Order of the canonical vector bundle over configuration spaces of projective spaces

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Abstract

The order of a vector bundle is the smallest positive integer n such that the vector bundle's n-fold self-Whitney sum is trivial. Since 1970's, the order of the canonical vector bundle over configuration spaces of Euclidean spaces has been studied by F.R. Cohen, R.L. Cohen, N.J. Kuhn and J.L. Neisendorfer [4], F.R. Cohen, M.E. Mahowald and R.J. Milgram [6], and S.W. Yang [17, 18]. And the order of the canonical vector bundle over configuration spaces of closed orientable Riemann surfaces with genus greater than or equal to one has been studied by F.R. Cohen, R.L. Cohen, B. Mann and R.J. Milgram [5]. In this paper, we study the order of the canonical vector bundle over configuration spaces of projective spaces as well as of the Cartesian products of a projective space and a Euclidean space.

1 Introduction

Let ξ be a vector bundle and let $\xi^{\oplus n}$ be its n-fold Whitney sum. If there exists a positive integer n such that $\xi^{\oplus n}$ is trivial, then we say that ξ has finite order. In this case, the smallest such n is called the order of ξ, denoted by $o(\xi)$.

Let M be a path-connected m-dimensional manifold without boundary, $m \geq 2$. Given a positive integer k, the configuration space $F(M, k)$ is the space of all k-tuples of distinct points in M. The symmetric group on k-letters, denoted by Σ_k, acts on $F(M, k)$ from the left by

$$\sigma(x_1, x_2, \cdots, x_k) = (x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(k)}), \quad \sigma \in \Sigma_k.$$

This action is free and induces a covering map from $F(M, k)$ to $F(M, k)/\Sigma_k$. The associated vector bundle of this covering map is

$$\xi_{M,k} : \mathbb{R}^k \longrightarrow F(M, k) \times_{\Sigma_k} \mathbb{R}^k \longrightarrow F(M, k)/\Sigma_k$$

where Σ_k act on $\mathbb{R}^k$ by permuting the coordinates from the right.

Since 1970's, the order of the canonical vector bundle over configuration spaces of manifolds has been extensively studied (cf. [4, 5, 6, 11, 17, 18]). For a positive integer t, let $\rho(t)$

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be the number of positive integers less than or equal to t that are congruent to $0, 1, 2$ or 4 mod 8. In 1978, F.R. Cohen, M.E. Mahowald and R.J. Milgram [6] initiated the study of the order of the canonical vector bundle over configuration spaces of manifolds, and they proved in [6, Theorem 1.2] that the order of $\xi_{\mathbb{R}^2,k}$ is 2. Later, [6, Theorem 1.2] was generalized from the case of the plane to the case of higher dimensional Euclidean spaces. It is proved by S.W. Yang [17] in 1981, F.R. Cohen, R.L. Cohen, N.J. Kuhn and J.L. Neisendorfer [4] in 1983 and the present author [11] in 2016 that the order of $\xi_{\mathbb{R}^m,k}$ is

$$a_{m,k} = 2^{\rho(m-1)} \prod_{3\leq p\leq k,\ p \text{ prime}} p^{[\frac{m-1}{2}]}.$$

Moreover, generalizing [6, Theorem 1.2] from the case of the plane to the case of closed orientable Riemann surfaces with genus greater than or equal to one, F.R. Cohen, R.L. Cohen, B. Mann and R.J. Milgram [5, Proposition 1.1] proved in 1989 that for any closed orientable Riemann surface S whose genus is greater than or equal to one, both $\xi_{S,k}$ and $\xi_{S\setminus\{\text{point}\},k}$ have order 4. Let S^m be the m-sphere. Recently, we proved in [11, Theorem 1.1] that the order of $\xi_{S^m,k}$ is either $a_{m,k}$ or $2^{\rho(m)-\rho(m-1)}a_{m,k}$; and generalizing [5, Proposition 1.1] from the case of Riemann surfaces with genus greater than or equal to one to the case of S^2, we also proved in [11, Theorem 1.1] that if k is even, then the order of $\xi_{S^2,k}$ is 4.

In this paper, we study the order of the canonical vector bundle over configuration spaces of projective spaces as well as of the Cartesian products of a projective space and a Euclidean space. Let $\mathbb{R}P^m$ be the projective space of dimension m. In Theorem 1.1, we give some estimations for the order of $\xi_{\mathbb{R}P^m,k}$. In particular, the order of $\xi_{\mathbb{R}P^2,k}$ is a multiple 4. In Theorem 1.4, we give some estimations for the order of $\xi_{\mathbb{R}P^m\times\mathbb{R}^n,k}$, $n \geq 1$. As a particular case, if n is congruent to 3 mod 8, then the order of $\xi_{\mathbb{R}P^2\times\mathbb{R}^n,k}$ is $a_{n+2,k}$. Throughout this paper, all maps are assumed to be continuous and all manifolds are assumed to be finite CW-complexes.

Let $k \geq 2$ and let $N(m)$ be the smallest integer N such that $\mathbb{R}P^m$ can be embedded into $\mathbb{R}^N$. The main results of this paper are listed as below.

Theorem 1.1 (Main Result I). *The order of $\xi_{\mathbb{R}P^m,k}$ is divisible by $a_{m,k}$ and divides*

$$2^{\rho(N(m)-1)} \prod_{3\leq p\leq k,\ p \text{ prime}} p^{[\frac{m}{2}]}.$$

Moreover, the order of $\xi_{\mathbb{R}P^m,k}$ is divisible by $2a_{m,k}$ if one of the followings happen: (i). $m = 2$; (ii). $m = 4$, $k = 2$; (iii). $m = 8$, $k = 2$.

It follows from Theorem 1.1 immediately that the order of $\xi_{\mathbb{R}P^2,k}$ is divisible by 4 and divides $4\prod_{3\leq p\leq k,\ p \text{ prime}} p$.

Corollary 1.2. *Let m be odd and r be an integer greater than or equal to $\rho(N(m)-1)$. Then the order of $(\xi_{\mathbb{R}P^m,k})^{\oplus 2^r}$ is*

$$\prod_{3\leq p\leq k,\ p\ prime} p^{\frac{m-1}{2}}.$$

Corollary 1.2 follows from the first assertion of Theorem 1.1 immediately.

Corollary 1.3. *Let $s\geq 2$. Then the order of $\xi_{\coprod_s \mathbb{R}P^m,k}$ over the configuration space of the disjoint union of s-copies of $\mathbb{R}P^m$ is divisible by $2a_{m,k}$ if $m=4$ or 8.*

Corollary 1.3 follows from the second assertion of Theorem 1.1.

Theorem 1.4 (Main Result II)**.** *Let m be even and let n be odd. Then the order of $\xi_{\mathbb{R}P^m\times\mathbb{R}^n,k}$ is divisible by $a_{m+n,k}$ and divides*

$$2^{\rho(N(m)+n-1)}\prod_{3\leq p\leq k,\ p\ prime} p^{\frac{m+n-1}{2}}.$$

Suppose n is congruent to 3 mod 8. Then as a particular case of Theorem 1.4, the order of $\xi_{\mathbb{R}P^2\times\mathbb{R}^n,k}$ is $a_{n+2,k}$.

Corollary 1.5. *Let m be even, n be odd and r be an integer greater than or equal to $\rho(N(m)+n-1)$. Then the order of $(\xi_{\mathbb{R}P^m\times\mathbb{R}^n,k})^{\oplus 2^r}$ is*

$$\prod_{3\leq p\leq k,\ p\ prime} p^{\frac{m+n-1}{2}}.$$

Corollary 1.5 follows from Theorem 1.4 immediately.

Finally, as by-products of the above results, we will give some periodicity properties of the k-adic constructions in the last section of this paper.

2 On the order of the canonical vector bundle and the k-adic construction

In this section, we review some lemmas about the order of the canonical vector bundle over configuration spaces as well as its relation to the k-adic construction.

Lemma 2.1. *The order of $\xi_{M,2}$ is a power of 2.*

Proof. By the Whitney embedding theorem, we can embed M into $\mathbb{R}^{2m+1}$. Then $\xi_{M,2}$ is a pull-back vector bundle of $\xi_{\mathbb{R}^{2m+1},2}$ through the embedding. Hence $o(\xi_{M,2})$ divides $o(\xi_{\mathbb{R}^{2m+1},2})$, which is a power of 2. The lemma follows. □

Let p be a prime. We denote $o_p(\xi_{M,k})$ to be the largest integer of the form p^l, $l \in \mathbb{Z}$, that divides $o(\xi_{M,k})$. We call $o_p(\xi_{M,k})$ the p-power of $o(\xi_{M,k})$.

Lemma 2.2. *[5] For any prime p and any $k \geq p$, $o_p(\xi_{M,k}) \leq o_p(\xi_{M,p})$.*

Proof. The proof follows from [5, p. 105]. □

The following lemma is a straight-forward generalization of [5, Lemma 2.1]. A detailed proof can be found in [11, Proof of Lemma 2.2].

Lemma 2.3. *[5, 11] Let M be a non-compact manifold. Then for any prime p and any $k \geq p$, $o_p(\xi_{M,k}) = o_p(\xi_{M,p})$.*

For a topological space X with a non-degenerate base-point $*$, the labelled configuration space (resp. the augmented labelled configuration space) is defined by

$$\begin{aligned} C(M;X) &= \coprod_{k\geq 1} F(M,k)\times_{\Sigma_k} X^k/\approx \\ (\text{resp. } C^*(M;X) &= \coprod_{k\geq 0} F(M,k)\times_{\Sigma_k} X^k/\approx) \end{aligned}$$

where the equivalent relation $\approx$ is generated by

$$(m_1,\cdots,m_k;x_1,\cdots,x_k) \approx (m_1,\cdots,m_{k-1};x_1,\cdots,x_{k-1})$$

if $x_k = *$, and the space $F(M,0)$ is defined to be a base-point. In [1, 10], such spaces occur as models for mapping spaces. We call the space $C(M;X)$ labelled configuration space for the reason that X plays the role as a label attached to each configuration $(m_1,m_2,\cdots,m_k)$ in $F(M,k)$, $k=1,2,\cdots$. The space $C(M;X)$ is filtered by closed subspaces

$$C_k(M;X) = \coprod_{j=1}^{k} F(M,j)\times_{\Sigma_j} X^j/\approx$$

with $C_0(M;X)$ defined to be the base-point and $C_1(M;X)$ defined to be the space $(M\vee S^0)\wedge X$. The inclusions of $C_{k-1}(M;X)$ into $C_k(M;X)$ are cofibrations [9, Theorem 7.1]. Their cofibres are denoted by

$$D_k(M;X) = C_k(M;X)/C_{k-1}(M;X),$$

called the k-adic construction. There is a splitting (cf. [15, Proposition 2.4])

$$\Sigma^\infty C(M;X) \simeq \Sigma^\infty \bigvee_{k=1}^{\infty} D_k(M;X).$$

Hence the k-adic constructions are stable wedge summands of the labelled configuration space.

The following lemma follows from an unpublished manuscript given by Professor Frederick R. Cohen. A copy of the proof can be found in [11, Lemma 2.7 and Corollary 2.8].

Lemma 2.4. *For any positive integer t, there is a homotopy equivalence*

$$\Sigma^{o(\xi_{M,k})kt}(D_k(M;X)) \longrightarrow D_k(M;\Sigma^{o(\xi_{M,k})t}X).$$

3 On the (co)homology of configuration spaces

In this section, we prove some lemmas about the (co)homology of configuration spaces. Let M be a closed manifold. Let p be a prime. Let $\mathbb{F}$ be either the field $\mathbb{Z}_p$ with p elements, or the rational numbers $\mathbb{Q}$.

3.1 (co)Homology of configuration spaces of projective spaces

In this subsection, we study the (co)homology of configuration spaces of $\mathbb{R}P^m$ and prove Lemma 3.2 and Lemma 3.4. Throughout this subsection, we assume that m is odd and n is even.

We define the graded algebra

$$\mathcal{C}(H_*(M;\mathbb{F});n) = \otimes_{q=0}^{m} \otimes_{\beta_q} H_*(\Omega^{m-q}S^{m+n};\mathbb{F}) \tag{3.1}$$

where $\beta_q = \dim_{\mathbb{F}} H_q(M;\mathbb{F})$ is the q-th Betti number. Each term $H_*(\Omega^{m-q}S^{m+n};\mathbb{F})$ in (3.1) has weights associated to its generators. Thus in the tensor product (3.1), we have an induced filtration $F_k\mathcal{C}(H_*(M;\mathbb{F});n)$ by weights. The filtration of $\mathcal{C}(H_*(M;\mathbb{F});n)$ by weights agrees with the filtration $C_k(M;S^n)$ of $C(M;S^n)$ via the following isomorphism (cf. [2, Theorem A, Theorem B])

$$H_*(C(M;S^n);\mathbb{F}) \cong \mathcal{C}(H_*(M;\mathbb{F});n).$$

For any $k \geq 1$, we let $\mathcal{D}_k(H_*(M;\mathbb{F});n)$ to be the quotient space

$$F_k\mathcal{C}(H_*(M;\mathbb{F});n)/F_{k-1}\mathcal{C}(H_*(M;\mathbb{F});n).$$

The next theorem gives the homology of unordered configuration spaces of M.

Theorem 3.1. *[2, Theorem C] There is an isomorphism of graded vector spaces*

$$H_*(F(M,k)/\Sigma_k;\mathbb{F}) \cong \Sigma^{-kn}\mathcal{D}_k(H_*(M;\mathbb{F});n).$$

As a result of Theorem 3.1, the next lemma follows.

Lemma 3.2. *If M is a rational homology sphere, then as graded vector spaces,*

$$H_*(F(M,k)/\Sigma_k;\mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{if } i = 0 \text{ or } m, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since M is a rational homology sphere, $H_*(M;\mathbb{Q})$ is isomorphic to $H_*(S^m;\mathbb{Q})$. It follows with the help of Theorem 3.1 that as graded vector spaces,

$$H_*(F(M,k)/\Sigma_k;\mathbb{Q}) \cong H_*(F(S^m,k)/\Sigma_k;\mathbb{Q}).$$

Consequently, with the help of the following fact (cf. [13, 14])

$$H^i(F(S^m,k)/\Sigma_k;\mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } i = 0 \text{ or } m, \\ 0, & \text{otherwise,} \end{cases}$$

the assertion follows. □

Remark 3.3. Since m is assumed odd, $\mathbb{R}P^m$ is a rational homology sphere. Hence the rational homology of the unordered configuration space of $\mathbb{R}P^m$ is given by Lemma 3.2.

Let p be an odd prime. In [2, Section 4], a precise description for $H_*(C(M;S^n);\mathbb{Z}_2)$ as well as its filtration was given in terms of Dyer-Lashof operations. In the following, we give a precise description for $H_*(C(M;S^n);\mathbb{Z}_p)$ and its filtration.

Firstly, for each $\alpha \in H_q(M;\mathbb{Z}_p)$, we introduce a generator u_α, and set its dimension and weight as

- dimension: $|u_\alpha| = |\alpha| + n$,
- weight: $\omega(u_\alpha) = 1$.

Secondly, for each u_α and index $I = (\epsilon_1, i_1, \epsilon_2, i_2, \cdots, \epsilon_r, i_r)$, $\epsilon_j = 0$ or 1 for each $j = 1, 2, \cdots, r$, there is an additional generator given by Dyer-Lashof operations (with lower indices) and Bockstein homomorphisms acting on u_α

$$Q_I u_\alpha = \beta^{\epsilon_1} Q_{i_1} \beta^{\epsilon_2} Q_{i_2} \cdots \beta^{\epsilon_r} Q_{i_r} u_\alpha$$

if the following condition hold (cf. [12, p.537]):

- $0 < i_1 \leq i_2 \leq \cdots \leq i_r < n$;
- i_r has the same parity as $|\alpha| + n$;
- the indices of two adjacent Q_i's have the same parity;
- the indices of two Q_i's separated by a β have opposite parity.

Such sequences of operations Q_I are called admissible. The dimension and weight of $Q_I u_\alpha$ are given inductively by

- dimension: $|Q_i x| = p|x| + i(p-1)$,

- weight: $\omega(Q_i x) = \omega(\beta Q_i x) = p\omega(x)$.

Thirdly, all the generators u_α and $Q_I u_\alpha$ are subject to the following relations

- $u_{\alpha+\beta} = u_\alpha + u_\beta$,
- $Q_I u_{\alpha+\beta} = Q_I u_\alpha + Q_I u_\beta$,
- $u_\alpha^2 = 0$ if $|\alpha| = m$.

Then $\mathcal{C}(H_*(M;\mathbb{Z}_p), n)$ can be described as the associative and commutative $\mathbb{Z}_p$-algebra generated by all u_α and $Q_I u_\alpha$.

As a result of the above description of $\mathcal{C}(H_*(M;\mathbb{Z}_p), n)$, the next lemma follows.

Lemma 3.4. *Let p be an odd prime. If M is a mod p homology m-sphere, then*

$$\begin{aligned} & Tor_p(H^k(F(M,p)/\Sigma_p;\mathbb{Z})) \\ = & \begin{cases} \mathbb{Z}_p, & \text{if } k = 2s(p-1), 1 \le s \le (m-1)/2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.2}$$

Proof. By Theorem 3.1 and the above description of $\mathcal{C}(H_*(M;\mathbb{Z}_p), n)$,

$$\begin{aligned} H_*(F(M,p)/\Sigma_p;\mathbb{Z}_p) &= \oplus_{s=1}^{(m-1)/2}(\mathbb{Z}_p Q_{2s} u_0 \oplus \mathbb{Z}_p \beta Q_{2s} u_0) \\ &\quad \oplus \mathbb{Z}_p u_0^p \oplus \mathbb{Z}_p u_m u_0^{p-1}. \end{aligned}$$

Here the dimensions of the generators are

$$\begin{aligned} |u_0^p| &= 0, \\ |u_m u_0^{p-1}| &= m, \\ |Q_{2s} u_0| &= 2s(p-1), \\ |\beta Q_{2s} u_0| &= 2s(p-1) - 1. \end{aligned}$$

Moreover, since $\beta^2 = 0$, we have

$$\beta(u_m u_0^{p-1}) = 0.$$

Consequently, by applying the Bockstein Spectral Sequence and the Universal Coefficient Theorem for cohomology (cf. [17, Proof of (3.2)], [18, p.17]), we obtain (3.2). □

Remark 3.5. Since m is assumed odd and p is an odd prime, $\mathbb{R}P^m$ is a mod p homology m-sphere. Hence the p-torsion part of the integral cohomology of $F(\mathbb{R}P^m, p)/\Sigma_p$ is given by Lemma 3.4.

3.2 (co)Homology of configuration spaces of the Cartesian products of a projective space and a Euclidean space

In this subsection, we study the (co)homology of configuration spaces of $\mathbb{R}P^m \times \mathbb{R}^n$ and prove Lemma 3.7 and Lemma 3.8. Throughout this subsection, we assume that m is even and $n \geq 1$ is odd.

We define the algebra

$$\mathcal{G}^{m+n}(H_*(M;\mathbb{F});X) = \otimes_{q=0}^{m+n} \otimes_{\beta_q} H_*(\Omega^{m+n-q}\Sigma^{m+n}X;\mathbb{F}) \tag{3.3}$$

where $\beta_q = \dim_{\mathbb{F}} H_q(M;\mathbb{F})$ is the q-th Betti number, and $\Omega^{m+n-q}\Sigma^{m+n}X = *$ if $q > m+n$. Each term $H_*(\Omega^{m+n-q}\Sigma^{m+n}X;\mathbb{F})$ in (3.3) has weights associated to its generators. Hence after taking tensor product, (3.3) is a filtered algebra. We denote the filtration of (3.3) by F_k, $k = 1, 2, \cdots$. We write $\mathcal{P}_k^{m+n}(H_*(M;\mathbb{Z}_p);X)$ for the $\mathbb{Z}_p$-module

$$F_k\mathcal{G}^{m+n}(H_*(M;\mathbb{F});X)/F_{k-1}\mathcal{G}^{m+n}(H_*(M;\mathbb{F});X).$$

Theorem 3.6. *[15, Theorem B] There is an isomorphism of $\mathbb{F}$-modules*

$$\tilde{H}_*(C(M \times \mathbb{R}^n;X);\mathbb{F}) \cong \oplus_{k=1}^{\infty}\Sigma^{-2k}\mathcal{P}_k^{m+n}(H_*(M;\mathbb{F});\Sigma^2 X).$$

By applying the filtrations, it follows from Theorem 3.6 that the graded vector space $H_*(F(M \times \mathbb{R}^n, k)/\Sigma_k;\mathbb{Q})$ only depends on the graded vector space $H_*(M;\mathbb{Q})$. Moreover, for any odd prime p, it is known (for example, [17, p.141]) that

$$\tilde{H}_*(F(\mathbb{R}^{m+n}, p)/\Sigma_p;\mathbb{Q}) = 0.$$

Consequently, we have the following lemma.

Lemma 3.7. *There is an isomorphism of graded vector spaces*

$$H_*(F(\mathbb{R}P^m \times \mathbb{R}^n, k)/\Sigma_k;\mathbb{Q}) \cong H_*(F(\mathbb{R}^{m+n}, k)/\Sigma_k;\mathbb{Q}).$$

Moreover, if p is an odd prime, then

$$\tilde{H}_*(F(\mathbb{R}P^m \times \mathbb{R}^n, p)/\Sigma_p;\mathbb{Q}) = 0.$$

The following lemma is obtained by setting M to be $\mathbb{R}P^m$ and setting $\mathbb{F}$ to be $\mathbb{Z}_p$, p an odd prime, in Theorem 3.6.

Lemma 3.8. *Let p be an odd prime. Then*

$$\begin{aligned} & Tor_p(H^k(F(\mathbb{R}P^m \times \mathbb{R}^n, p)/\Sigma_p;\mathbb{Z})) \\ = & \begin{cases} \mathbb{Z}_p, & \text{if } k = 2s(p-1), 1 \leq s \leq (m+n-1)/2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.4}$$

Proof. Since m is assumed even, we have

$$\begin{aligned}\tilde{H}_*(\mathbb{R}P^m \times \mathbb{R}^n;\mathbb{Z}_p) &\cong \tilde{H}_*(\mathbb{R}P^m;\mathbb{Z}_p)\\ &= 0.\end{aligned}$$

Hence it follows with the help of (3.3) that

$$\mathcal{G}^{m+n}(H_*(\mathbb{R}P^m \times \mathbb{R}^n;\mathbb{Z}_p);X) = H_*(\Omega^{m+n}\Sigma^{m+n}X;\mathbb{Z}_p). \tag{3.5}$$

By setting M to be $\mathbb{R}P^m$ and setting X to be S^0 in Theorem 3.6, we have

$$\tilde{H}_*(C(\mathbb{R}P^m \times \mathbb{R}^n;S^0);\mathbb{Z}_p) \cong \oplus_{k=1}^{\infty}\Sigma^{-2k}\mathcal{P}_k^{m+n}(\mathbb{Z}_p;S^2). \tag{3.6}$$

It follows from [1, Example 13] and [9, Theorem 2.7] that $\Omega^{m+n}S^{m+n+2}$ is weak homotopy equivalent to $C^*(\mathbb{R}^{m+n};S^2)$. Hence with the help of (3.5),

$$\begin{aligned}&\mathcal{P}_k^{m+n}(\mathbb{Z}_p;S^2)\\ \cong\;& F_k(H_*(\Omega^{m+n}S^{m+n+2};\mathbb{Z}_p))/F_{k-1}(H_*(\Omega^{m+n}S^{m+n+2};\mathbb{Z}_p))\\ \cong\;& F_k(H_*(C^*(\mathbb{R}^{m+n};S^2);\mathbb{Z}_p))/F_{k-1}(H_*(C^*(\mathbb{R}^{m+n};S^2);\mathbb{Z}_p))\\ \cong\;& H_*(D_k(\mathbb{R}^{m+n};S^2);\mathbb{Z}_p).\end{aligned} \tag{3.7}$$

By [2, Section 1.6, Section 2.6], $D_k(\mathbb{R}^{m+n};S^2)$ is the Thom space of $\xi^{\oplus 2}_{\mathbb{R}^{m+n},k}$. It follows from the Thom isomorphism that

$$H_*(F(\mathbb{R}^{m+n},k)/\Sigma_k;\mathbb{Z}_p) \cong \Sigma^{-2k}H_*(D_k(\mathbb{R}^{m+n};S^2);\mathbb{Z}_p). \tag{3.8}$$

Therefore, it follows from (3.6) - (3.8) that

$$\tilde{H}_*(C(\mathbb{R}P^m \times \mathbb{R}^n;S^0);\mathbb{Z}_p) \cong \oplus_{k=1}^{\infty}H_*(F(\mathbb{R}^{m+n},k)/\Sigma_k;\mathbb{Z}_p).$$

By the filtration of the lengths of configurations, it follows that as graded vector spaces,

$$H_*(F(\mathbb{R}P^m \times \mathbb{R}^n,k)/\Sigma_k;\mathbb{Z}_p) \cong H_*(F(\mathbb{R}^{m+n},k)/\Sigma_k;\mathbb{Z}_p). \tag{3.9}$$

In particular, we let $k = p$ in (3.9). By applying an argument analogous to [17, Proposition 3.1 and Proposition 3.2], we obtain (3.4). □

4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. In order to do this, we first prove the following proposition.

Proposition 4.1. *Let m be odd and let p be an odd prime. Then for any $k \geq p$,*

$$o_p(\xi_{\mathbb{R}P^m,k}) = p^{\frac{m-1}{2}}. \tag{4.1}$$

Proof. Let p be an odd prime and $k \geq p$. Let $K(-)$ denote the abelian group associated with the abelian semi-group of isomorphism classes of complex vector bundles under the Whitney sum operation and $\tilde{K}(-)$ the reduced generalized cohomology group associated to $K(-)$. Then we have an Atiyah-Hirzebruch Spectral Sequence with E_2-page

$$E_2^{i,j} = H^i(F(\mathbb{R}P^m, p)/\Sigma_p; K^j(*)). \tag{4.2}$$

This spectral sequence converges to a filtration of $K^{i+j}(F(\mathbb{R}P^m, p)/\Sigma_p)$ in the E_∞-page. Hence with the help of Lemma 3.2, the only differential whose domain and target are possible to have $\mathbb{Z}$-summands at the same time is

$$d_m : E_m^{0,2t} \longrightarrow E_m^{m,2t-m+1}, \quad t \in \mathbb{Z}.$$

Since $m + (2t - m + 1)$ is odd, d_m does not create new torsion parts of $K^0(\mathbb{R}P^m, p)/\Sigma_p)$. Hence all the differentials of the spectral sequence do not create new torsion parts of $K^0(\mathbb{R}P^m, p)/\Sigma_p)$, and the p-torsion part of (4.2), with $i + j = 0$, converges to

$$\mathrm{Tor}_p(K^0(F(\mathbb{R}P^m, p)/\Sigma_p)).$$

By the Atiyah-Hirzebruch Spectral Sequence and Lemma 3.4,

$$|\mathrm{Tor}_p(\tilde{K}(F(\mathbb{R}P^m, p)/\Sigma_p))| \leq p^{\frac{m-1}{2}}.$$

Thus it follows with the help of Lemma 2.2 that

$$\begin{aligned} o_p(\xi_{\mathbb{R}P^m,k}) &\leq o_p(\xi_{\mathbb{R}P^m,p}) \\ &= o_p(\xi_{\mathbb{R}P^m,p} \otimes \mathbb{C}) \\ &\leq |\mathrm{Tor}_p(\tilde{K}(F(\mathbb{R}P^m, p)/\Sigma_p))| \\ &\leq p^{\frac{m-1}{2}}. \end{aligned} \tag{4.3}$$

On the other hand, since $\mathbb{R}^m$ can be embedded into $\mathbb{R}P^m$, it follows that

$$o_p(\xi_{\mathbb{R}P^m,k}) \geq o_p(\xi_{\mathbb{R}^m,k}). \tag{4.4}$$

Therefore, (4.1) follows from (4.3) and (4.4). ☐

Now we prove Theorem 1.1.

Proof of Theorem 1.1. Let p be an odd prime and $k \geq p$. We first prove

$$o_p(\xi_{\mathbb{R}P^m,k}) = p^{[\frac{m}{2}]}. \tag{4.5}$$

When m is odd, Proposition 4.1 gives (4.5). When m is even, with the helps of the canonical embedding of $\mathbb{R}P^m$ into $\mathbb{R}P^{m+1}$, we have $o_p(\xi_{\mathbb{R}^m,k}) \leq o_p(\xi_{\mathbb{R}P^m,k}) \leq o_p(\xi_{\mathbb{R}P^{m+1},k})$. Thus (4.5) follows. On the other hand, since $\mathbb{R}P^m$ can be embedded into $\mathbb{R}^{N(m)}$, we have $o_2(\xi_{\mathbb{R}^m,k}) \leq o_2(\xi_{\mathbb{R}P^m,k}) \leq o_2(\xi_{\mathbb{R}^{N(m)},k})$. Therefore, by (4.5), the first assertion of Theorem 1.1 follows.

To prove the second assertion, we consider the following two cases.

Case 1. $m = 2$.

It follows from [3] or [7, Proposition 16] that

$$\begin{aligned} & H_1(F(\mathbb{R}P^2, k)/\Sigma_k; \mathbb{Z}) \\ = \; & B_k(\mathbb{R}P^2)/[B_k(\mathbb{R}P^2), B_k(\mathbb{R}P^2)] \\ \cong \; & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{aligned} \tag{4.6}$$

where $B_k(\mathbb{R}P^2)$ is the k-stranded braid group of $\mathbb{R}P^2$. By the Universal Coefficient Theorem and (4.6), there is no element of order 4 in $H^1(F(\mathbb{R}P^2, k)/\Sigma_k; \mathbb{Z}_4)$. Consequently, by a direct computation, for any non-zero element x in $H^1(F(\mathbb{R}P^2, k)/\Sigma_k, \mathbb{Z}_2)$, we have $x^2 = Sq^1 x = \beta x \neq 0$ where β is the Bockstein homomorphism associated to the short exact sequence $\mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$. Therefore, with the help of [11, Lemma 3.5],

$$w(\xi_{\mathbb{R}P^2,k}^{\oplus 2}) \neq 0.$$

Hence $\xi_{\mathbb{R}P^2,k}^{\oplus 2}$ is not trivial, and $o_2(\xi_{\mathbb{R}P^2,k}) \geq 4$. Finally, since $N(2) = 4$, it follows with the help of the first assertion of Theorem 1.1 that $o_2(\xi_{\mathbb{R}P^2,k}) = 4$.

Case 2. $m = 4$ or 8, $k = 2$.

With the help of [8, Theorem 2.2] or [16, p. 380], it follows from a direct computation that the largest integer λ such that

$$[w_1(\xi_{\mathbb{R}P^m,2})]^\lambda \neq 0$$

is $\lambda = 2^{[\log_2 m]+1} - 1$. Hence it follows that

$$w(\xi_{\mathbb{R}P^m,2}^{\oplus 2^{[\log_2 m]+1}-1}) = 1 + \cdots + [w_1(\xi_{\mathbb{R}P^m,2})]^{2^{[\log_2 m]+1}-1},$$

which is non-trivial. This implies

$$o(\xi_{\mathbb{R}P^m,2}) \geq 2^{[\log_2 m]+1}. \tag{4.7}$$

Since $m = 4$ or 8, we have $2a_{m,2} = 2^{[\log_2 m]+1}$. It follows from (4.7) that the order of $\xi_{\mathbb{R}P^m,2}$ can be divided by $2a_{m,2}$. □

Proof of Corollary 1.3. We observe that for any $s \geq 2$, the order of $\xi_{\coprod_s \mathbb{R}P^m, k}$ equals to the smallest common multiple of $o(\xi_{\mathbb{R}P^m,t})$, $1 \leq t \leq k$ (cf. [11, Lemma 3.3]). Let $m = 4$ or 8. Since $2a_{m,2}$ divides $o(\xi_{\mathbb{R}P^m,2})$ and $a_{m,k}$ divides $o(\xi_{\mathbb{R}P^m,k})$, the smallest common multiple of $2a_{m,2}$ and $a_{m,k}$ divides the order of $\xi_{\coprod_s \mathbb{R}P^m,k}$. The corollary follows. $\square$

5 Proof of Theorem 1.4

In this section, we prove Theorem 1.4. In order to do this, we prove the next proposition first.

Proposition 5.1. *Let m be even, n be odd and p be an odd prime. Then for any $k \geq p$,*

$$o_p(\xi_{\mathbb{R}P^m \times \mathbb{R}^n, k}) = p^{\frac{m+n-1}{2}}. \tag{5.1}$$

Proof. Let p be an odd prime. Then we have an Atiyah-Hirzebruch Spectral Sequence with E_2-page

$$E_2^{i,j} = H^i(F(\mathbb{R}P^m \times \mathbb{R}^n, p)/\Sigma_p; K^j(*)). \tag{5.2}$$

This spectral sequence converges to a filtration of $K^{i+j}(F(\mathbb{R}P^m \times \mathbb{R}^n, p)/\Sigma_p)$ in the E_∞-page. With the help of Lemma 3.7, there is no differential whose domain and target are possible to have $\mathbb{Z}$-summands at the same time. Hence all the differentials of the above spectral sequence do not create new torsion parts of $K^0(\mathbb{R}P^m \times \mathbb{R}^n, p)/\Sigma_p)$, and the p-torsion part of (5.2), with $i + j = 0$, converges to

$$\mathrm{Tor}_p(K^0(F(\mathbb{R}P^m \times \mathbb{R}^n, p)/\Sigma_p)).$$

Consequently, it follows from Lemma 3.8 that

$$|\mathrm{Tor}_p(\tilde{K}(F(\mathbb{R}P^m \times \mathbb{R}^n, p)/\Sigma_p))| \leq p^{\frac{m+n-1}{2}}.$$

Hence for any $k \geq p$,

$$\begin{aligned} o_p(\xi_{\mathbb{R}P^m \times \mathbb{R}^n, k}) &\leq o_p(\xi_{\mathbb{R}P^m \times \mathbb{R}^n, p}) \\ &\leq |\mathrm{Tor}_p(\tilde{K}(F(\mathbb{R}P^m \times \mathbb{R}^n, p)/\Sigma_p))| \\ &\leq p^{\frac{m+n-1}{2}}. \end{aligned} \tag{5.3}$$

On the other hand, since $\mathbb{R}^{m+n}$ can be embedded into $\mathbb{R}P^m \times \mathbb{R}^n$, we have

$$o_p(\xi_{\mathbb{R}P^m \times \mathbb{R}^n, k}) \geq o_p(\xi_{\mathbb{R}^{m+n}, k}). \tag{5.4}$$

Therefore, (5.1) follows from (5.3) and (5.4). $\square$

Now we prove Theorem 1.4 and Corollary 1.5.

Proof of Theorem 1.4. The order of $\xi_{\mathbb{R}^{m+n},k}$ divides the order of $\xi_{\mathbb{R}P^m\times\mathbb{R}^n,k}$. Since $\mathbb{R}P^m\times\mathbb{R}^n$ can be embedded into $\mathbb{R}^{N(m)+n}$, we see that the order of $\xi_{\mathbb{R}P^m\times\mathbb{R}^n,k}$ divides the order of $\xi_{\mathbb{R}^{N(m)+n},k}$. Hence $o_2(\xi_{\mathbb{R}P^m\times\mathbb{R}^n,k})$ is divisible by $2^{\rho(m+n-1)}$ and divides $2^{\rho(N(m)+n-1)}$. With the help of Proposition 5.1, Theorem 5.1 follows. □

6 Stable homotopy types of the k-adic construction

With the help of the order of the canonical vector bundle over configuration spaces, the stable homotopy types of $D_k(M;\Sigma^n X)$ exhibit a natural periodic behavior as n varies. In this section, as by-products of our main results, we give the following propositions.

Proposition 6.1. *Let X be a topological space with a non-degenerate base-point. Let m be odd and r be an integer greater than or equal to $\rho(N(m)-1)$. Then*

$$\Sigma^{(2^r\prod_{3\le p\le k,\ p\ prime} p^{\frac{m-1}{2}})k} D_k(\mathbb{R}P^m;X) \simeq D_k(\mathbb{R}P^m;\Sigma^{2^r\prod_{3\le p\le k,\ p\ prime} p^{\frac{m-1}{2}}}X).$$

Proposition 6.1 follows from Corollary 1.2 and Lemma 2.4.

Proposition 6.2. *Let X be a topological space with a non-degenerate base-point. Let m be even, n be odd and r be an integer greater than or equal to $\rho(N(m)+n-1)$. Then*

$$\begin{aligned} & \Sigma^{(2^r\prod_{3\le p\le k,\ p\ prime} p^{\frac{m+n-1}{2}})k} D_k(\mathbb{R}P^m\times\mathbb{R}^n;X) \\ \simeq\ & D_k(\mathbb{R}P^m\times\mathbb{R}^n;\Sigma^{2^r\prod_{3\le p\le k,\ p\ prime} p^{\frac{m+n-1}{2}}}X). \end{aligned}$$

Proposition 6.2 follows from Corollary 1.5 and Lemma 2.4.

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References

[1] C.F. Bödigheimer, *Stable splittings of mapping spaces.* Lecture Notes in Math. **1286** (1985), 174-187.

[2] C.F. Bödigheimer, F.R. Cohen and L. Taylor, *On the homology of configuration spaces.* Topology **28** (1989), 111-123.

[3] J.V. Buskirk, *Braid groups of compact 2-manifolds with elements of finite order.* Trans. Amer. Math. Soc. **122** (1966), 81-97.

[4] F.R. Cohen, R.L. Cohen, N.J. Kuhn and J.L. Neisendorfer, *Bundles over configuration spaces.* Pacific J. Math. **104** (1983), 47-54.

[5] F.R. Cohen, R.L. Cohen, B. Mann and R.J. Milgram, *Divisors and configurations on a surface.* Contemp. Math. **96** (1989), 103-108.

[6] F.R. Cohen, M.E. Mahowald and R.J. Milgram, *The stable decomposition for the double loop space of a sphere.* Proc. Sympos. Pure Math. **32** (1978), 225-228.

[7] D.L. Gonçalves and J. Guaschi, *The braid groups of the projective plane.* Algebr. Geom. Topol. **4** (2004), 757-780.

[8] D. Handel, *Some existence and nonexistence theorems for k-regular maps.* Fund. Math. **109** (1980), 229-233.

[9] J.P. May, *The geometry of iterated loop spaces.* Lecture Notes in Math. **271** (1972).

[10] D. McDuff, *Configuration spaces of positive and negative particles.* Topology **14** (1975), 91-107.

[11] S. Ren, *Order of the canonical vector bundle over configuration spaces of disjoint unions of spheres.* arXiv/abs/1605.04737.

[12] P. Salvatore, *Configuration spaces on the sphere and higher loop spaces.* Math. Z. **248** (2004), 527-540.

[13] M.B. Sevryuk, *Cohomology of projectively compactified complex swallowtails and their complements.* Russian Math. Surveys **39** (1984), 285-286.

[14] O. Randal-Williams, *"Topological chiral homology" and configuration spaces of spheres.* Morfismos **17** (2013), 57-70.

[15] J. Wu, *On the homology of configuration spaces $C((M) \times \mathbf{R}^n; X)$.* Math. Z. **22** (1998), 235-248.

[16] W. Wu, *On the realization of complexes in Euclidean space, II.* Scientia Sinica **7** (1958), 365-387.

[17] S.W. Yang, *Order of the canonical vector bundle on $C_n(k)/\Sigma_k$.* Illinois J. Math. **25** (1981), 136-146.

[18] S.W. Yang, *Order of the canonical vector bundle on $C_n(k)/\Sigma_k$.* Ph.D Dissertation, Brandeis University, 1978.

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