ON A THEOREM OF MURAI

by

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ABSTRACT

If B is a p-block of a finite group G, then the intersection of the kernels of the height zero characters in B has a normal p-complement.

Dedicated to M. Murai

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If B is a p-block of a finite group G, it was already known to Richard Brauer himself that the group

$$\bigcap_{\chi \in \mathrm{Irr}(B)} \ker(\chi)$$

has order not divisible by p. (See for instance Theorem 6.10 of [N].) If $\mathrm{Irr}_0(B)$ is the subset of height zero characters in $\mathrm{Irr}(B)$, we show the following.

THEOREM A. *If B is a Brauer p-block of a finite group G, then*

$$\bigcap_{\chi \in \mathrm{Irr}_0(B)} \ker(\chi)$$

has a normal p-complement.

In fact, Theorem A is an easy consequence of a nice theorem of M. Murai (Theorem 4.4 of [M]) to whom this note is dedicated. The contributions of Murai to block theory are not easily overstated.

Proof of Theorem A. Let K be the intersection of the kernels of the height zero characters in B. Let b be a block of K covered by B. Let D be a common defect group of B and of the Fong-Reynolds correspondent of B over b (Theorem 9.14 of [N]). By Lemma 2.2 of [M], we have that the unique block $\hat{b}$ of KD that covers b has defect group D. Let $\tau \in \mathrm{Irr}_0(\hat{b})$. By Corollary 9.18 of [N], we have that $\tau_K = \theta \in \mathrm{Irr}(K)$, and θ has height zero in b. Now, by Theorem 4.4 of [M], we have that θ lies below some $\chi \in \mathrm{Irr}_0(B)$. By the definition of K, we have that $\theta = 1_K$, and therefore τ is linear. Fix some (linear) $\tau \in \mathrm{Irr}_0(\hat{b})$, and consider

$$\tilde{b} = \{\bar{\tau}\gamma \mid \gamma \in \mathrm{Irr}(\hat{b})\} \subseteq \mathrm{Irr}(KD)\,,$$

where $\bar{\tau}$ is the complex conjugate of τ. By using Theorem 3.19 of [N], for instance, we check that $\tilde{b}$ is a block of KD. Furthermore, every height zero character in $\tilde{b}$ is linear. Since 1_{KD} is in $\tilde{b}$, we have that $\tilde{b}$ is the principal block of KD. Now we use a result of Isaacs-Smith, Corollary 3 of [IS], to conclude that KD has a normal p-complement. Hence, K has a normal p-complement, as desired. ∎

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