TYPE NUMBERS OF QUATERNION HERMITIAN FORMS AND SUPERSINGULAR ABELIAN VARIETIES

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Abstract. The word *type number* of an algebra means classically the number of isomorphism classes of maximal orders in the algebra, but here we consider quaternion hermitian lattices in a fixed genus and their right orders. Instead of inner isomorphism classes of right orders, we consider isomorphism classes realized by similitudes of the quaternion hermitian forms. The number T of such isomorphism classes are called *type number* or *G-type number*, where G is the group of quaternion hermitian similitudes. We express T in terms of traces of some special Hecke operators. This is a generalization of the result announced in [5] (I) from the principal genus to general lattices. We also apply our result to the number of isomorphism classes of any polarized superspecial abelian varieties which have a model over $\mathbb{F}_p$ such that the polarizations are in a "fixed genus of lattices". This is a generalization of [8] and has an application to the number of components in the supersingular locus which is defined over $\mathbb{F}_p$.

1. Introduction

First we review shortly the classical theory of Deuring and Eichler, and then explain how this will be generalized to quaternion hermitian cases. Let B be a quaternion algebra central over an algebraic number field F and fix a maximal order $\mathfrak{O}$ of B. The class number H of B is the number of equivalence classes of left $\mathfrak{O}$-ideals $\mathfrak{a}$ up to right multiplication by $B^\times$. Any maximal order of B is isomorphic (equivalently $B^\times$-conjugate) to the right order of some left $\mathfrak{O}$-ideal $\mathfrak{a}$, and the number of such isomorphism classes is called the type number T. Obviously $T \leq H$ and the formula for H and T are known by Eichler, Deuring, Peters, and Pizer, as a part of the trace formula for Hecke operators on the adelization $B_A^\times$ (called Brandt matrices traditionally), and also several explicit formulas have been written down (See [1], [3], [2], [12], [13]). Now for a fixed prime p, an elliptic curve E defined over a field of characteristic p is called supersingular if $End(E)$ is a maximal order of a definite quaternion algebra B over $\mathbb{Q}$ with discriminant p. The class number of B is equal to the number of isomorphism classes

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of supersingular elliptic curves E over an algebraically closed field. All such curves E have a model defined over $\mathbb{F}_{p^2}$ and the number of E which have a model over $\mathbb{F}_p$ is known to be equal to $2T - H$ (Deuring [1]). But for $n \geq 2$, the class number of $M_n(B)$ is one if $F = \mathbb{Q}$ by the strong approximation theorem and all the maximal orders of $M_n(B)$ are conjugate to $M_n(\mathfrak{O})$, so there is nothing to ask. Instead, we define G to be the group of similitudes of a quaternion hermitian form, and G_A the adelization. We fix a left $\mathfrak{O}$-lattice L in B^n and consider the G_A-orbit of L in B^n. Such a set of global lattices is called a genus $\mathcal{L}(L)$ determined by L. The number $h(\mathcal{L})$ of G-orbits in $\mathcal{L} = \mathcal{L}(L)$ is called the class number of $\mathcal{L}$ and this is a complicated object. (For some explicit formulas, see [5] (I), (II)). Now take a complete set of representatives of classes $L = L_1, \ldots, L_h$ in $\mathcal{L}(L)$. Define the right order R_i of $M_n(B)$ by

$$R_i = \{g \in M_n(B); L_i g \subset L_i\}.$$

These are maximal orders. We say that R_i and R_j have the same type if $R_i = a^{-1} R_j a$ for some $a \in G$. We denote this relation by $R_i \cong_G R_j$. The number T of types in $\{R_i : 1 \leq i \leq h\}$ is called a type number of $\mathcal{L}(L)$. We give a formula to express T in terms of traces of Hecke operators defined by some two sided ideals of R_1 (Theorem 3.6) under a general setting on F, B, and quaternion hermitian forms.

Now let E be a supersingular elliptic curve defined over $\mathbb{F}_p$. (Such a curve always exists.) The abelian variety $A = E^n$ is called superspecial, and it has a standard principal polarization ϕ_X associated with a divisor $X = \sum_{a+b=n-1} E^a \times \{0\} \times E^b$. For any polarization λ of A, the map $\phi_X^{-1}\lambda$ gives a positive definite quaternion hermitian matrix in $End(A) = M_n(\mathfrak{O})$ for a maximal order $\mathfrak{O}$ of the definite quaternion algebra B over $\mathbb{Q}$ with discriminant p, and we can define a genus $\mathcal{L}(\phi_X^{-1}\lambda)$ of lattices to which $\phi_X^{-1}\lambda$ belongs. We denote by $\mathcal{P}(\lambda)$ the set of polarizations μ of A such that $\phi_X^{-1}\mu \in \mathcal{L}(\phi_X^{-1}\lambda)$. We fix λ and denote the class number and the type number of $\mathcal{L}(\phi_X^{-1}\lambda)$ by H and T respectively. Then the number of isomorphism classes of polarized abelian varieties (E^n, μ) with $\mu \in \mathcal{P}(\lambda)$ is H and the number of those which have models over $\mathbb{F}_p$ is equal to $2T - H$ (Theorem 4.3). As an application, we can show that the number of irreducible components of the supersingular locus $S_{n,1}$ in the moduli of principally polarized abelian varieties $\mathcal{A}_{n,1}$ which have models over $\mathbb{F}_p$ is equal to $2T-H$ where H and T are class numbers and type numbers of the principal genus (resp. the non-principal genus) when n is odd (resp. n is even) (Theorem 4.6).

By the way, for a prime discriminant, an explicit formula for T for the principal genus for $n = 2$ has been given in [8]. The formulas for T for the non-principal genus for $n = 2$ will be given in a separate paper [6]. Together with the formula in [5] (I), (II), an explicit formula for $2T-H$ for $n = 2$ for any genera of maximal lattices will be given there.

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2. Fundamental definitions

We review several fundamental things about quaternion hermitian forms. For the claims without proofs, see [14]. Let F be an algebraic number field which is a finite extension of $\mathbb{Q}$. Let B be any quaternion algebra over F, not necessarily totally definite. For any $\alpha \in B$, we denote by $Tr(\alpha)$ and $N(\alpha)$ the reduced trace and the reduced norm over F, respectively. We denote by $\overline{\alpha}$ the main involution of B over F, so $Tr(\alpha) = \alpha + \overline{\alpha}$, $N(\alpha) = \alpha\overline{\alpha}$. A non-degenerate quaternion hermitian form f on B^n over B is defined to be a map $f : B^n \times B^n \to B$ such that $f(ax + by, z) = af(x, z) + bf(y, z)$ for $a, b \in B$, $\overline{f(y, x)} = f(x, y)$, and $f(x, B^n) = 0$ implies $x = 0$. For any $n_1 \times n_2$ matrix $b = (b_{ij}) \in M_{n_1 n_2}(B)$, we write ${}^t\overline{b} = (\overline{b_{ji}})$. It is well-known that, by a base change over B, we may assume that

$$f(x, y) = xJy^* \qquad (x, y \in B^n),$$

where $J = diag(\epsilon_1, \ldots, \epsilon_n)$ is a non-degenerate diagonal matrix in $M_n(F)$. For any place v of F, we denote by F_v the completion at v. We denote by $\mathbb{H}$ the division quaternion algebra over $\mathbb{R}$. Equivalence classes of non-degenerate quaternion hermitian forms over $\mathbb{H}$ are determined by the signature of the forms. More precisely, if we denote by $v_1, \ldots, v_r$ the set of all infinite places of F such that $B_v = B \otimes_F F_v$ is a division algebra, then the forms f on B^n are equivalent under the base change over B if and only if their embeddings to the maps on $B^n_{v_i}$ are equivalent over B_{v_i} for all v_i $(1 \leq i \leq r)$. If v is a finite place of F, then any non-degenerate quaternion hermitian forms are equivalent under the base change over B_v. So for a finite v, we may change to $J = 1_n$ locally by a base change over B_v. We fix f once and for all. We define a group of similitudes with respect to f by

$$G = \{g \in GL_n(B) = M_n(B)^\times ; gJ\,{}^t\overline{g} = n(g)J \text{ for some } n(g) \in F^\times\}$$

and call this a quaternion hermitian group with respect to f. If we write $g^\sigma = Jg^*J^{-1}$, then the condition $g \in G$ is written simply as $gg^\sigma = n(g)1_n$. For any place v, we put

$$G_v = \{g \in M_n(B_v); gg^\sigma = n(g)1_n, n(g) \in F_v^\times\}$$

where $B_v = B \otimes_F F_v$. We denote by F_A and G_A the adelizations of F and G, respectively. For $c \in F$ or F_A, it is clear that $c1_n \in G$ or G_A.

We denote by $\mathfrak{o}$ the ring of integers of F. We fix a maximal order $\mathfrak{O}$ of B. An $\mathfrak{o}$-module L in B^n such that $L \otimes_{\mathfrak{o}} F = B^n$ is called a

left $\mathfrak{O}$-lattice if it is a left $\mathfrak{O}$-module. For any finite place v of F, we denote by $\mathfrak{o}_v$ the v-adic completion of $\mathfrak{o}$ and put $L_v = L \otimes_{\mathfrak{o}} \mathfrak{o}_v$. We say that left $\mathfrak{O}$-lattices L_1 and L_2 belong to the same class if $L_1 = L_2 g$ for some $g \in G$. We say that L_1 and L_2 belong to the same genus if $L_{1,v} = L_{2,v} g_v$ for some $g_v \in G_v$ for all finite places v of F. We fix a left $\mathfrak{O}$-lattice L and denote by $\mathcal{L}(L)$ the set of left $\mathfrak{O}$-lattices belonging to the same genus as L and call this a genus of L. In other words, if we put

$$Lg = \bigcap_{v\text{: finite places}} (L_v g_v \cap B^n)$$

for any $g = (g_v) \in G_A$, then we have

$$\mathcal{L}(L) = \{Lg;\ g \in G_A\}.$$

We fix a left $\mathfrak{O}$-lattice L. For any finite place v, we define

$$U_v = U(L_v) = \{u \in G_v; L_v = L_v u\}$$

and write $U = G_\infty \prod_{v<\infty} U_v$, where G_∞ is the product of all G_v over the archimedean places v. Then the class number h of $\mathcal{L}(L)$ is equal to $|U \backslash G_A / G|$, which is known to be finite. Now we write $G_A = \bigcup_{i=1}^{h} U g_i G$ (disjoint), where we assume that $g_1 = 1$. We write $\mathfrak{O}_v = \mathfrak{O} \otimes_{\mathfrak{o}} \mathfrak{o}_v$. For $1 \leq i \leq h$, we define left $\mathfrak{O}$-lattices L_i by $L_i = L g_i$. The ring

$$R_i = \{b \in M_n(B); L_i b \subset L_i\}$$

is called the right order of L_i. This is an maximal order of $M_n(B)$, since for any prime v, we have $M_v = \mathfrak{O}_v^n h_p$ for some $h_p \in GL_n(B_v)$ (where we can take $h_v = 1$ for almost all v), so $R_{i,v} = R_i \otimes_{\mathfrak{o}} \mathfrak{o}_v = h_v^{-1} M_n(\mathfrak{O}_v) h_v$ are maximal orders for any finite places v. For any order R of $M_n(B)$ and $g = (g_v) \in G_A$, we define $g^{-1}Rg$ by

$$g^{-1}Rg = \bigcap_{v<\infty} (g_v^{-1} R_v g_v \cap M_n(B)).$$

So if we write $R = R_1$ (where we chose $g_1 = 1$), then $R_i = g_i^{-1} R g_i$. We say that R_i and R_j have the same type (or G-type) if $a^{-1} R_i a = R_j$ for some $a \in G$. We denote this relation by $R_i \cong_G R_j$. The number of equivalence classes in $\{R_1, \ldots, R_h\}$ in this sense is called the type number T of $\mathcal{L}(L)$. When $n = 1$, since $G = B^\times$ and $G_A = B_A^\times$, this is nothing but the type number in the classical sense.

Now we give a complete set of representatives of local equivalence classes of quaternion hermitian lattices for finite places. First we show an easy result that for a finite place v, left $\mathfrak{O}_v$-lattices correspond to quaternion hermitian matrices. We denote by $GL_n(O_v)$ the group of nonsingular elements u in $M_n(O_v)$ such that $u^{-1} \in M_n(O_v)$. We say that $X \in M_n(B)$ is a quaternion hermitian matrix if $X = X^*$. We say that two hermitian matrices X_1, $X_2 \in M_n(B_v)$ are equivalent if there exists a $u \in GL_n(O_v)$ such that $u X_1 u^* = m X_2$ for some $m \in F_v^\times$. We

say that two left $\mathfrak{O}_v$-lattices L_1 and L_2 are G_v-equivalent if $L_1 g = L_2$ for some $g_v \in G_v$.

Lemma 2.1. *The set of G_v-equivalence classes of left $\mathfrak{O}_v$-lattices and the set of equivalence classes of hermitian matrices in $M_n(B_v)$ correspond bijectively.*

Proof. Take J as before. Since $N(B_v^\times) = F_v^\times$ for any finite place v, there exists a diagonal matrix $J_1 \in GL_n(B_v)$ such that $J = J_1\, {}^t\overline{J_1}$ and we may assume that $J = 1_n$. But to avoid any likely confusion, we keep using a general J here in the proof. For any finite place v, it is clear that any $\mathfrak{O}_v$-lattice L_v may be written as $L_v = \mathfrak{O}_v^n h$ with $h \in GL_n(B_v)$ by the elementary divisor theorem. We define a map ϕ by $\phi(L_v) = hJ\,{}^t\overline{h}$. The equivalence class of the image does not depend on the choice of h. If $\mathfrak{O}_v^n h_1 g = \mathfrak{O}_v^n h_2$ for $g \in G_v$, then we have $uh_1 g = h_2$ for some $u \in GL_n(O_v)$. This means that

$$n(g) u h_1 J h_1^* u^* = u h_1 g J g^* h_1^* u^* = h_2 J h_2^*.$$

So ϕ induces a map from a G_v-equivalence class to a class of hermitian matrices. The map is surjective. Indeed for any hermitian matrix $X \in GL_n(B_v)$, there exists an $x \in GL_n(B_v)$ such that $X = xx^*$, so if we put $hJ_1 = x$ for J_1 such that $J_1 J_1^* = J$, then we have $\phi(O_v^n h) = X$. The map is injective. Indeed, if $uh_1 J h_1^* u^* = m h_2 J h_2^*$ for some $m \in F_v$, then $g = h_2^{-1} u h_1 \in G_v$ with $n(g) = m$ and we have $\mathfrak{O}_v^n h_2 g = \mathfrak{O}_v^n h_1$. □

For a finite place v, we denote by p_v a prime element of $\mathfrak{o}_v$. First we consider the case when B_v is division. When B_v is a division quaternion algebra, let O_v be the maximal order of B_v and π a fixed prime element of O_v such that $N_{B_v/F_v}(\pi) = p_v$ and $\pi^2 = -p_v$.

Proposition 2.2. *Let B_v be a division quaternion algebra and $H = H^* \in M_n(B_v)$ be a quaternion hermitian matrix. Then there exists a $u \in GL_n(O_v)$ such that*

$$uHu^* = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & A_r \end{pmatrix}$$

where $A_i = p_v^{e_i}$ or

$$A_i = p_v^{e_i} \begin{pmatrix} 0 & \pi \\ \overline{\pi} & 0 \end{pmatrix}.$$

Proof. We prove this by induction of the size of H. Multiplying by a power of p_v, we may assume that $H \in M_n(\mathfrak{O}_v)$. Assume that the $\mathfrak{O}_v$ ideal spanned by the components h_{ij} of $H = (h_{ij})$ is $\pi^e \mathfrak{O}_v$. By replacing H by $p_v^{-[e/2]} H$, we may assume that $e = 0$ or $e = 1$. First assume that $e = 0$. Then some component of H is in $O_v^\times$. If a diagonal

component belongs to $O_v^\times$, then by permuting the rows and columns, we may assume that the $(1,1)$ component h_{11} belongs to $O_v^\times$. Since $H = H^*$, this means $h_{11} \in \mathfrak{o}_v^\times$. Since we have $N(O_v^\times) = \mathfrak{o}_p^\times$, by changing H to $\epsilon H \epsilon^*$ for $\epsilon \in O_v^\times$ with $N(\epsilon) = h_{11}^{-1}$, we may assume that $h_{11} = 1$. Denote by e_{ij} the $n \times n$ matrix whose (i,j) component is 1 and whose other components are 0. Then if we put $u_1 = 1_n - \sum_{i=2}^n h_{i1} e_{i1}$, where we write $H = (h_{ij})$, obviously $u_1 \in GL_n(O_v)$ and we have

$$u_1 H u_1^* = \begin{pmatrix} 1 & 0 \\ 0 & H_1 \end{pmatrix}.$$

So we reduce to the matrix H_1 of size $n-1$. If all the diagonal components belong to $p_v \mathfrak{o}_v$ and there exists some off-diagonal component belonging to $O_v^\times$, then, by permuting the rows and columns, we may assume that the $(1,2)$ component is $h_{12} = \epsilon \in O_v^\times$. We write $h_{11} = p_v t$ and $h_{22} = p_v s$ with t, $s \in \mathfrak{o}_v$. If we put $u_2 = 1_n + b e_{12}$ with $b \in O_v$, then $u_2 \in GL_n(O_v)$ and the $(1,1)$ component of uHu^* is given by

$$p_v t + p_v s N(b) + Tr(b\overline{\epsilon}).$$

Since it is well known that $Tr(O_v) = \mathfrak{o}_v$ (e.g. the unramified extension of F_v contains an integral element whose trace is one), we take $b = \epsilon_0 \overline{\epsilon}^{-1}$ for an element $\epsilon_0 \in O_v$ such that $tr(\epsilon_0) = 1$. Since $1 + p_v t + p_v s n(b) \in \mathfrak{o}_v^\times$, we reduce to the previous case. Secondly we assume that $e = 1$. Then all the diagonal components belong to $p_v \mathfrak{o}_v$ and changing rows and columns, we may assume that $h_{12} = \pi\epsilon$ with $\epsilon \in \mathfrak{O}_v^\times$. We assume that $h_{11} = p_v^e t_0$ with $e \geq 1$ and $t_0 \in \mathfrak{o}_v^\times$ and $h_{22} = p_v s$ with $s \in \mathfrak{o}_v$. Again by $v_1 = 1_n + b_1 e_{12}$, the $(1,1)$ component of $v_1 H v_1^*$ is given by $p_v^e t_0 + p_v s N(b_1) + Tr(\pi\epsilon\overline{b_1})$. If we put $\overline{b_1} = p_v^{e-1}\epsilon^{-1}\overline{\pi}\epsilon_0$ with $\epsilon_0 \in \mathfrak{O}_v$ such that $Tr(\epsilon_0) = -t_0$, then we have

$$p_v^e t_0 + p_v s N(b_1) + Tr(\pi\epsilon\overline{b_1}) = p_v^e(t_0 + Tr(\epsilon_0)) + s p_v^{2e} N(\epsilon^{-1}\epsilon_0) = p_v^{2e} s N(\epsilon^{-1}\epsilon_0).$$

This is divisible by p_v^{2e}. Since $Tr(\pi\mathfrak{O}_v) = p_v \mathfrak{o}_v$, we see that $\epsilon_0 \in \mathfrak{O}_v^\times$ and $b_1 \in p^{e-1}\pi\mathfrak{O}_v^\times$. Repeating the same process, we can take $v_i = 1 + b_i e_{12}$ such that the $(1,1)$ component of $v_i v_{i-1} \cdots v_1 H v_1^* \cdots v_i^*$ is of arbitrary high p_v-adic order. Since the π-adic order of b_i monotonically increases, the limit $\lim_{i\to\infty} v_i \cdots v_1$ converges to $v \in GL_n(\mathfrak{O}_v)$ and we see that the $(1,1)$ component of vHv^* is zero. By these changes, the $(1,2)$ components always belong to $\pi\mathfrak{O}_v^\times$, so we may assume that $h_{11} = 0$ and $h_{12} = \pi\epsilon_2 \in \pi\mathfrak{O}_v^\times$. By taking the diagonal matrix $A_0 = diag(1, \epsilon_2^{-1}, 1 \ldots, 1) \in GL_n(O_v)$ and $A_0^* H A_0$, we may assume that $h_{12} = \pi$. So now we can assume that the diagonal block of H of (i,j) components with $1 \leq i, j \leq 2$ is given by

$$\begin{pmatrix} 0 & \pi \\ \overline{\pi} & p_v s \end{pmatrix}$$

We have

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} 0 & \pi \\ \overline{\pi} & p_v s \end{pmatrix}\begin{pmatrix} 1 & \overline{b} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \pi \\ \overline{\pi} & p_v s + Tr(b\pi) \end{pmatrix}.$$

Since $Tr(\pi\mathfrak{O}_v) = p_v\mathfrak{o}_v$, we can take $b \in \mathfrak{O}_v$ such that $p_v s + Tr(b\pi) = 0$, so we may assume that $s = 0$. Now we will show that we can change H so that the components of the first and the second row vanish except for the $(1,2)$ and $(2,1)$ components. Since we assumed that $e = 1$, all the components belong to $\pi\mathfrak{O}_v$, and if we put

$$w = 1_n - \sum_{j=3}^{n} \overline{\pi}^{-1} h_{2j} e_{1j} - \sum_{j=3}^{n} \pi^{-1} h_{1j} e_{2j},$$

then $w \in GL_n(\mathfrak{O}_v)$ and we have

$$w^* H w = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}$$

with $H_1 = \begin{pmatrix} 0 & \pi \\ \overline{\pi} & 0 \end{pmatrix}$, so the claim for H reduces to the claim for H_2. □

For any subset W of G_A, we put

$$n(W) = \{n(w) \in F_A^\times; \quad w \in W\}.$$

Corollary 2.3. *For any finite place v, let L_v be a left $\mathfrak{O}_v$-lattice and define U_v as before as a group of elements $g \in G_v$ such that $L_v g = L_v$. Then we have $n(U_v) = \mathfrak{o}_v^\times$.*

Proof. First we show that $n(U_v) \subset \mathfrak{o}_v^\times$. Assume that $g \in U_v$ and $gg^\sigma = n(g)1_n$. Since $L_v g = L_v$ and L_v is a free o_v-module of finite rank, the characteristic polynomial of the representation of g is monic integral if we identify B_v with F_v^4. Since the characteristic polynomial of $g^\sigma = Jg^*J^{-1}$ is the same as that of g, this is also monic integral. In particular, the determinants of g and g^σ in this representation are integral. So $n(g)^{4n}$ is integral, and so $n(g)$ is also integral. Since $L_v = L_v g^{-1}$, this is also true for $n(g)^{-1}$. So we have $n(g) \in O_v^\times$. Next we show the converse. First we assume that B_v is division. We take $h \in GL_v(B_p)$ such that $L_v = \mathfrak{O}^n h_v$ and put $H = h_v J\, {}^t\overline{h_v}$. Then for any $m \in \mathfrak{o}_v^\times$, we have an element $\alpha \in GL_n(\mathfrak{O}_v)$ such that $\alpha H \alpha^* = mH$. Indeed, we have $uHu^* = \mathrm{diag}(A_1, \ldots, A_r)$ for some $u \in GL_n(\mathfrak{O}_v)$ as in Proposition 2.2. Take $b_i \in \mathfrak{O}_v^\times$ such that $N(b_i) = m$, then if $A_i = p_v^e$, we have $b_i A_i b_i^* = mA_i$. If $A_i = \begin{pmatrix} 0 & \pi \\ \overline{\pi} & 0 \end{pmatrix}$, then $\mathfrak{O}_v$ is realized as $\mathfrak{O}_v = \mathfrak{o}_v^{un} + \mathfrak{o}_v^{un}\pi$ where $\pi^2 = -p$ and $\mathfrak{o}_v^{un}$ is a subring of $\mathfrak{O}_v$, which is isomorphic to the maximal order of the unique unramified quadratic extension of F_v. Here for $b \in \mathfrak{o}_v^{un}$, we have $b\pi = \pi\overline{b}$. We have $N((o_v^{un})^\times) = \mathfrak{o}_v^\times$ by local

class field theory. So taking $b \in (\mathfrak{o}_v^{un})^\times \subset \mathfrak{O}_v^\times$ with $N(b) = m$, put

$$C_i = \begin{pmatrix} b & 0 \\ 0 & \overline{b} \end{pmatrix}.$$

Then

$$C_i \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix} C_i^* = \begin{pmatrix} 0 & b\pi b \\ -\overline{b}\pi\overline{b} & 0 \end{pmatrix} = m \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}.$$

So taking a diagonal matrix v consisting of diagonal blocks b_i and C_i, we have $vuHu^*v^* = muHu^*$. So by $H = h_v J h_v^*$, we have $h_v^{-1}vuh_v \in G_p$ and $n(h^{-1}vuh) = m$. We also have $L_v h^{-1}vuh_v = O_v^n vuh_v = O_v^n h_v = L_v$, so $h_v^{-1}vuh_v \in U_v$. Next assume that $B_v = M_2(F_v)$. In this case, by virtue of Shimura [14] Proposition 2.10, there exists an element $X \in GL_n(B_v)$ satisfying $XX^* = 1_n$ and fractional left O_v-ideals $\mathfrak{b}_i$ such that $L_v = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n)X$. Let m be any element in $\mathfrak{o}_v^\times$. we take $J_1 = diag(u_1, \ldots, u_n)$ such that $J_1\, {}^t\overline{J_1} = J$. Since the right orders $\mathfrak{O}_i$ of $\mathfrak{b}_i$ are again maximal orders which are all conjugate to $M_2(o_v)$, there exist $\alpha_i \in u_i \mathfrak{O}_i^\times u_i^{-1}$ for each $1 \leq i \leq n$ such that $N(\alpha_i) = m$. Put $g = X^{-1}J_1^{-1}diag(\alpha_1, \ldots, \alpha_n)J_1 X$. Then we have

$$L_v g = (\mathfrak{b}_1 u_1^{-1}\alpha_1, \ldots, \mathfrak{b}_n u_n^{-1}\alpha_n)J_1 X = (\mathfrak{b}_1 u_1^{-1}, \ldots, \mathfrak{b}_n u_n^{-1})J_1 X = L_v.$$

So we have $g \in U_v$ and $gJg^* = mJ$. So $m \in n(U_v)$. □

3. G-type numbers and Hecke operators

3.1. **A formula for a type number.** We fix a left $\mathfrak{O}$-lattice L in B^n. We define $U \subset G_A$ by the group of stabilizers of L as before and fix representatives $L_1, \ldots, L_h$ of classes in $\mathcal{L}(L)$ and right orders R_i of L_i. We set $L_1 = L$ and $R_1 = R$. We denote by L_v and R_v the tensor of L and R over $\mathfrak{o}$ and $\mathfrak{o}_v$, respectively. First, to define some good Hecke operators, we see there exist some special elements in $R_v \cap G_v$. When B_v is division, we fix an element $\pi \in \mathfrak{O}_v$ with $\pi^2 = -p_v$ as before. First we recall the following well-known fact.

Lemma 3.1. *When B_v is division, any two sided ideal of $M_n(\mathfrak{O}_v)$ in $M_n(\mathfrak{O}_v)$ is given by $\pi^e M_n(\mathfrak{O}_v)$ for some integer $e \geq 0$. When $B_v = M_2(F_v)$, then any two sided ideal of $M_n(\mathfrak{O}_v) \cong M_{2n}(\mathfrak{o}_v)$ in $M_n(\mathfrak{O}_v)$ is given by $p_v^e M_n(\mathfrak{O}_v)$ for some integer $e \geq 0$.*

The proof is well-known and straightforward by using the elementary divisor theorem in both cases and omitted here. It is also clear that for any u_1, $u_2 \in GL_n(\mathfrak{O}_v)$, we have $u_1\pi^e u_2 M_n(O_p) = \pi^e M_n(O_p)$ when B_p is division.

Proposition 3.2. *When B_v is division, there exists an element $\omega_v \in R_v \cap G_v$ such that $\omega_v^2 = -p_v 1_n$, $\omega_v\omega_v^* = p_v 1_n$ and any two sided ideal of R_v in R_v is given by $\omega_v^e R_v$ for some $e \geq 0$.*

Proof. First we show that there exists an element $\omega_v \in R_v$ such that $\omega_v^2 = -p_v 1_n$, $\omega_v \omega_v^\sigma = p_v 1_n$, and $\omega_v R_v = R_v \omega_v$. Take $h_v \in GL_n(B_v)$ such that $L_v = \mathfrak{O}_v^n h_v$ and put $H = h_v J\, {}^t\overline{h_v}$. By changing a representative of the G_v-equivalence class of L_v by multiplying an element of $\mathfrak{o}_v$, we may assume that $L_v \subset O_v^n$ and $H \in M_n(\mathfrak{O}_v)$. Then by Proposition 2.2, there exists some $u \in GL_n(O_v)$ such that all the components of uHu^* are in $\mathfrak{o}_v \cup \pi\mathfrak{o}_v$. So we have $\pi(uHu^*) = (uHu^*)\pi$, so $\pi(uHu^*)\overline{\pi} = puHu^*$. So if we put $\omega_v = h_v^{-1}u^{-1}\pi u h_v$, then we have $\omega_v J \omega_v^* = p_v J$ and $\omega_v^2 = -p_v 1_n$. We also have $\mathfrak{O}_v^n h_v \omega_v = \mathfrak{O}_v^n u^{-1}\pi u h_v = \mathfrak{O}_v^n \pi u h_v \subset \mathfrak{O}_v^n u h_v = \mathfrak{O}_v h_v$, so $\omega_v \in h_v^{-1} M_n(\mathfrak{O}_v) h_v = R_v$. We also have $R_v\omega_v = h_v^{-1}M_n(\mathfrak{O}_v)u^{-1}\pi u h_v = h_v^{-1}u^{-1}M_n(\mathfrak{O}_v)\pi u h = h_v^{-1}u^{-1}\pi u M_n(\mathfrak{O}_v)h_v = \omega_v R_v$, so $R_v\omega_v$ is a two sided ideal. By using Lemma 3.1, any two sided ideal of R_v is given by $h_v^{-1}u_1\pi^e u_2 h_v R_v$ for some $e \geq 0$ and any u_1, $u_2 \in GL_n(\mathfrak{O}_v)$ and this is equal to $\omega_v^e R_v$. □

We denote by $\mathfrak{d}$ the $\mathfrak{o}_v$-ideal defined as the product of the prime ideals p_v of $\mathfrak{o}_v$ such that B_v is division. This is called the discriminant of B. We say that p_v is ramified when B_v is division and split when $B_v = M_2(F_v)$. We fix ω_v for $p_v|D$ as above and for any integral ideal $\mathfrak{m}|\mathfrak{d}$ of $\mathfrak{o}_v$, we define $\omega(\mathfrak{m}) = (g_v) \in G_A$ by setting $g_v = 1$ for all archimedean places v and finite places v such that $p_v \nmid \mathfrak{m}$, and $g_v = \omega_v$ for any places v such that $p_v|\mathfrak{m}$. We put $F_\infty = \prod_{v\,:\,\text{infinite}} F_v$ where v runs over all archimedean places of F. We choose a complete set $\mathfrak{c}_1$, …, $\mathfrak{c}_{h_0}$ of representatives of $F_A^\times / F^\times \cdot F_\infty^\times \prod_v \mathfrak{o}_v^\times$. This set of course corresponds to a complete set of representatives of ideal classes of F and h_0 is the class number of F. By embedding $F_A 1_n \subset G_A$, we regard $\mathfrak{c}_i$ as an element of G_A. We also have $(F_\infty^\times \prod_v \mathfrak{o}_v^\times)1_n \subset U$ for any $\mathfrak{O}$-lattice L. We have

Proposition 3.3. *(1) R_i and R_j have the same G-type if and only if $\mathfrak{c}_l^{-1}\omega(\mathfrak{m})^{-1}g_i \in Ug_jG$ for some $\mathfrak{m}|\mathfrak{d}$ and some $\mathfrak{c}_l$.*
(2) Assume that the class number of F is one. Then for a fixed $\mathfrak{m}|\mathfrak{d}$, if $\omega(\mathfrak{m})^{-1}g_i \in Ug_jG$, then $\omega(\mathfrak{m})^{-1}g_j \in Ug_iG$.

Proof. First we assume that $R_i \cong_G R_j$, so we have $a^{-1}R_i a = R_j$ for some $a \in G$. This means that $a^{-1}g_i^{-1}Rg_i a = g_j^{-1}Rg_j$, so by definition, we have $a^{-1}g_{i,v}^{-1}R_p g_{i,v} a = g_{j,v}R_p g_{j,v}$, where $g_{i,v}$ and $g_{j,v}$ are v-adic components of g_i and g_j. So $R_v g_{i,v} a g_{j,v}^{-1}$ is a two sided ideal of R_v. So if B_v is division, then $g_{i,v}ag_{j,v} = \omega_v^{e_v}u$ with $u \in U_v$. If $B_v = M_2(F_v)$, then $g_{i,v}ag_{j,v}^{-1} = p_v^{e_v}u$ with $u \in U_v$. Since $g_{i,v}ag_{j,v}^{-1}$ is the v-component of an element in G_A, we have $g_{i,v}ag_{j,v}^{-1} \in U_v$ for almost all v. So $e_v \neq 0$ only for the finitely many v. We denote by m_1 an element of $F_A^\times$ such that v component is $p_v^{e_v}$ for split primes p_v, and $p_v^{[e_v/2]}$ for ramified primes p_v, where $[x]$ is the least integer which does not exceed x. For some l with $1 \leq l \leq h_0$, we have $m_1 = u_0\mathfrak{c}_l c$ with $u_0 \in F_\infty \prod_v \mathfrak{o}_v^\times$ and $c \in F^\times$. If we define $\mathfrak{m}$ as a product of ramified p_v such that e_v is odd, we see

$g_i a c^{-1} g_j^{-1} \in \omega(\mathfrak{m})\mathfrak{c}_l U$, so $\mathfrak{c}_l^{-1}\omega(\mathfrak{m})^{-1} g_i \in U g_j G$. Next we prove the converse. We assume that $\mathfrak{c}_l^{-1}\omega(\mathfrak{m})^{-1} g_i \in U g_j G$ for some $\mathfrak{m}|\mathfrak{d}$ and l. Then $g_i = \omega(\mathfrak{m})\mathfrak{c}_l u g_j a$ for some $u \in U$ and $a \in G$. Then we have

$$R_i = g_i^{-1} R g_i = a^{-1} g_j^{-1} u^{-1} \mathfrak{c}_l^{-1} \omega(\mathfrak{m})^{-1} R\, \omega(\mathfrak{m}) \mathfrak{c}_l u g_j a.$$

We have $\omega(m)^{-1} R \omega(m) = R$ since conjugation is defined locally. Since $\mathfrak{c}_l 1_n$ is in the center of $M_n(B_A)$ and $u^{-1} R u = R$ by definition of U, we have $a^{-1} R_j a = R_i$, hence we have proved (1). Now if $\omega(\mathfrak{m})^{-1} g_i \in U g_j G$ for some $\mathfrak{m}|\mathfrak{d}$, then since $\omega(\mathfrak{m}) U = U \omega(\mathfrak{m})$ by definition of $\omega(\mathfrak{m})$, we have $g_i \in \omega(\mathfrak{m}) U g_j G = U \omega(\mathfrak{m}) g_j G$, hence $\omega(\mathfrak{m}) g_j \in U g_i G$. Since $\omega(\mathfrak{m})^2 \in F_A 1_n$ and we assumed that the class number of F_A is one, we see that $\omega(\mathfrak{m})^2 = u_0 c$ for some $u_0 \in F_\infty \prod_v \mathfrak{o}_v^\times$ and $c \in F^\times$. We have $\omega(\mathfrak{m}) = \omega(\mathfrak{m})^{-1} u_0 c$ and we have $\omega(\mathfrak{m})^{-1} g_j \in u_0^{-1} U g_i G c^{-1} = U g_i G$. □

Now we review the definition of the action of Hecke operators on functions on the double coset $U \backslash G_A / G$. In particular when G_∞ is compact, this is nothing but the space of automorphic forms of trivial weight (See [4] and [5] (I)). We define the space $\mathfrak{M}_0(U)$ by

$$\mathfrak{M}_0(U) = \{f : G_A \to \mathbb{C}; f(uga) = f(g) \text{ for any } u \in U, a \in G, g \in G_A\}.$$

Then for any $z \in G_A$ and $UzU = \bigcup_{i=1}^d z_i U$, the double coset acts on $f(g) \in \mathfrak{M}_0(U)$ by

$$([UzU]f)(g) = \sum_{i=1}^d f(z_i^{-1} g) \qquad (g \in G_A).$$

For the class number $h = h(\mathcal{L})$ of $\mathcal{L} = \mathcal{L}(L)$ and $1 \leq i \leq h$, we denote by f_i the element in $M_0(U)$ such that $f_i(g) = 1$ for any $g \in U g_i G$ and $= 0$ for any $g \in U g_j G$ with $j \neq i$. Then since $\mathfrak{M}_0(U)$ is the set of functions on G_A which are constant on each double coset $U g_i G$, we see that $\{f_1, \ldots, f_h\}$ is a basis of $\mathfrak{M}_0(U)$ and $h = \dim \mathfrak{M}_0(U)$. To count the type number by traces of Hecke operators, we define Hecke operators $R(\mathfrak{m}\mathfrak{c}_l^2)$ for $\mathfrak{m}|\mathfrak{d}$ and $\mathfrak{c}_l$ for $1 \leq l \leq h_0$ by

$$R(\mathfrak{m}\mathfrak{c}_l^2) = U \omega(\mathfrak{m}) \mathfrak{c}_l U.$$

(Here we write $\mathfrak{c}_l^2$ in $R(*)$ just because $\mathfrak{c}_l^2 \in F_A^\times$ gives the multiplicator of the similitude $\mathfrak{c}_l 1_n$ and fits the notation $\mathfrak{m}$.) If we denote by t the number of prime divisors of $\mathfrak{d}$, then there are $2^t h_0$ such operators. Since $\omega_v R_v = R_v \omega_v$, we have $\omega_v R_v^\times = R_v^\times \omega_v$ and $\omega_v U_v = U_v \omega_v$. Also $\mathfrak{c}_l 1_n$ is in the center of G_A. So it is clear that $U\omega(m)\mathfrak{c}_l U = \omega(m)\mathfrak{c}_l U$. So these operators are obviously commutative. By definition, this acts on $\mathfrak{M}_0(U)$ by

$$R(\mathfrak{m}\mathfrak{c}_l^2) f = [U \omega(\mathfrak{m}\mathfrak{c}_l^2) U] f = f(\omega(\mathfrak{m})^{-1} \mathfrak{c}_l^{-1} g).$$

By definition, we have $R(\mathfrak{m}\mathfrak{c}_l^2) f_i = f_j$ for the unique j such that $\omega(\mathfrak{m})^{-1}\mathfrak{c}_l^{-1} g_i \in U g_j G$. So $R(\mathfrak{m}\mathfrak{c}_l^2)$ induces a permutation of $\{f_1, \ldots, f_h\}$. If $c \in F_A$ belongs to the trivial ideal class, then we have $U(c1_n)U =$

$(c1_n)U$ with $c \in F^\times$ and this acts trivially on $\mathfrak{M}_0(U)$, so the definition of $R(\mathfrak{m}\mathfrak{c}_l^2)$ depends only on $\mathfrak{m}$ and the class of $\mathfrak{c}_l$. We have $(U\omega(\mathfrak{m})\mathfrak{c}_l U)^2 = Um\mathfrak{c}_l^2$ for some $m \in F_A^\times$ and this also acts as a permutation on $\{f_1, \ldots, f_h\}$. We also see by this that the image of the action of the algebra of $R(\mathfrak{m}\mathfrak{c}_l^2)$ for all $\mathfrak{m}$ and $\mathfrak{c}_l$ is a finite abelian group. As a whole, the action of the semi-group spanned by $R(\mathfrak{m}\mathfrak{c}_l^2)$ on $\mathfrak{M}_0(U)$ is regarded as an action of a finite abelian group Γ of order $2^t h_0$.

Now we review an easy general theory of group actions. Let Γ be a finite abelian group acting on a finite set X (faithful or not.) We would like to count the number of the transitive orbits of X under Γ. We denote by ρ the linear representation on the formal sum $\oplus_{x\in X}\mathbb{C}x$ associated to the action of Γ on the set X.

Lemma 3.4. *The number T of transitive orbits of X by Γ is given by*

$$T = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} Tr(\rho(g)).$$

Proof. Let $X = \bigcup_{i=1}^T X_i$ be the decomposition into the disjoint union of transitive orbits of Γ. Then Γ acts on X_i transitively. Fix $x_i \in X_i$ for each i and denote by Γ_i the stablizer of x_i in Γ. Then we have $|X_i| = |\Gamma/\Gamma_i|$. The stablizer of any other point $\gamma x_i \in X_i$ for $\gamma \in \Gamma$ is $\gamma\Gamma_i\gamma^{-1}$, but since Γ is abelian, this is equal to Γ_i. So Γ_i acts trivially on X_i. Also, any $\gamma \in \Gamma$ with $\gamma \notin \Gamma_i$ has no fixed point in X_i. So if we denote by ρ_i the linear representation of Γ associated with the action on X_i, then we have

$$Tr(\rho_i(g)) = \begin{cases} |X_i| & \text{if } g \in \Gamma_i, \\ 0 & \text{if } g \notin \Gamma_i. \end{cases}$$

In other words, we have

$$\sum_{g\in\Gamma} Tr(\rho_i(g)) = |X_i||\Gamma_i| = |\Gamma|.$$

Since we have $\rho = \sum_{i=1}^T \rho_i$, we have

$$\sum_{g\in\Gamma} Tr(\rho(g)) = \sum_{i=1}^T |\Gamma| = |\Gamma| \times T.$$

Hence we prove the lemma. □

Now we come back to the G-type number.

Proposition 3.5. *We have $R_i \cong_G R_j$ if and only if f_i and f_j are in the same orbit of the action of the semi-group spanned by $\{R(\mathfrak{m}\mathfrak{c}_l^2); \mathfrak{m}|\mathfrak{d}, 1 \leq l \leq h_0\}$.*

Proof. This claim is obvious from Proposition 3.3. □

Theorem 3.6. *The G-type number T is given by*

$$T = \sum_{l=1}^{h_0} \sum_{\mathfrak{m}|\mathfrak{d}} \frac{Tr(R(\mathfrak{m}\mathfrak{c}_l^2))}{2^t h_0},$$

where Tr means the trace of the action of the U-double cosets on $M_0(U)$.

3.2. **Relation with global integral elements.** Interpretation of the above results in terms of global quaternion hermitian matrices is important for a geometric interpretation. For that purpose, we specialize the situation. From now on, we assume that $F = \mathbb{Q}$ and B is a definite quaternion algebra over $\mathbb{Q}$. We assume that the quaternion hermitian form is positive definite, so $J = 1_n$. Then $g^\sigma = g^* = {}^t\overline{g}$ and $n(g) > 0$ for $g \in G$. For a left $\mathfrak{O}$-lattice L, we define $U = U(L)$ as before. For $G_A = \cup_{i=1}^h U g_i G$ with $g_1 = 1$, we may assume that $n(g_i) = 1$ since the class number of $\mathbb{Q}$ is one and we have $n(G_A) = n(U)n(G)$. The set of lattices $L_i = Lg_i$ $(1 \leq i \leq h)$ is a complete set of representatives of the classes in $\mathcal{L}(L)$. We assume $n \geq 2$. Then by the strong approximation theorem on $GL_n(B)$, we can show easily that any left $\mathfrak{O}$-lattice L may be written as $L = \mathfrak{O}^n h$ for some $h \in GL_n(B)$. We define the associated quaternion hermitian matrix by $H = hh^*$. This is positive definite. We say that two quaternion hermitian matrices H_1 and H_2 are equivalent if there exists $u \in GL_n(O)$ and $0 < m \in \mathbb{Q}^\times$ such that $uH_1u^* = mH_2$.

Lemma 3.7. *Assume that $n \geq 2$. By the above mapping, the set of G equivalence classes of left O-lattices and the set of equivalence classes of positive definite quaternion hermitian matrices correspond bijectively.*

A proof is the same as in Lemma 2.1 and omitted here. For representatives $L = L_1, \ldots, L_h$ of the genus $\mathcal{L}(L)$, where $L_i = Lg_i$, we can take $h_i \in GL_n(B)$ such that $L_i = \mathfrak{O}^n h_i$ $(1 \leq i \leq h)$. So we have $L_i = Lg_i = O^n h_1 g_i$. Then we have $uh_i = h_1 g_i$ for some $u \in G_\infty \prod_p GL_n(\mathfrak{O}_p)$, and $uh_i h_i^* u^* = h_1 h_1^*$. This means that the reduced norms of $h_i h_i^*$ and $h_1 h_1^*$ are the same. Denote by D the discriminant of B. For $m|D$, we define $\omega(m)$ as before. We denote by R the right order of L as before.

Proposition 3.8. *For $0 < m$ with $m|D$, the following conditions (1) and (2) are equivalent.*
(1) $\omega(m)^{-1} g_i \in U g_j G$.
(2) There exists $\alpha \in M_n(\mathfrak{O})$ such that $\alpha M_n(\mathfrak{O}) = M_n(\mathfrak{O})\alpha$ and $\alpha h_j h_j^* \alpha^* = m h_i h_i^*$.

Proof. Assume (1). We have $\omega(m)^{-1} g_i = u g_j a$ for some $u \in U$, $a \in G$, and $g_i = \omega(m) u g_j a$. Since all the p-adic components of $\omega(m)$ are in R_p, we have $L\omega(m) \subset L$. Hence

$$L_i = Lg_i = L\omega(m) u g_j a \subset Lg_j a = L_j a.$$

Since $L_i = \mathfrak{O}^n h_i$ and $L_j = \mathfrak{O}^n h_j$, we have $\mathfrak{O}^n h_i \subset \mathfrak{O}^n h_j a$. Hence if we put $\alpha = h_i a^{-1} h_j^{-1}$ then $\mathfrak{O}^n \alpha \subset \mathfrak{O}^n$, so $\alpha \in M_n(\mathfrak{O})$ and $\alpha h_j h_j^* \alpha^* = n(a)^{-1} h_i h_i^*$. Since we assumed $n(g_i) = n(g_j) = 1$, we have $n(a)n(u) = n(\omega(m)^{-1})$. Since $n(u) \in \mathbb{R}_+^\times \prod_p \mathbb{Z}_p^\times$, $n(\omega(m)) \in m\mathbb{R}_+^\times \prod_p \mathbb{Z}_p^\times$, and $n(a) \in \mathbb{Q}_+^\times$, we have $n(a) = m^{-1}$, and $\alpha h_j h_j^* \alpha^* = m h_i h_i^*$. By definition of a, we have $a^{-1} = g_i^{-1} \omega(m) u g_j$, so

$$\begin{aligned} a^{-1} R_j &= g_i^{-1} \omega(m) u g_j (g_j^{-1} R g_j) \\ &= g_i^{-1} \omega(m) u R g_j = g_i^{-1} R \omega(m) u g_j = g_i^{-1} R g_i a^{-1} = R_i a^{-1}. \end{aligned}$$

Since we have $R_k = h_k^{-1} M_n(\mathfrak{O}) h_k$ for any k, we have $a^{-1} h_j^{-1} M_n(\mathfrak{O}) h_j = h_i^{-1} M_n(\mathfrak{O}) h_i a^{-1}$, and $h_i a^{-1} h_j^{-1} M_n(\mathfrak{O}) = M_n(\mathfrak{O}) h_i a^{-1} h_j^{-1}$. Since $\alpha = h_i a^{-1} h_j^{-1}$ by definition, we see that $\alpha M_n(\mathfrak{O})$ is a two-sided ideal. Hence we have (2). Now assume (2) and define a by $a^{-1} = h_i^{-1} \alpha h_j$. Then $a \in G$ and $n(a^{-1}) = m$. By $\alpha M_n(\mathfrak{O}) = M_n(\mathfrak{O})\alpha$, $n(g_i a^{-1} g_j^{-1}) = m$, and Lemma 3.1, we have $g_i a^{-1} g_j^{-1} = \omega(m) u$ with $u \in U$. So $\omega(m)^{-1} g_i = u g_j a \in U g_j G$. So we have (1). □

Now for a fixed i, if there exists no $j \neq i$ such that $R_j \cong_G R_i$, then by Proposition 3.3, for any $j \neq i$ and $m|D$, we have $\omega(m)^{-1} g_i G \notin U g_j G$. But $\omega(m)^{-1} g_i \in G_A = \bigcup_{j=1}^h U g_j G$, so we have $\omega(m)^{-1} g_i \in U g_i G$ for all $m|D$. If we assume that $D = p$ is a prime, then $R_i \cong_G R_j$ if and only if $\omega(m)^{-1} g_i \in U g_j G$ for $m = 1$ or p. So we have

Lemma 3.9. *Assume that $D = p$ is a prime. We fix i. Then there exists at most one $j \neq i$ such that $R_j \cong_G R_i$. If there exist such $j \neq i$, then we have $\omega(p)^{-1} g_i \in U g_j G$. If $R_i \cong_G R_j$ only for $j = i$, then $\omega(p)^{-1} g_i \in U g_i G$.*

Proof. If there exist j and k such that $j \neq i$ and $k \neq i$, then $g_i \notin U g_j G$ and $g_i \notin U g_k G$, and if $R_i \cong_G R_j \cong_G R_k$ besides, then by Proposition 3.3, we have $\omega(p)^{-1} g_i \in U g_j G$ and $\omega(p)^{-1} g_i \in U g_k G$, hence $U g_j G = U g_k G$ so $j = k$. If there exist no $j \neq i$ such that $R_i \cong_G R_j$, then we have $\omega(p)^{-1} g_i \notin U g_j G$ for any $j \neq i$. This means that $\omega(p)^{-1} g_i \in U g_i G$. □

So, when $D = p$ is a prime, then the G-type of any genus is either a subset of a pair of maximal orders or a subset of single element in $\{R_i; 1 \leq i \leq h\}$.

4. Models of polarizations defined over $\mathbb{F}_p$

4.1. **Polarizations on superspecial abelian varieties.** Let A be an abelian variety and A^t the dual of A. For an effective divisor D of A, we define an isogeny ϕ_D from A to A^t by

$$\phi_D(t) = Cl(D_t - D) \qquad (t \in A),$$

where D_t is the translation of D by t and Cl denotes the linear equivalence class of the divisor. We say that an isogeny λ from A to A^t is a

polarization if there exists an effective divisor D such that $\lambda = \phi_D$. We say that a polarization λ is a principal polarization if λ is an isomorphism. Two polarized abelian varieties (A_1, λ_1) and (A_2, λ_2) are said to be isomorphic if there exists an isomorphism $\phi : A_1 \to A_2$ such that $\lambda_1 = \phi^t \lambda_2 \phi$, where ϕ^t is the dual map from A_2^t to A_1^t associated with ϕ.

Let p be a prime. An elliptic curve E over a field of characteristic p such that $End(E)$ is a maximal order of a definite quaternion algebra B with discriminant p is called supersingular. There exists a supersingular elliptic curve defined over $\mathbb{F}_p$ such that $End(E)$ contains an element π with $\pi^2 = -p \cdot id_E$. We fix such an E once and for all. Then we can regard π as the Frobenius endomorphism of E and every element of $End(E)$ is defined over $\mathbb{F}_{p^2}$. An abelian variety A which is isogenous to E^n is called supersingular. An abelian variety which is isomorphic to E^n is called superspecial. It is well known that any product of various supersingular elliptic curves are all isomorphic (Shioda, Deligne). The superspecial abelian variety E^n has a principal polarization defined over $\mathbb{F}_p$ (See [7]). Indeed, if we take a divisor X defined by

$$X = \sum_{i=0}^{n-1} E^i \times \{0\} \times E^{n-1-i},$$

then the n-fold self-intersection $X^n = n!$, so $\det \phi_X = 1$, and this is defined over $\mathbb{F}_p$. We put $O = End(E)$. Then we have identifications $End(E^n) = M_n(O)$ and $Aut(E^n) = M_n(O)^\times = GL_n(O)$. For any $\phi \in End(E^n)$, the Rosati involution is defined by $\phi_X^{-1} \phi^t \phi_X$. Then this is equal to ϕ^* under the identification of $End(E^n)$ with $M_n(O)$. In particular, if we put $H_\lambda = \phi_X^{-1} \lambda$ for a polarization λ, then $H_\lambda^* = H_\lambda$ and H_λ is a positive definite quaternion hermitian matrix in $M_n(O)$. It is easy to show that two polarized abelian varieties (E^n, λ_1) and (E^n, λ_2) are isomorphic if and only if there exists an $\alpha \in GL_n(O)$ such that $\alpha H_{\lambda_1} \alpha^* = H_{\lambda_2}$.

Any polarization λ of E^n is defined over $\mathbb{F}_{p^2}$ since ϕ_X is defined over $\mathbb{F}_p$ and any endomorphism of E is defined over $\mathbb{F}_{p^2}$ by the choice of our E. We also see that if polarized abelian varieties (E^n, λ_1) and (E^n, λ_2) are isomorphic, then they are isomorphic over $\mathbb{F}_{p^2}$ since any element of $Aut(E^n)$ is defined over $\mathbb{F}_{p^2}$. Now we denote by σ the Frobenius automorphism of the algebraic closure $\overline{\mathbb{F}_p}$ over $\mathbb{F}_p$.

Lemma 4.1. *Notation being as before, a polarized abelian variety (E^n, λ) has a model defined over $\mathbb{F}_p$ if and only if (E^n, λ) and (E^n, λ^σ) are isomorphic.*

Proof. Assume that there is a model (A, η) of (E^n, λ) defined over $\mathbb{F}_p$. We write an isomorphism $(A, \eta) \to (E^n, \lambda)$ by ψ. Here ψ is defined over the algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$. Anyway, for any element $\tau \in$

$Gal(\overline{\mathbb{F}_p}/\mathbb{F}_p)$, we have

$$(E^n, \lambda) \cong (A, \tau) = (A^\tau, \eta^\tau) \cong (E^n, \lambda^\tau).$$

So the condition is necessary. On the other hand, if ψ gives an isomorphism $(E^n, \lambda) \cong (E^n, \lambda^\sigma)$, then $\psi \in Aut(E^n)$ is defined over $\mathbb{F}_{p^2}$ and $\psi^\sigma\psi$ is an automorphism of (E^n, λ) since $\lambda^{\sigma^2} = \lambda$. Since $\psi^\sigma\psi$ fixes a polarization (corresponding to a positive definite lattice), it is well-known that this is of finite order. So $(\psi^\sigma\psi)^r = (\psi\psi^\sigma)^r = 1$ for some positive integer r, where 1 means the identity map of E^n. Now we regard σ as a generator of the Galois group $Gal(\mathbb{F}_{p^{2r}}/\mathbb{F}_p)$. Since ψ is defined over $\mathbb{F}_{p^2}$, we have $\psi^{\sigma^2} = \psi$ and $(\psi^\sigma\psi)^{\sigma^{2i}} = \psi^\sigma\psi$. So if we put $f_1 = 1$, $f_\sigma = \psi$, and $f_{\sigma^i} = \psi^{\sigma^{i-1}}\psi^{\sigma^{i-2}}\cdots\psi$ for $1 \leq i \leq 2r-1$, then we have

$$f_{\sigma^i}^{\sigma^j} f_{\sigma^j} = \psi^{\sigma^{i+j-1}} \cdots \psi^{\sigma^j}\psi^{\sigma^{j-1}} \cdots \psi = f_{\sigma^{i+j}}.$$

This is obvious if $i + j < 2r$. If $2r \leq i + j \leq 4r - 1$, then this is equal to

$$\psi^{\sigma^{i+j-1-2r}} \cdots \psi^\sigma\psi,$$

since we have

$$\psi^{\sigma^{i+j-1}} \cdots \psi^{i+j-2r} = (\psi^\sigma\psi)^{\sigma^{i+j-2}}(\psi^\sigma\psi)^{\sigma^{i+j-4}} \cdots = ((\psi^\sigma\psi)^r)^{\sigma^\delta} = 1,$$

where $\delta = 0$ or 1 according as $i + j$ is even or odd. So we have $f_{\sigma^{i+j-2r}} = f_{\sigma^{i+j}}$ and the set of maps $\{f_{\sigma^i}; 0 \leq i \leq 2r-1\}$ satisfies the descent condition for $Gal(\mathbb{F}_p^{2r}/\mathbb{F}_p)$ (See [15]). So we have a model over $\mathbb{F}_p$. □

Proposition 4.2. *Notation being the same as before, the polarized abelian varieties (E^n, λ) and (E^n, λ^σ) are isomorphic if and only if $\alpha^* H_\lambda \alpha = pH_\lambda$ for some $\alpha \in End(E^n) = M_n(O)$ such that $\alpha M_n(O) = M_n(O)\alpha$.*

Proof. Let F be the Frobenius endomorphism of E^n over $\mathbb{F}_p$ and set $F = \pi 1_n$ where π is a prime element of O over p with $\pi^2 = -p$. Let F_1 be the Frobenius map of $(E^n)^t$ over $\mathbb{F}_p$. (Actually it is the same as F if we identify $(E^n)^t$ with E^n.) For a polarization λ of E^n, we have $\lambda^\sigma F = F_1\lambda$ by definition. In particular, since ϕ_X is defined over $\mathbb{F}_p$, we have $\phi_X F = F_1\phi_X$. So we have $(\phi_X^{-1}\lambda^\sigma)F = F(\phi_X^{-1}\lambda)$. Now assume that (E^n, λ^σ) and (E^n, λ) are isomorphic. This means that there exists an automorphism ϕ of E^n such that $\lambda^\sigma = \phi^t\lambda\phi$. So we have $\phi_X^{-1}\lambda^\sigma = \phi_X^{-1}\phi^t\phi_X\phi_X^{-1}\lambda\phi$. We have $\phi_X^{-1}\phi^t\phi_X = \phi^*$, identifying $End(E^n)$ with $M_n(O)$ and writing $g^* = {}^t\overline{g}$ for any $g \in M_n(B)$. So if we put $H_\lambda = \phi_X^{-1}\lambda$, then we have $FH_\lambda = \phi^* H_\lambda \phi F$. Since $F^*F = p1_n$, we have $pH_\lambda = \alpha^* H_\lambda \alpha$ for $\alpha = \phi F$. We have $\alpha M_n(O) = \phi F M_n(O) = \phi M_n(O)F = M_n(O)F = M_n(O)\phi F = M_n(O)\alpha$. So we have proved the "only if" part. Conversely, assume that $pH_\lambda = \alpha^* H_\lambda \alpha$ for some $\alpha \in M_n(O)$ such that $\alpha M_n(O)$ is a two sided ideal. Since we assumed

that the two sided prime ideal of O over p is generated by $F \in O$, it is classically well-known that any two sided ideal of $M_n(O)$ is given by $bF^r M_n(O)$ with positive rational number b and some non-negative integer r. So we have $\alpha = bF^r\epsilon$ for some $\epsilon \in GL_n(O) = M_n(O)^\times$. By taking the reduced norm of the both sides of $pH_\lambda = \alpha^* H_\lambda \alpha$, we see that the reduced norm $N(\alpha)$ of α is p^n. Since $N(F) = p^n$ and $N(\epsilon) = 1$, we see that $p^n = b^{2n}p^{nr}$, so $b = p^{n(1-r)/2}$. Since $F^2 = -p$, this is equal to $\pm F^{n(1-r)}$, and $\alpha = \phi F^s$ for some $\phi \in GL_n(O)$. Here comparing the reduced norm, we have $s = 1$ and this ϕ gives an isomorphism of (E^n, λ) to (E^n, λ^σ). □

4.2. **Relation to the type number.** For any polarization λ of E^n, $\phi_X^{-1}\lambda$ is a positive definite quaternion hermitian matrix in $M_n(O)$. If $\mathcal{L}$ is the genus of quaternion hermitian lattices to which $\phi_X^{-1}\lambda$ belongs, we write $\mathcal{L} = \mathcal{L}(\lambda)$ and we say that λ belongs to $\mathcal{L}$ by abuse of language. We denote by $\mathcal{P}(\lambda)$ the set of polarizations of E^n which belong to the same genus as λ belongs to. We denote by $H(\lambda)$ and $T(\lambda)$ the class number and the type number of $\mathcal{L}(\lambda)$, respectively.

Theorem 4.3. *Assume that $n \geq 2$ and fix a polarization λ of E^n. Then the number of isomorphism classes of polarizations in $\mathcal{P}(\lambda)$ is equal to $H(\lambda)$. The number of isomorphism classes of (E^n, μ) with $\mu \in \mathcal{P}(\lambda)$ which have a model over $\mathbb{F}_p$ is equal to $2T(\lambda) - H(\lambda)$.*

Proof. The first assertion is obvious so we prove the second assertion. We define U as the stabilizer in G_A of a lattice corresponding to $H_\lambda = \phi_X^{-1}\lambda$ and write $G_A = \bigcup_i U g_i G$. The isomorphism classes of $\mu \in \mathcal{P}(\lambda)$ correspond bijectively to the set $\{g_i\}$, so assume that μ corresponds to g_i. Write $H_\mu = \phi_X^{-1}\mu$ as before. The condition that $\alpha H_\mu \alpha^* = pH_\mu$ for some $\alpha \in M_n(O)$ with $\alpha M_n(O) = M_n(O)\alpha$ is equivalent to the condition that $\omega(p)g_i \in Ug_iG$ by Proposition 3.8. The number of isomorphism classes of such μ is equal to $Tr(R(p))$ by Lemma 3.9. Since $T(\lambda) = (Tr(R(p)) + Tr(R(1)))/2 = (Tr(R(p)) + H(\lambda))/2$, we prove the assertion. □

We note that even if (E^n, λ) has a model over $\mathbb{F}_p$, it is not necessarily true that E^n has a polarization equivalent to λ defined over $\mathbb{F}_p$. We give such an example below. If a polarization λ of E^n is defined over $\mathbb{F}_p$, this means that $F(\phi_X^{-1}\lambda) = (\phi_X^{-1}\lambda)F$, so the quaternion hermitian matrix associated with λ should be realized as a matrix which commutes with π. Now when the discriminant of B is a prime p, there are two genera of quaternion hermitian maximal left O-lattices in B^n, the one which contains O^n, and the other which does not contain O^n. We call the former a principal genus, denoted by $\mathcal{L}_{pr}$, and the latter a non-principal genus denoted by $\mathcal{L}_{npr}$. Now we consider the case $\mathcal{L}_{npr}$. If $n = 2$ and O contains π, then any quaternion hermitian matrix associated with a

lattice in $\mathcal{L}_{npr}$ is given by

$$H_1 = m\begin{pmatrix} pt & \pi r \\ \overline{\pi r} & ps \end{pmatrix}$$

with $0 < m \in \mathbb{Q}$, t, $s \in \mathbb{Z}$ and $r \in O$ such that $pts - N(r) = 1$. If $p = 3$, the maximal order O of B is concretely given up to conjugation by

$$O = \mathbb{Z} + \mathbb{Z}\frac{1+\pi}{2} + \mathbb{Z}\beta + \mathbb{Z}\frac{(1+\pi)\beta}{2},$$

where $\pi^2 = -3$, $\beta^2 = -1$, $\pi\beta = -\beta\pi$. If H_1 commutes with π, then r should be in $\mathbb{Q}(\pi)$. So we should have $3ts - N(r) = 1$ for some positive integers t, s and an element $r = (a + b\pi)/2$ with a, $b \in \mathbb{Z}$, $a \equiv b \bmod 2$. Here $N(r) = (a^2 + 3b^2)/4$ but we should have $N(r) \equiv -1 \bmod 3$ by the above relation. This means that $a^2 \equiv -1 \bmod 3$ but this is impossible. So there is no such polarization. On the other hand, since the class number H is 1 for this genus, and hence the type number T is also 1, we have $2T - H = 1$. More concretely, if we put

$$H = \begin{pmatrix} 3 & \pi(1+\beta) \\ -\pi(1+\beta) & 3 \end{pmatrix},$$
$$\alpha = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix} = \begin{pmatrix} \beta\pi & 0 \\ 0 & \pi \end{pmatrix},$$

then H corresponds with a lattice in $\mathcal{L}_{npr}$, and we have $\alpha H \alpha^* = 3H$ and $\alpha M_2(O) = M_2(O)\alpha$. This means that the corresponding polarized abelian surface has a model over $\mathbb{F}_3$. Besides, for any $n \geq 2$, if we take $[n/2]$ copies of H and take

$$H_n = H \perp \cdots \perp H \perp p$$

where p appears only when n is odd, then the corresponding n-dimensional polarized abelian variety also has a model over $\mathbb{F}_3$. By the way, for $n = 2$, we will see in [6] that $2T(\mathcal{L}_{npr}) - H(\mathcal{L}_{npr}) > 0$ for all p. So in the same argument, we see that

Proposition 4.4. *For all primes p, there exists a polarized abelian variety, whose polarization belongs to $\mathcal{L}_{npr}$, that has a model over $\mathbb{F}_p$.*

4.3. **Components of the supersingular locus which have models over $\mathbb{F}_p$.** We denote by $\mathcal{A}_{n,1}$ the moduli of principally polarized abelian varieties and by $\mathcal{S}_{n,1}$ the locus of principally polarized supersingular abelian varieties in $\mathcal{A}_{n,1}$. The author learned the following theorem from Professor F. Oort.

Theorem 4.5 (Li-Oort[10], Oort [11], Katsura-Oort [9])**.** *(1) The set of irreducible components of $\mathcal{S}_{n,1}$ corresponds bijectively with equivalence classes of polarizations of E^n belonging to $\mathcal{L}_{pr}$ if n is odd, and to $\mathcal{L}_{npr}$ if n is even, respectively.*
(2) The locus $\mathcal{S}_{n,1}$ is defined over $\mathbb{F}_p$. Each irreducible component of

$\mathcal{S}_{n,1}$ is defined over $\mathbb{F}_{p^2}$. The irreducible component corresponding to the polarization λ in the sense of (1) has a model defined over $\mathbb{F}_p$ if and only if (E^n, λ) has a model over $\mathbb{F}_p$.

For any genus $\mathcal{L}$ of quaternion hermitian lattices, we denote by $H(\mathcal{L})$ and $T(\mathcal{L})$ the class number and the type number of $\mathcal{L}$ as before. As a corollary of our previous Theorems 4.3 and 4.5 and Proposition 4.4, the following theorem is obvious.

Theorem 4.6. *Assume that $n \geq 2$. Then the number of irreducible components of $\mathcal{S}_{n,1}$ which have models over $\mathbb{F}_p$ is equal to $2T(\mathcal{L}_{pr}) - H(\mathcal{L}_{pr})$ when n is odd and to $2T(\mathcal{L}_{npr}) - H(\mathcal{L}_{npr})$ when n is even. In particular, there always exists an irreducible component of $\mathcal{S}_{n,1}$ defined over $\mathbb{F}_p$.*

Proof. Except for the last claim, the assertion has been already proved. It is obvious that $2T(\mathcal{L}_{pr}) - H(\mathcal{L}_{pr}) > 0$ for all n, since E^n has a principal polarization defined over $\mathbb{F}_p$. So by Proposition 4.4 and Theorem 4.5, we have the claim. □

When $n = 2$, the number $2T(\mathcal{L}_{npr}) - H(\mathcal{L}_{npr})$ is concretely given in [6] and is always positive, as we remarked.

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