

Answer to a Question by Nakamura, Nakanishi, and Satoh involving crossing numbers of knots ^{*}

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Abstract

In this paper we give a positive answer to a question raised by Nakamura, Nakanishi, and Satoh concerning an inequality involving crossing numbers of knots. We show it is an equality only for the trefoil and for the figure-eight knots.

Keywords: Knot colorings; Minimum number of colors; Crossing number; Determinants of knots.

1 Introduction

A Fox m -coloring [4] is an assignment of elements from $\{0, 1, \dots, m-1\}$ to the arcs of a link diagram such that at each crossing twice the integer assigned to the over-arc equals to the sum of the integers assigned to the two under-arcs mod m . For each link diagram and each modulus $m > 1$, there are always m trivial colorings, namely by assigning the same integer mod m to every arc of the diagram. A coloring with at least two distinct colors (i.e., two distinct integers mod m assigned to two arcs) is called a non-trivial coloring. It is easy to check that if one diagram of a link has a non-trivial m -coloring, then each diagram of that link has a non-trivial m -coloring. A link is called m -colorable if it admits a diagram with non-trivial m -colorings. The following well-known theorem (see Exercise 8, page 133 of [3]) presents a criterion for checking if a given link is m -colorable.

Theorem 1. [3] *A link L is m -colorable if and only if the determinant of L ($\det L$) and m are not relatively prime.*

For the proof of Theorem 1, for example, refer to [6].

The following definition was introduced by Harary and Kauffman in [5].

Definition 1. Given an integer m greater than 1. Let L be a link admitting non-trivial m -colorings. Let D be a diagram of L , and let $n_{m,D}$ be the minimum number of colors

^{*} Mathematics Subject Classification 2010: 57M27

mod m which takes to construct a non-trivial m -coloring on D . Set

$$\text{mincol}_m L \doteq \min\{n_{m,D} \mid D \text{ is a diagram of } L\}.$$

We call $\text{mincol}_m L$ the minimum number of colors of L , mod m .

We call any non-trivial m -coloring of L using $\text{mincol}_m L$ colors a minimal m -coloring of L .

Nakamura, Nakanishi, and Satoh proved the following theorem in [7].

Theorem 2. *Let p be an odd prime. Any p -colorable knot K satisfies*

$$\text{mincol}_p(K) \geq \lfloor \log_2 p \rfloor + 2 \tag{1}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x .

Let $c(K)$ denote the crossing number of K . Since $c(K) \geq \text{mincol}_p(K)$, any p -colorable knot K satisfies $c(K) \geq \lfloor \log_2 p \rfloor + 2$. In Remark 3.3 (iii) on page 96 of [7], Nakamura, Nakanishi, and Satoh ask if the equality only holds for the trefoil knot ($p = 3$) and the figure-eight knot ($p = 5$). We give a positive answer to this question for classical knots.

2 Answering the Question

We recall that Theorem 2 which states that $\text{mincol}_p(K) \geq \lfloor \log_2 p \rfloor + 2$ is proved in Nakamura et al.' [7]. Since the crossing number of knot K , $c(K)$, satisfies $c(K) \geq \text{mincol}_p(K)$, for any p -colorable knot K , these authors wonder if the equality $c(K) = \lfloor \log_2 p \rfloor + 2$ only holds for the trefoil and the figure-eight knots, see (iii) in Remark 3.3 on page 96 of [7]. Here we settle this matter with Theorem 3.

Theorem 3. *Let p be an odd prime. Let K be a p -colorable classical knot. Then the equality in $c(K) \geq \lfloor \log_2 p \rfloor + 2$ only holds for the trefoil knot ($p = 3$) and the figure-eight knot ($p = 5$).*

Let D be a link diagram. Let

$$d_n^\infty := \max\{\det(D) \mid D \text{ is a link diagram of } n \text{ crossings}\}.$$

In [8], Stoimenow showed

$$d_n^\infty \leq d_{n-1}^\infty + d_{n-2}^\infty + d_{n-3}^\infty \quad (n > 2), \tag{2}$$

and then proved the following theorem.

Theorem 4. [8] Let $\delta \approx 1.83929$ be the real positive root of $x^3 - x^2 - x - 1 = 0$. There exists a constant $C > 0$ such that for any link diagram D of $c(D)$ crossings

$$\det(D) \leq C \cdot \delta^{c(D)}.$$

We now prove that $C = 2/\delta^2 \approx 0.59120$ is always valid for non-trivial link diagrams.

Theorem 5. Let $\delta \approx 1.83929$ be the real positive root of $x^3 - x^2 - x - 1 = 0$. Then for any non-trivial link diagram D of $c(D)$ crossings,

$$\det(D) \leq \frac{2}{\delta^2} \cdot \delta^{c(D)}.$$

Proof. We prove it by induction.

Since $d_1^\infty = 1$, $d_2^\infty = 2$, $d_3^\infty = 3$, it is easy to check that $\det(D) \leq \frac{2}{\delta^2} \cdot \delta^{c(D)}$ holds for any diagram D with $1 \leq c(D) \leq 3$.

For any given integer $n \geq 3$, suppose that for any link diagram D with $1 \leq c(D) \leq n$, $\det(D) \leq \frac{2}{\delta^2} \cdot \delta^{c(D)}$ holds. Then by the definition of d_k^∞ , we have $d_k^\infty \leq \frac{2}{\delta^2} \cdot \delta^k$ for each $1 \leq k \leq n$.

So for any link diagram D' with $n + 1$ crossings, we have

$$\begin{aligned} \det(D') &\leq d_{n+1}^\infty \\ &\leq d_n^\infty + d_{n-1}^\infty + d_{n-2}^\infty \\ &\leq \frac{2}{\delta^2} \cdot \delta^{n-2} \cdot (\delta^2 + \delta + 1) \\ &= \frac{2}{\delta^2} \cdot \delta^{n+1}. \end{aligned}$$

□

Proof of Theorem 3. Unknot is not p -colorable for any prime p , so we only need to consider non-trivial knots.

Let \tilde{D} be a minimal diagram of K . Since K is a p -colorable knot, we have $p \mid \det(K)$ and $\det K > 0$. By Theorem 4,

$$\log_2 p \leq \log_2 \det(K) = \log_2 \det(\tilde{D}) \leq c(\tilde{D}) \log_2 \delta + \log \frac{2}{\delta^2} < 0.87915 \cdot c(K) - 0.5256.$$

It is easy to see, for $c(K) \geq 13$,

$$c(K) > 0.87915 \cdot c(K) - 0.5256 + 2 > \log_2 p + 2 > \lfloor \log_2 p \rfloor + 2.$$

Table 1 shows the numerical results of d_n^∞ and $\lfloor \log_2 d_n^\infty \rfloor + 2$ for $3 \leq n \leq 12$. The first 9 values of d_n^∞ ($n \geq 3$) were obtained by using KnotInfo [1] and LinkInfo [2] and the last value is estimated by formula (2).

Table 1: d_n^∞ and $\lfloor \log_2 d_n^\infty \rfloor + 2$ for $3 \leq n \leq 16$.

n	3	4	5	6	7	8	9	10	11	12
d_n^∞	3	5	8	16	24	45	75	130	224	≤ 429
$\lfloor \log_2 d_n^\infty \rfloor + 2$	3	4	5	6	6	7	8	9	9	≤ 10

Hence, for any knot K with crossing number between 7 and 16, we obtain

$$c(K) > \lfloor \log_2 d_n^\infty \rfloor + 2 \geq \lfloor \log_2 \det(K) \rfloor + 2 \geq \lfloor \log_2 p \rfloor + 2.$$

For any knot K with crossing number 5 or 6, it is easy to check that $c(K) > \lfloor \log_2 p \rfloor + 2$.

The proof is complete. \square

ACKNOWLEDGEMENTS. J. Ge is supported by the National Natural Science Foundation of China (No. 11626163). X. Jin is supported by the National Natural Science Foundation of China (No. 11671336). P. Lopes acknowledges partial funding from FCT (Portugal) through projects *PEst – OE/EEI/LA0009/2013*, and *EXCL/MAT – GEO/0222/2012* (“Geometry and Mathematical Physics”). L. Zhang is supported by the National Natural Science Foundation of China (Nos. 11471273 and 11671186). We thank Prof. Satoh for kindly sending us [7].

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