_1)$ with $a(1)=1$, we denote by $L(s,f)$ the classical Hecke L function defined by $\sum_{n=1}^{\infty} a(n)n^{-s}$. For a Siegel modular form F of degree n, we denote by $L(s,F,\mathrm{Sp})$ for $n \geq 2$ or $L(s,F,\mathrm{St})$ for $n \geq 1$ the spinor, or the standard L function of F. For the sake of simplicity, we assume that these L functions are normalized so that the functional equations (conjectural in the spinor case) is for $s \to 1-s$. This is different from the classical or geometrically natural setting of Andrianov, but since we will use several different L functions in this paper, this unified normalization seems suitable to avoid confusion. If we write Satake parameters at a prime p of a Siegel modular form F of any weight of degree n by $\alpha_{0,p}$, $\alpha_{1,p}$, …, $\alpha_{n,p}$, then both L functions are defined by $L(s,F,\mathrm{St}) = \prod_p H_p(F,\mathrm{St})^{-1}$ and $L(s,F,\mathrm{Sp}) = \prod_p H_p(F,\mathrm{Sp})^{-1}$ where the Euler p-factors $H_p(s,\mathrm{St})$ and $H_p(s,\mathrm{Sp})$ are defined by

$$H_p(F,\mathrm{St}) = (1-p^{-s})\prod_{i=1}^{n}(1-\alpha_{i,p}p^{-s})(1-\alpha_{i,p}^{-1}p^{-s}),$$
$$H_p(F,\mathrm{Sp}) = (1-\alpha_{0,p}p^{-s})\prod_{r=1}^{n}\prod_{1\leq i_1<i_2<\ldots<i_r\leq n}(1-\alpha_{0,p}\alpha_{i_1,p}\cdots\alpha_{i_r,p}p^{-s}).$$

In our setting, we have $\alpha_{0,p}^2\alpha_{1,p}\cdots\alpha_{n,p} = 1$ (cf. Asgari and Schmid [4].) If F is an eigenform of the Hecke operators in $GSp(n,\mathbb{Q}) \cap M_{2n}(\mathbb{Z})$, then it is also a Hecke eigenform of $Sp(n,\mathbb{Q}) \cap M_{2n}(\mathbb{Z})$ and both L functions $L(s,F,\mathrm{Sp})$ and $L(s,F,\mathrm{St})$ are defined. But $L(s,F,\mathrm{Sp})$ is not determined by $L(s,F,\mathrm{St})$ in general and there remains sign ambiguity. We will explain this relation in the Appendix (section 4).

We review the classical style definition of the Spinor L function when the degree is two. For any natural number l, we define the Hecke operator $T(l)$ as a formal sum of the Γ_2 double cosets in the set $\{g \in M_4(\mathbb{Z}) : {}^t g J_2 g = l J_2\}$, where $J_2 = \left(\begin{smallmatrix} 0 & -1_2 \\ 1_2 & 0 \end{smallmatrix}\right)$. For any Siegel modular form $F(Z,u) \in A_{k,j}(\Gamma_2)$, the action of the Hecke operator $T(l)$ is defined by

$$F|_{k,j}T(l) = l^{2k+j-3}\sum_{i=1}^{d}\det(c_iZ+d_i)^{-k}F(g_iZ, u(c_iZ+d_i)^{-1}),$$

where $T(l) = \cup_{i=1}^{d}\Gamma_2 g_i$ and $g_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$. For a Hecke eigenform $F \in A_{k,j}(\Gamma_2)$, we denote by $\lambda(l) = \lambda(l,F)$ the eigenvalue of the Hecke operator $T(l)$. We define

$$L^{cls}(s,F,\mathrm{Sp}) = \prod_p H_p^{cls}(F,\mathrm{Sp})^{-1},$$

where

(2.1) $H_p^{cls}(F,\mathrm{Sp}) =$
$1-\lambda(p)p^{-s}+(\lambda(p)^2-\lambda(p^2)-p^{2k+j-4})p^{-2s}-\lambda(p)p^{2k+j-3-3s}+p^{4k+2j-6-4s}.$

Then we have

$$L^{cls}(s,F,\mathrm{Sp}) = L\left(s-\frac{2k+j-3}{2},F,\mathrm{Sp}\right).$$

More generally, for any $F \in A_m(\Gamma_n)$, the *classical* L function is given by $L^{cls}(s,F,\mathrm{Sp}) = L(s-nm/2+n(n+1)/4,F,\mathrm{Sp})$ (cf. [4]).
We give one more definition, a convolution product of L functions. For any eigenform $f \in A_l(\Gamma_1)$ and any prime p, define Satake parameters β_p by

$$L(s,f) = \prod_p (1-\beta_p p^{l-1/2}p^{-s})^{-1}(1-\beta_p^{-1}p^{l-1/2}p^{-s})^{-1}.$$

For example $\beta_p+\beta_p^{-1}$ is $p^{-(l-1)/2}$ times the eigenvalue of f at p. For $g \in A_{k,j}(\Gamma_2)$, denoting by $\alpha_{i,p}$ ($i = 0,\ 1,\ 2$) the Satake parameters of g as before, we define the convolution product L function of f and g by

$$(2.2)\quad L(s,f\otimes g) = \prod_p \prod_{i=1,-1}(1-\beta_p^i\alpha_{0,p}p^{-s})^{-1}(1-\beta_p^i\alpha_{0.p}\alpha_{1,p}p^{-s})^{-1}$$
$$\times(1-\beta_p^i\alpha_{0,p}\alpha_{2,p}p^{-s})^{-1}(1-\beta_p^i\alpha_{0,p}\alpha_{1,p}\alpha_{2,p}p^{-s})^{-1}$$

2.2. **Conjectures.** We give two conjectures for degree 4 case.

Conjecture 2.1. *For any even natural number $m \geq 4$, let f be a vector valued Siegel eigenform in $A_{4,2m-8}(\Gamma_2)$. Then there should exist a Siegel eigenform $F \in A_m(\Gamma_4)$ such that the following relations hold.*

(2.3)
$$L(s,F,\mathrm{St}) = \zeta(s)L(s-\frac{1}{2},f,\mathrm{Sp})L(s+\frac{1}{2},f,\mathrm{Sp}),$$
(2.4)
$$L(s,F,\mathrm{Sp}) = \zeta(s-1)\zeta(s)\zeta(s+1)L(s,f,\mathrm{St})L(s+\frac{1}{2},f,\mathrm{Sp})L(s-\frac{1}{2},f,\mathrm{Sp}).$$

Here we can deduce (2.3) from (2.4) (cf. Appendix). This conjecture is based on the following three reasons.
(1) A numerical example given in the next section. When $m = 16$ and $f \in S_{4,24}(\Gamma_2)$, then $F \in S_{16}(\Gamma_4)$ should be h_7 in [16].
(2) The comparison of the gamma factors and functional equations.
(3) The case when f is a non-cusp form is explained by the Ikeda lift.
The reasons (2) and (3) will be explained in a more general setting in section 3.2 and 3.3.

Conjecture 2.2. *For any even natural number $m > 2$, let g be a vector valued Siegel eigenform in $A_{m-2,2}(\Gamma_2)$ and f an elliptic eigenform in $A_{2m-4}(\Gamma_1)$. Then there should exist a Siegel eigenform $F \in A_m(\Gamma_4)$ such that*

$$L(s, F, \mathrm{St}) = L(s + m - 2, f)L(s + m - 3, f)L(s, g, \mathrm{St}), \tag{2.5}$$

$$L(s, F, \mathrm{Sp}) = L(s - \frac{1}{2}, g, \mathrm{Sp})L(s + \frac{1}{2}, g, \mathrm{Sp})L(s, f \otimes g). \tag{2.6}$$

Here we have similar reasons as in Conjecture 2.1. For example, when $m = 16$, $g \in A_{14,2}(\Gamma_2)$ and $f \in A_{28}(\Gamma_1)$, we should have $F = h_3$ or h_4, according to the choice of f since $\dim A_{28}(\Gamma_1) = 2$. This will be explained in the next section. The coincidence of the gamma factors in the general case and the relation in the case of non-cuspidal g to the Ikeda-Miyawaki lift will be explained in 3.2 and 3.3.

2.3. **Numerical examples.** Here we calculate the Euler two factors of our conjectures for the cases which should correspond with the examples in [16]. These Euler 2 factors have been also calculated by J. Bergström based on work by Faber and van der Geer, but the method here is independent and to give modular forms in $A_{4,24}(\Gamma_2)$ or $A_{14,2}(\Gamma_2)$ more directly by theta functions.

We define the inner product (u, v) of $u = (u_i)$, $v = (v_i) \in \mathbb{C}^8$ by $(u, v) = \sum_{i=1}^{8} u_i v_i$ and put $\mathrm{n}(u) = (u, u)$. We denote by E_8 the even unimodular lattice of rank 8 in $\mathbb{Q}^8$ which is unique up to isometry. We write the variable $Z \in \mathfrak{H}_2$ as $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}$. For any $v \in \mathbb{C}^8$ with $(v, v) = 0$ and any integer μ, we put

$$\vartheta_{v,\mu}(Z) = \sum_{x,y \in E_8} (x, v)^{24-\mu}(y, v)^{\mu}\exp(\pi i(\mathrm{n}(x)\tau + 2(x, y)z + \mathrm{n}(y)\omega))$$

and

$$\vartheta_v(Z) = \sum_{\mu=0}^{24} \vartheta_{v,\mu}(Z)\binom{24}{\mu} u_1^{24-\mu}u_2^{\mu}.$$

Then this is a vector valued Siegel modular form in $A_{4,24}(\Gamma_2)$. The image of $A_{4,24}(\Gamma_2)$ under the Siegel Φ-operator is two dimensional, spanned by $f\,u_1^{24}$ with $f \in S_{28}(\Gamma_1)$ (cf. [2]). We have

$$\Phi(\vartheta_v) = \left(\sum_{x \in E_8} (x, v)^{24}\exp(\pi i \mathrm{n}(x)\tau)\right) u_1^{24}.$$

Since we would like to have a cusp form, we take a linear combination of theta functions to erase this image. We prepare three vectors a, b, $c \in \mathbb{C}^8$ defined by

$$\begin{array}{rcl} a &=& (1,\ i,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0), \\ b &=& (2,\ i,\ i,\ i,\ i,\ 0,\ 0,\ 0), \\ c &=& (3,\ 2i,\ i,\ i,\ i,\ i,\ i,\ 0), \end{array}$$

where we write $i=\sqrt{-1}$. We put

$$\begin{aligned} f_{4,24}(Z) &= (418770277377874329600000)^{-1} \\ &\quad \times(-1112395251995136\,\vartheta_a(Z)+549963945\,\vartheta_b(Z)-54784\,\vartheta_c(Z)) \end{aligned}$$

Then we see that this is a non-zero cusp form in $S_{4,24}(\Gamma_2)$.

Now when $j>0$, the dimension formula for $S_{k,j}(\Gamma_2)$ is known by Tsushima [18] only for $k>4$, so the formula for $\dim S_{4,24}(\Gamma_2)$ is not in his paper. But there is a very strong evidence that, for $\dim S_{k,j}(\Gamma_2)$ with $k\geq 3$, the same formula as for $k>4$ gives the true dimensions (cf. [9], [10]). In fact, very recently, the dimension formula for $S_{4,j}(\Gamma_2)$ is announced in [5], and the result is as expected. So we have $\dim S_{4,24}(\Gamma_2)=1$ and this means that $f_{4,24}$ is a Hecke eigenform. By using the Fourier coefficients of $f_{4,24}$, we can show

$$\lambda(2,f_{4,24})=5280, \qquad \lambda(4,f_{4,24})=439542784,$$

$$H_2^{cls}(f_{4,24},\mathrm{Sp})=1-5280X-680099840X^2-5280\cdot 2^{29}X^3+2^{58}X^4,$$

$$H_2(f_{4,24},\mathrm{St})=1+53523\cdot 2^{-14}X+2404121\cdot 2^{-19}X^2+53523\cdot 2^{-14}X^3+X^4,$$

where $X=2^{-s}$. Indeed, if we denote by $C(a,c,b)$ the Fourier coefficients of $f_{4,24}$ at $T=\begin{pmatrix} a & b/2 \\ b/2 & c\end{pmatrix}$, then by a computer calculation we have

$$\begin{aligned} C(1,1,1) &= -2652\binom{24}{18}u_1^{18}u_2^{6}-9282\binom{24}{17}u_1^{17}u_2^{7}-15652\binom{24}{16}u_1^{16}u_2^{8} \\ &\quad -14742\binom{24}{15}u_1^{15}u_2^{9}-4368\binom{24}{14}u_1^{14}u_2^{10}+9009\binom{24}{13}u_1^{13}u_2^{11} \\ &\quad +15114\binom{24}{12}u_1^{12}u_2^{12}+9009\binom{24}{11}u_1^{11}u_2^{13}-4368\binom{24}{10}u_1^{10}u_2^{14} \\ &\quad -14742\binom{24}{9}u_1^{9}u_2^{15}-15652\binom{24}{8}u_1^{8}u_2^{16}-9282\binom{24}{7}u_1^{7}u_2^{17} \\ &\quad -2652\binom{24}{6}u_1^{6}u_2^{18}, \end{aligned}$$

$$\begin{aligned} C(2,2,2) &= -14002560\binom{24}{18}u_1^{18}u_2^{6}-49008960\binom{24}{17}u_1^{17}u_2^{7}-82642560\binom{24}{16}u_1^{16}u_2^{8} \\ &\quad -77837760\binom{24}{15}u_1^{15}u_2^{9}-23063040\binom{24}{14}u_1^{14}u_2^{10}+47567520\binom{24}{13}u_1^{13}u_2^{11} \\ &\quad +79801920\binom{24}{12}u_1^{12}u_2^{12}+47567520\binom{24}{11}u_1^{11}u_2^{13}-23063040\binom{24}{10}u_1^{10}u_2^{14} \\ &\quad -77837760\binom{24}{9}u_1^{9}u_2^{15}-82642560\binom{24}{8}u_1^{8}u_2^{16}-49008960\binom{24}{7}u_1^{7}u_2^{17} \\ &\quad -14002560\binom{24}{6}u_1^{6}u_2^{18}, \end{aligned}$$

$$\begin{aligned}
C(4,4,4) = &-1165667463168\binom{24}{18}u_1^{18}u_2^{6} - 4079836121088\binom{24}{17}u_1^{17}u_2^{7}\\
&- 6879723655168\binom{24}{16}u_1^{16}u_2^{8} - 6479739721728\binom{24}{15}u_1^{15}u_2^{9}\\
&- 1919922880512\binom{24}{14}u_1^{14}u_2^{10} + 3959840941056\binom{24}{13}u_1^{13}u_2^{11}\\
&+ 6643249637376\binom{24}{12}u_1^{12}u_2^{12} + 3959840941056\binom{24}{11}u_1^{11}u_2^{13}\\
&- 1919922880512\binom{24}{10}u_1^{10}u_2^{14} - 6479739721728\binom{24}{9}u_1^{9}u_2^{15}\\
&- 6879723655168\binom{24}{8}u_1^{8}u_2^{16} - 4079836121088\binom{24}{7}u_1^{7}u_2^{17}\\
&- 1165667463168\binom{24}{6}u_1^{6}u_2^{18}.
\end{aligned}$$

Denote by $C(p;(a,c,b))$ the Fourier coefficient of $T(p)(f_{4,24})$ at T. Then as written in [9] p. 126, we have

$$\begin{aligned}
C(2;(1,1,1)) &= C(2,2,2),\\
C(4;(1,1,1)) &= C(4,4,4).
\end{aligned}$$

Hence by calculating $C(2,2,2)/C(1,1,1)$ and $C(4,4,4)/C(1,1,1)$, we have our result for $\lambda(2)$ and $\lambda(4)$. Then the Euler 2 factors of the *classical* spinor L function of $f_{4,24}$ is calculated by definition (2.1) and the standard L function is also calculeted by using formulas in the Appendix from this. Besides, by $L(s, h_7, \mathrm{St})$ and the numerical value $T(2) = 230400000$ of h_7 in [16], we can determine $L(s, h_7, \mathrm{Sp})$ as explained in the Appendix, since $T(2)/2^{27}$ is the coefficient of 2^{-s} of the Euler 2 factor of $L(s, h_7, \mathrm{Sp})$. So we see that the Euler two factors of

$$\zeta(s)L(s+\frac{1}{2}, f_{4,24}, \mathrm{Sp})L(s-\frac{1}{2}, f_{4,24}, \mathrm{Sp})$$

and

$$\zeta(s-1)\zeta(s)\zeta(s+1)L(s, f_{4,24}, \mathrm{St})L(s+\frac{1}{2}, f_{4,24}, \mathrm{Sp})L(s-\frac{1}{2}, f_{4,24}, \mathrm{Sp})$$

coincide with those of $L(s, h_7, \mathrm{St})$ and $L(s, h_7, \mathrm{Sp})$ respectively by virtue of numerical data in [16]. Hence we may expect that $f_{4,24}$ is lifted to h_7 in the sense of Conjecture 2.1.

By the way, we have $\dim S_{4,2m-8}(\Gamma_2) = 0$ for all $m < 16$, so it is natural that $m = 16$ is the first example of a lifted cusp form, and this is compatible with our conjecture. By [18] and [5], the generating functions of $\dim A_{4,s}(\Gamma_2)$ and $\dim S_{4,s}(\Gamma_2)$ are given by

$$\sum_{j=0}^{\infty} \dim A_{4,j}(\Gamma_2)s^j = \frac{s^8+s^{12}-s^{18}-s^{20}-s^{22}+s^{28}+s^{30}+s^{32}}{(1-s^6)(1-s^8)(1-s^{10})(1-s^{12})},$$
$$\sum_{j=0}^{\infty} \dim S_{4,j}(\Gamma_2)s^j = \frac{s^{24}(1+s^4+s^8-s^{10})}{(1-s^6)(1-s^8)(1-s^{10})(1-s^{12})}.$$

We give a table of dimensions for small s for reference.

j	8	10	12	14	16	18	20	22	24	26	28	30	32
$\dim A_{4,j}(\Gamma_2)$	1	0	1	1	1	1	2	1	3	2	3	3	5
$\dim S_{4,j}(\Gamma_2)$	0	0	0	0	0	0	0	0	1	0	1	1	2

Since we are assuming that m is even, we have $j = 2m-8 \equiv 0 \bmod 4$ in our conjecture.

Next we consider a numerical example of Conjecture 2.2. By Tsushima [18], we have $\dim S_{14,2}(\Gamma_2) = 1$. By virtue of [17], we can give $S_{k,2}(\Gamma_2)$ explicitly for any k. Indeed, let φ_4 be the Eisenstein series of degree two normalized so that the constant term is one and χ_{10} the unique cusp form of weight 10 of degree two such that the Fourier coefficients at $(1,1,1)$ is 1. We put

$$g_{14,2} = \{\varphi_4, \chi_{10}\}_{Sym(2)} = (2\pi i)^{-1}\left(\left(4\varphi_4\frac{\partial\chi_{10}}{\partial\tau} - 10\chi_{10}\frac{\partial\varphi_4}{\partial\tau}\right)u_1^2 + \left(4\varphi_4\frac{\partial\chi_{10}}{\partial z} - 10\chi_{10}\frac{\partial\varphi_4}{\partial z}\right)u_1u_2 + \left(4\varphi_4\frac{\partial\chi_{10}}{\partial\omega} - 10\chi_{10}\frac{\partial\varphi_4}{\partial\omega}\right)u_2^2\right).$$

Then we have $g_{14,2} \in S_{14,2}(\Gamma_2)$. The Fourier coefficients of $g_{14,2}$ is given by

$$C(1,1,1) = 4(u_1^2+u_1u_2+u_2^2),$$
$$C(2,2,2) = -76800(u_1^2+u_1u_2+u_2^2),$$
$$C(4,4,4) = 141819904(u_1^2+u_1u_2+u_2^2).$$

Hence we have $\lambda(2) = -19200$ and $\lambda(4) = 35454976$ for $g_{14,2}$ and

$$H_2^{cls}(g_{14,2}, \mathrm{Sp}) = 1 + 19200X + 266076160X^2 + 2^{27}\cdot 19200X^3 + 2^{54}X^4$$
$$= 1 + 2^8\cdot 3\cdot 5^2X + 2^{18}\cdot 5\cdot 7\cdot 29X^2 + 2^{35}\cdot 3\cdot 5^2X^3 + 2^{54}X^4,$$

where $X = 2^{-s}$. By this result and the formula in the Appendix we also have

$$H_2(g_{14,2}, \mathrm{St}) = 1 + 9\cdot 2^{-9}X + 1601\cdot 2^{-11}X^2 + 9\cdot 2^{-9}X^3 + X^4.$$

If we denote by $f \in S_{28}(\Gamma_1)$ any eigenform, then we see that the Euler 2 factors of $L(s+13, f)L(s+14, f)L(s, g_{14,2}, \mathrm{St})$ and $L(s-\frac{1}{2}, g_{14,2}, \mathrm{Sp})L(s+\frac{1}{2}, g_{14,2}, \mathrm{Sp})L(s, g_{14,2}\otimes f)$ are equal to those of $L(s, h_i, \mathrm{St})$ and $L(s, h_i, \mathrm{Sp})$ respectively, where $i = 3$ or 4 according to a choice of two eigenforms $f \in S_{28}(\Gamma_1)$. (Note that there is a typo in the formula of $Q_2(h_3, st)$ in p. 68 of [16]. The number 2^{-15} should read 2^{-13} there.) Here the Euler 2 factor of the spinor L function of h_i for $i = 3$ or 4 can be calculated by the numerical value $T(2) = -230400(1703 \pm 9\sqrt{18209})$ in [16] and the Euler 2 factors of the standard L function of h_i. So we see that h_3 or h_4 should be a lift from a pair $\{g_{14,2}, f\}$ in the sense of Conjecture 2.2. Although we do not need explicit numerical values of the convolution product L function defined by (2.2) in order to see the equality to the spinor L functions of h_3 or h_4, we give the Euler 2 factor for the sake of completeness. For a primitive form $f \in S_{28}(\Gamma_1)$ such that the eigenvalue at 2 is $-4140 + 108\beta$ where $\beta = \sqrt{18209}$ or $\beta = -\sqrt{18209}$, the Euler 2 factor of $L(s, f \otimes g_{14,2})$ is given by

$$\begin{aligned}
&1 - 675(115 - 3\beta) \cdot 2^{-17}x + 25(186567511 - 1134567\beta) \cdot 2^{-31}x^2 \\
&-675(930299515-5963979\beta)\cdot 2^{-38}x^3+3(5940783667003-42329083275\beta)2^{-43}x^4 \\
&-675(930299515-5963979\beta)\cdot 2^{-38}x^5+25(186567511-1134567\beta)\cdot 2^{-31}x^6 \\
&\qquad - 675(115 - 3\beta)2^{-17}x^7 + x^8,
\end{aligned}$$

where $x = 2^{-s}$. For Conjecture 2.2, we also have $\dim S_{k,2}(\Gamma_2) = 0$ for $k < 14$ and weight 16 of degree 4 should be the first example of the lift. This fits the conjecture. In our conjecture we are assuming that $k = m - 2$ is even. For reference we reproduce generating functions of dimensions in [17].

$$\sum_{k:even}^{\infty} \dim A_{k,2}(\Gamma_2)t^k = \frac{t^{10} + t^{14} + 2t^{16} + t^{18} - t^{20} - t^{26} - t^{28} + t^{32}}{(1 - t^4)(1 - t^6)(1 - t^{10})(1 - t^{12})}$$

$$\sum_{k:even}^{\infty} \dim S_{k,2}(\Gamma_2)t^k = \frac{t^{14} + 2t^{16} + t^{18} + t^{22} - t^{26} - t^{28}}{(1 - t^4)(1 - t^6)(1 - t^{10})(1 - t^{12})}$$

3. Conjectures for general degree

3.1. **Conjectures.** We generalize the conjectures in the previous section to a lift to higher degrees. We describe only the standard L functions since the spinor L functions should be much more complicated.

Conjecture 3.1. *Let n and m be any even natural numbers with $2m \geq 3n + 2$ and $n \geq 2$. For any vector valued Hecke eigenform $f \in A_{n+2,2m-3n-2}(\Gamma_2)$, there should exist a scalar valued Siegel eigenform $F \in A_m(\Gamma_{2n})$ such that*

$$L(s, F, \mathrm{St}) = \zeta(s) \prod_{i=1}^{n} L(s + \frac{n+1}{2} - i, f, \mathrm{Sp}).$$

In the classical normalization, the right hand side is given by

$$\zeta(s) \prod_{i=1}^{n} L^{cls}(s + m - i, f, \mathrm{Sp}).$$

This is a generalization of Conjecture 2.1. Note that when n is odd, we have $A_{n+2,2m-3n-2}(\Gamma_2) = 0$ and the conjecture tells nothing.

Conjecture 3.2. *Let n and m be natural numbers and assume that m is even with $m > 2n-2$. For any elliptic eigenform $f \in S_{2m-2n}(\Gamma_1)$ and any vector valued Siegel eigenform $g \in A_{m-2n+2,2n-2}(\Gamma_2)$, there should exist a Siegel eigenform $F \in A_m(\Gamma_{2n})$ such that*

$$L(s,F,\mathrm{St}) = L(s,g,\mathrm{St}) \prod_{i=1}^{2n-2} L(s+m-1-i,f).$$

This is a generalization of Conjecture 2.2. When $n=1$, we can take $F = g$ and f is irrelevant.

Beyond the numerical evidence that we have already given in the previous section, we give two grounds for these conjectures below.

3.2. **Coincidence of gamma factors.** The gamma factor part of $L(s,F,\mathrm{St})$ for Siegel modular form $F \in A_m(\Gamma_{2n})$ is given by

$$\Gamma_{\mathbb{R}}(s) \prod_{i=1}^{2n} \Gamma_{\mathbb{C}}(s+m-i)$$

and for $f \in A_{k,j}(\Gamma_2)$ it is given by

$$\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s+k+j-1)\Gamma_{\mathbb{C}}(s+k-2),$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ (cf. [6] for the scalar valued case and [13] for the vector valued case). The gamma factor of $L^{cls}(s,f,\mathrm{Sp})$ for $f \in A_{k,j}(\Gamma_2)$ is given by $\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k+2)$ (cf. Andrianov [1] and Arakawa [2]), so the gamma factor for $L(s,f,\mathrm{Sp})$ is $\Gamma_{\mathbb{C}}(s+(2k+j-3)/2)\Gamma_{\mathbb{C}}(s+(j+1)/2)$. Now we see the gamma factors of both sides of our conjectures. For Conjecture 3.1, the gamma factor of the right hand side is as follows. We have $\Gamma_{\mathbb{R}}(s)$ for $\zeta(s)$, and since $k = n+2$ and $j = 2m-3n-2$ for $f \in A_{k,j}(\Gamma_2)$ in this conjecture, we have $2k+j-3 = 2m-n-1$ and the gamma factor for $L(s,f,\mathrm{Sp})$ is $\Gamma_{\mathbb{C}}(s+m-(n+1)/2)\Gamma_{\mathbb{C}}(s+m-(3n+1)/2)$. Hence for $\prod_{i=1}^{n} L(s-i+(n+1)/2,f,\mathrm{Sp})$, the gamma factor is $\prod_{i=1}^{n}\Gamma_{\mathbb{C}}(s+m-i)\Gamma_{\mathbb{C}}(s+m-n-i) = \prod_{i=1}^{2n}\Gamma_{\mathbb{C}}(s+m-i)$. Hence the gamma factors for both sides of the conjecture coincide. Next we see the gamma factors of Conjecture 3.2. Here the gamma factor of $L(s,g,\mathrm{St})$ is given by $\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s+m-1)\Gamma_{\mathbb{C}}(s+m-2n)$ and that of $L(s,f)$ is $\Gamma_{\mathbb{C}}(s)$, so the gamma factor of $\prod_{i=1}^{2n-2} L(s+m-1-i,f)$ is $\prod_{i=1}^{2n-2}\Gamma_{\mathbb{C}}(s+m-1-i)$. So the gamma factor of the right hand side is $\Gamma_{\mathbb{R}}(s)\prod_{i=1}^{2n}\Gamma_{\mathbb{C}}(s+m-i)$ which coincides the gamma factor of the left hand side.

3.3. **Relations with the Ikeda lift and the Ikeda-Miyawaki lift.** Ikeda proved a certain generalization of Saito-Kurokawa lift to Siegel modular forms of even degrees in [11]. This was first conjectured by Duke and Imamoglu (and also by the present author independently in terms of Koecher-Maass series). Ikeda also applied this to construct another type of lifts in [12] related with Miyawaki's conjecture. We call these lifts as Ikeda lift and Ikeda-Miyawaki lift and here we study relations between our conjectures and these lifts. We assume that $2m \geq 3n+2$, $n > 2$ and that n and m are even. Then for any primitive form $f_0 \in S_{2m-2n}(\Gamma_1)$, we have the Klingen type Eisenstein series $E(f_0) \in A_{n+2,2m-3n-2}(\Gamma_2)$ such that $E(f_0)$ is a Hecke eigenform with $\Phi(E(f_0)) = f_0$ (cf. [2]). When $n = 2$, the weight in question is $\det^4 \mathrm{Sym}(2m-8)$, and since the power of the determinant is small, the Klingen type Eisenstein series itself might not converge. But since the Siegel Φ operator is surjective to $S_{2m-4}(\Gamma_1)$ even from $A_{4,2m-8}(\Gamma_2)$ (cf. [10]), we have also a Hecke eigenform $E(f_0) \in A_{4,2m-8}(\Gamma_2)$ defined by theta series such that $\Phi(E(f_0)) = f_0$ and we take this in that case. By [19] or [2], we have

$$L^{cls}(s, E(f_0), \mathrm{Sp}) = L(s, f_0)L(s-n, f_0).$$

So we have

$$\begin{aligned} L(s, E(f_0), \mathrm{Sp}) &= L^{cls}\left(s + \frac{2m-n-1}{2}, E(f_0), \mathrm{Sp}\right) \\ &= L\left(s + \frac{2m-n-1}{2}, f_0\right) L\left(s + \frac{2m-3n-1}{2}, f_0\right). \end{aligned}$$

Hence we have

$$\begin{aligned} &\prod_{i=1}^{n} L\left(s + \frac{n+1}{2} - i, E(f_0), \mathrm{Sp}\right) \\ &= \prod_{i=1}^{n} L(s+m-i, f_0)L(s+m-n-i, f_0) = \prod_{i=1}^{2n} L(s+m-i, f_0). \end{aligned}$$

Since we assumed that m is even, and so $m - n \equiv n \bmod 2$, we have the Ikeda lift $I(f_0) \in S_m(\Gamma_{2n})$ of f_0. Then we have $L(s, I(f_0), \mathrm{St}) = \zeta(s)\prod_{i=1}^{2n} L(s+m-i, f_0)$. So this satisfies the relation of the L functions in Conjecture 3.1 for $f = E(f_0)$ and $F = I(f_0)$, and the conjectured lift is nothing but the Ikeda lift in this case. Here f is not a cusp form but F is a cusp form. Note that here we are assuming that n is even and $2n \equiv 0 \bmod 4$, so the Ikeda lifts for odd n are not covered by our Conjecture. By the way, we also see the compatibility with Saito-Kurokawa lift. For any $f_0 \in S_{2k-2}(\Gamma_1)$ with even k, we have the Saito-Kurokawa lift $f \in S_k(\Gamma_2) = S_{k,0}(\Gamma_2)$ and we have $L^{cls}(s, f, \mathrm{Sp}) = \zeta(s-k+1)\zeta(s-k+2)L(s, f_0)$. If we take this f as the origin of our lifting, then we should put $n = k-2$, $m = (3n+2)/2 = 3k/2 - 2$. Since we are assuming that m is even, we should assume here that $k \equiv 0 \bmod 4$. Then we have $(k-1) + (k-2)/2 \equiv 0 \bmod 2$ and this is the parity condition of the existence of the Ikeda lift $I(f_0) \in S_m(\Gamma_{k-2})$ with $L(s, I(f_0), \mathrm{St}) = \zeta(s)\prod_{i=1}^{k-2} L(s+3k/2-2-i, f_0)$. If there is a

Hecke eigenform $F \in A_m(\Gamma_{2k-4})$ such that $\Phi^{k-2}(F) = I(f_0)$, then we have $L(s, F, \mathrm{St}) = L(s, I(f_0), \mathrm{St}) \prod_{i=1}^{k-2} \zeta(s+k/2-i)\zeta(s-k/2+i)$. So we have $L(s, F, \mathrm{St}) = \zeta(s) \prod_{i=1}^{k-2} L^{cls}(s+m-i, f, \mathrm{Sp})$ and F satisfies the demand of the conjecture and the situation is compatible. Since the weight of $I(f_0)$ is small compared with the degree, we cannot construct F by the Klingen Eisenstein series, but if the weight of $I(f_0)$ is 0 mod 4 and it is a theta series, then we can prolong it to F. Note that in this example, f is a cusp form but F is not.

Next we consider Conjecture 3.2. By a part of the results in Ikeda [12], for any primitive forms $g_0 \in S_m(\Gamma_1)$ and $f \in S_{2m-2n}(\Gamma_1)$, we can construct $F_0 = \mathcal{F}_{f,g_0} \in S_m(\Gamma_{2n-1})$ such that if $F_0 \neq 0$, then

$$L(s, F_0, \mathrm{St}) = L(s, g_0, \mathrm{St}) \prod_{i=1}^{2n-2} L(s+m-1-i, f).$$

Here by definition we have

$$L(s, g_0, \mathrm{St}) = \zeta(s) \prod_p (1-(a(p)^2 p^{-m+1}-2)p^{-s}+p^{-2s})^{-1},$$

where $g_0(\tau) = \sum_{n=1}^{\infty} a(n) q^n$ with $q = \exp(2\pi i \tau)$, $\tau \in \mathfrak{H}_1$. For this Ikeda-Miyawaki lift $F_0 = \mathcal{F}_{f,g_0}$, we take the Klingen type Eisenstein series $E(F_0) \in A_m(\Gamma_{2n})$ above F_0 under the assumption that $m > 4n$ for convergence. Then the spinor L function of $E(F_0)$ is by [19] given by

$$L^{cls}(s, E(F_0), \mathrm{Sp}) = L^{cls}(s, F_0, \mathrm{Sp}) L^{cls}(s-m+2n, F_0, \mathrm{Sp}),$$

so we have

$$L(s, E(F_0), \mathrm{St}) = \zeta(s-m+2n)\zeta(s+m-2n) L(s, F_0, \mathrm{St}).$$

On the other hand, if we take the Klingen Eisenstein series $g = E(g_0) \in A_{m-2n+2,2n-2}(\Gamma_2)$ above g_0, then we have

$$L^{cls}(s, g, \mathrm{Sp}) = L(s, g_0) L(s-m+2n, g_0),$$

so

$$L(s, g, \mathrm{St}) = \zeta(s-m+2n)\zeta(s+m-2n) L(s, g_0, \mathrm{St}).$$

So we have

$$\begin{aligned} L(s, E(F_0), \mathrm{St}) &= \zeta(s-m+2n)\zeta(s+m-2n) L(s, g_0, \mathrm{St}) \prod_{i=1}^{2n-2} L(s+m-1-i, f) \\ &= L(s, E(g_0), \mathrm{St}) \prod_{i=1}^{2n-2} L(s+m-1-i, f). \end{aligned}$$

So if we put $F = E(F_0)$, then this satisfies the relation of the L functions in Conjecture 3.2 for the pair f and $g = E(g_0)$. On the contrary, if there is a lift from the pair f and $E(g_0)$ to $F \in A_m(\Gamma_{2n})$ which satisfies the relation in the conjecture and $\Phi(F) = F_0$, then F_0 may be regarded as an Ikeda-Miyawaki lift for f and g_0.

3.4. **Homomorphisms between L-groups.** Langlands philosophy predicts that if there is a homomorphism between L groups of different algebraic groups, then there should exist a correspondence between automorphic representations of both groups(cf. [7]). So in this section we see what are homomorphisms which explain the lifting conjectures above. First we explain a morphism related to Conjecture 3.1. We fix a prime p and denote the Satake parameters of an eigenform $f \in A_{n+2,2m-3n-2}(\Gamma_2)$ at p by α_0, α_1, α_2. Here we note that $\alpha_0^2\alpha_1\alpha_2 = 1$ and the Euler p factor of the Spinor L function of f is given by

$$(1-\alpha_0 p^{-s})^{-1}(1-\alpha_0\alpha_1 p^{-s})^{-1}(1-\alpha_0\alpha_2 p^{-s})^{-1}(1-\alpha_0\alpha_1\alpha_2 p^{-s})^{-1}.$$

Then for a lifting $F \in S_m(\Gamma_{2n})$ from f in Conjecture 3.1, the parameters for the standard L function of F are given by

$$\left\{1, p^r\alpha_0, p^r\alpha_0\alpha_1, p^r\alpha_0^{-1}, p^r(\alpha_0\alpha_1)^{-1}; r \in \left\{\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots, \pm\frac{n-1}{2}\right\}\right\}.$$

Here we assumed that n is even. These parameters are given by the diagonal components of the tensor of $\mathrm{Sym}^{n-1}\begin{pmatrix} p^{1/2} & 0 \\ 0 & p^{-1/2}\end{pmatrix}$ and $\mathrm{diag}(\alpha_0, \alpha_0\alpha_1, \alpha_0\alpha_2, \alpha_0\alpha_1\alpha_2)$, where Sym^{n-1} is the symmetric tensor representation of $\mathrm{SL}_2(\mathbb{C})$ of degree $n-1$ and $\mathrm{diag}(a_i)$ is the diagonal matrix whose diagonal components are a_i. Since n is even, we have an alternating form invariant by $\mathrm{Sym}^{n-1}(\mathrm{SL}_2(\mathbb{C}))$ given by $(u_1v_2 - u_2v_1)^{n-1}$ regarding $u_1^i u_2^{n-1-i}$ and $v_1^j v_2^{n-1-j}$ as bases and taking two copies of this, we have a morphism of $\mathrm{Sym}^{n-1}(\mathrm{SL}_2(\mathbb{C})) \times \mathrm{Sp}(2,\mathbb{C})$ into $\mathrm{SO}(4n)$. More precisely, if we take the metric $h(x,y)$ on $\mathbb{C}^n \times \mathbb{C}^n$ defined by $h(\omega_i, \omega_j) = (-1)^i\binom{n}{i}\delta_{i,n-j}$ where ω_i $(0 \le i \le n-1)$ is a natural basis of $\mathbb{C}^n$ above and δ is the Kronecker delta, then we have $h(\mathrm{Sym}^{n-1}(g)x, \mathrm{Sym}^{n-1}(g)y) = h(x,y)$ for $h \in \mathrm{SL}_2(\mathbb{C})$. If we take the metric $H = h(x_1,y_1) + h(x_2,y_2)$ and define an action of $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2,\mathbb{C})$ on $\mathbb{C}^{4n}$ regarded as four row vectors in $\mathbb{C}^n$ by

$$\begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{pmatrix} = G \times \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix},$$

then we have $h(X_1,Y_1) + h(X_2,Y_2) = h(x_1,y_1) + h(x_2,y_2)$. So regarding H as a quadratic form of $4n$ variables, we have a homomorphism $\mathrm{Sym}^{n-1}(\mathrm{SL}_2(\mathbb{C})) \times \mathrm{Sp}(2,\mathbb{C}) \to \mathrm{SO}(4n)$. Adding the identity, this is prolonged to a mapping to $\mathrm{SO}(4n+1)$ which is the L group of Sp$(2n)$ (of rank $2n$, i.e. of matrix size $4n$). This explains Conjecture 3.1. Next we consider Conjecture 3.2. For any eigenform $f=\sum_{n=1}^{\infty}a(n)q^n\in S_{2m-2n}(\Gamma_1)$, we define the Satake parameter β, β^{-1} of f at p of by

$$\sum_{\nu=0}^{\infty}a(p^{\nu})p^{-\nu s}=(1-\beta p^{(2m-2n-1)/2-s})^{-1}(1-\beta^{-1}p^{(2m-2n-1)/2-s})^{-1}.$$

We write the parameters of an eigenform $g\in A_{m-2n+2,2n-2}(\Gamma_2)$ at a prime p in $SO(5)$ by $\{1,\alpha_1,\alpha_1^{-1},\alpha_2,\alpha_2^{-1}\}$ such that the Euler p-factor of $L(s,g,\mathrm{St})$ is given by

$$(1-p^{-s})^{-1}(1-\alpha_1p^{-s})^{-1}(1-\alpha_1^{-1}p^{-s})^{-1}(1-\alpha_2p^{-s})^{-1}(1-\alpha_2^{-1}p^{-s})^{-1}.$$

Then the parameters for the standard L function of the lifting F in the conjecture 3.2 is given by

$$\left\{1,\alpha_1,\alpha_2,\alpha_1^{-1},\alpha_2^{-1},p^r\beta,p^r\beta^{-1};r\in\left\{\pm\frac{1}{2},\pm\frac{3}{2},\ldots,\pm\frac{2n-3}{2}\right\}\right\}.$$

Again there exists an alternating form of degree $2n-2$ invariant by $\mathrm{Sym}^{2n-3}(\mathrm{SL}_2(\mathbb{C}))$ and in the same way as before, we have a homomorphism

$$\mathrm{SO}(5)\times\mathrm{Sym}^{2n-3}(\mathrm{SL}_2(\mathbb{C}))\times\mathrm{SL}_2(\mathbb{C})\to\mathrm{SO}(5)\times\mathrm{SO}(4n-4)\to\mathrm{SO}(4n+1).$$

So Conjecture 3.2 can be explained by this morphism of the corresponding L groups.

4. Appendix. Spinor L functions and Standard L functions

We give examples of relations between the standard L functions and the spinor L functions for small n for the convenience for the readers. We use L functions which have the functional equation (conjectural for spinor) for $s\to 1-s$. Then for a Siegel eigenform F of degree n (withour character), (the inverse of) the Euler p-factors $H_p(p^{-s},F,\mathrm{St})$ or $H_p(p^{-s},F,\mathrm{Sp})$ of each L functions has the following definition.

$$\begin{aligned}H_p(x,F,\mathrm{St}) &= (1-x)\prod_{i=1}^{n}(1-\alpha_ix)(1-\alpha_i^{-1}x),\\ H_p(x,F,\mathrm{Sp}) &= (1-\alpha_0x)\prod_{r=1}^{n}\prod_{1\le i_1<i_2<\cdots<i_r\le n}(1-\alpha_0\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_r}x),\end{aligned}$$

where α_0, α_i $(1\le i\le n)$ are the Satake parameters of F. In our setting, we have $\alpha_0^2\alpha_1\alpha_2\cdots\alpha_n=1$. This means that the polynomial $H_p(x,F,\mathrm{Sp})$ is reciprocal, i.e. the coefficients of x^i and x^{2^n-i} are the same for any i. Also we see that the coefficient of x^{2i} of $H_p(x,F,\mathrm{Sp})$ is a symmetric polynomial of $\alpha_j+\alpha_j^{-1}$ with $1\le j\le n$ and if we put $b_1=\alpha_0\prod_{i=1}^{n}(1+\alpha_i)$ which is (-1) times the coefficient of x of $H_p(x,F,\mathrm{Sp})$, then for any i, the coefficient of x^{2i+1} is b_1 times a symmetric polynomial of $\alpha_j + \alpha_j^{-1}$. On the other hand, we have

$$b_1^2 = \alpha_0^2 \prod_{i=1}^{n} (1+\alpha_i)^2 = \prod_{i=1}^{n} (\alpha_i + \alpha_i^{-1} + 2),$$

so this is also written by the coefficients of $H_p(x, F, \mathrm{St})$. So $H_p(x, F, \mathrm{Sp})$ is determined by $H_p(x, F, \mathrm{St})$ up to the sign of b_1. It seems that there is no way to predict this sign in general (cf. Ryan, Cris and Yuen [16]). The procedure to express coefficients of $H_p(x, F, \mathrm{Sp})$ by the coefficients of $H_p(x, F, \mathrm{St})$ and b_1 is not completely straight forward but this is an elementary algebra to express symmetric polynomials by fundamental symmetric polynomials. Concrete results for $2 \le n \le 4$ are given below.

4.1. **The case $n = 2$.** We write

$$\begin{aligned} H_p(x, F, \mathrm{Sp}) &= 1 - b_1 x + b_2 x^2 - b_1 x^3 + x^4, \\ H_p(x, F, \mathrm{St}) &= (1-x)(1 - c_1 x + c_2 x^2 - c_1 x^3 + x^4) \end{aligned}$$

for $x = p^{-s}$. Then we have

$$\begin{aligned} b_1^2 &= 2 + 2c_1 + c_2, \\ b_2 &= 2 + c_1. \end{aligned}$$

By the way, if we write the Euler p factor of $L(s, f)$ for a primitive form $f \in S_k(\Gamma_1)$ as $1 - ap^{(k-1)/2-s} + p^{k-1-2s}$, then the Euler p factor of $L(s, f \otimes F)$ defined by (2.2) is given by

$$\begin{aligned} H_p(x, f \otimes F) = 1 - ab_1 x + (b_1^2 + (a^2-2)b_2)x^2 - b_1(a^3 - 3a + ab_2)x^3 \\ + ((a^2-2)^2 + (a^2-2)b_1^2 + b_2^2 - 2)x^4 - b_1(a^3 - 3a + ab_2)x^5 \\ + (b_1^2 + b_2(a^2-2))x^6 - ab_1 x^7 + x^8. \end{aligned}$$

4.2. **The case $n = 3$.** We write

$$\begin{aligned} H_p(x, F, \mathrm{Sp}) &= 1 - b_1 x + b_2 x^2 - b_3 x^3 + b_4 x^4 - b_3 x^5 + b_2 x^6 - b_1 x^7 + x^8, \\ H_p(x, F, \mathrm{St}) &= (1-x)(1 - c_1 x + c_2 x^2 - c_3 x^3 + c_2 x^4 - c_1 x^5 + x^6). \end{aligned}$$

Then we have

$$\begin{aligned} b_1^2 &= 2 + 2c_1 + 2c_2 + c_3, \\ b_2 &= 1 + 2c_1 + c_2, \\ b_3 &= b_1(1 + c_1), \\ b_4 &= 2 + 2c_1 + c_1^2 + c_3. \end{aligned}$$

4.3. **The case $n = 4$.** We write

$$\begin{aligned} H_p(x, F, \mathrm{Sp}) &= 1 - b_1x + b_2x^2 - b_3x^3 + b_4x^4 - b_5x^5 + b_6x^6 - b_7x^7 + b_8x^8 \\ &\quad -b_7x^9 + b_6x^{10} - b_5x^{11} + b_4x^{12} - b_3x^{13} + b_2x^{14} - b_1x^{15} + x^{16}, \\ H_p(x, F, \mathrm{St}) &= (1-x)(1 - c_1x + c_2x^2 - c_3x^3 + c_4x^4 - c_3x^5 + c_2x^6 - c_1x^7 + x^8). \end{aligned}$$

Then we have

$$\begin{aligned} b_1^2 &= 2 + 2c_1 + 2c_2 + 2c_3 + c_4, \\ b_2 &= c_3 + 2c_2 + c_1, \\ b_3 &= b_1(c_2 + c_1 - 1), \\ b_4 &= -2 + 2c_1^2 + c_2^2 + 2c_1c_2 + c_1c_4 + c_4 - 2c_3 - 2c_2, \\ b_5 &= b_1(c_4 + 2c_3 + c_1c_2 + c_1^2 - 3c_3 - c_2 - 1), \\ b_6 &= -c_1 + c_1^3 + 2c_1^2c_2 - 2c_2^2 - c_3 + c_1^2c_3 + c_2c_4, \\ b_7 &= b_1(1 - c_1 + c_1^3 - c_1c_2 + c_3 + c_1c_3 - c_4), \\ b_8 &= 2 - c_1^2 + 2c_1^3 + c_1^4 - 4c_1c_2 - 2c_1^2c_2 + 2c_2^2 + 4c_3 \\ &\quad +2c_1c_3 + 2c_1^2c_3 + c_3^2 - 2c_4 - 2c_1c_4 + c_1^2c_4 - 2c_2c_4. \end{aligned}$$

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