

Global existence of solutions to an n -dimensional parabolic-parabolic system for chemotaxis with logistic-type growth and superlinear production

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Dedicated to Professor Masayasu Mimura on the occasion of his 75th birthday.

Abstract

We study the global existence of solutions to an n -dimensional parabolic-parabolic system for chemotaxis with logistic-type growth. We introduce superlinear production of a chemoattractant. We then show the global existence of solutions in L_p space ($p > n$) under certain relations between the degradation and production orders.

1 Introduction

In the present paper we study a chemotaxis system with logistic growth:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u) & \text{in } \Omega \times (0, \infty), \\ \tau \frac{\partial v}{\partial t} = \Delta v - v + g(u) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) & \text{in } \Omega. \end{cases} \quad (\text{E})$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and the space dimension $n \in \mathbb{N}$ is an arbitrary positive integer. The unknown functions $u(x, t)$ and $v(x, t)$ are the population density of bacteria and the concentration of a chemical substance at the position x and time t , respectively. The term $-\chi \nabla \cdot (u \nabla v)$ expresses the advection of bacteria due to chemotaxis. The coefficient χ is a positive constant, which shows chemotactic intensity. The function $f(u)$ is the proliferation and the reduction in numbers due to death of bacteria (we refer to the combined effects of proliferation and reduction in numbers simply as growth). Typical $f(u)$'s are quadratic $u(1 - u)$ and cubic $u(1 - u)(u - \gamma)$, $0 < \gamma < 1/2$, logistic growth functions [12]. The coefficient τ is a positive constant, which shows the time scale of reaction and diffusion of v . The function

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$g(u)$ is the secretion of chemical substance v by bacteria. A typical $g(u)$ is a linear function; and some nonlinear forms of $g(u)$ have been proposed, such as the saturating function $u/(1 + \gamma u)$, as used in the nonlinear signal kinetics model. For these topics, see the book by Murray [15], and the review articles by Hillen and Painter [4] and by Tindall, Maini, Porter and Armitage [23].

We consider the global existence of solutions to (E). In the context of global existence, the degradation of the growth $f(u)$ can be considered as an inhibitory effect on the increase of u . Indeed, if there is no growth ($f(u) \equiv 0$) and the production $g(u)$ is linear, then the system (E) reduces to the classical parabolic-parabolic Keller-Segel system [10]. In the Keller-Segel system, it is known that when $n = 2$, a finite-time blow-up with a δ -function singularity of u occurs if $\chi \|u_0\|_{L^1}$ is sufficiently large [3, 7]. In contrast, when $n \geq 3$, no restriction on χ and $\|u_0\|_{L^1}$ is necessary for the occurrence of blow-up [26]. For other topics on the Keller-Segel system, see Horstmann's review papers [5, 6] and the references therein. On the contrary, if $f(u)$ is quadratic and $g(u)$ is linear, then blow-up does not occur and global existence of solutions is assured even if $\|u_0\|_{L^1}$ and χ are large. This has been shown for $n = 2$ by one of the authors et al. [19] and for $n \geq 1$ with convex Ω and large μ by Winkler [27]. See also the recent related works [1, 11, 14].

We henceforth assume that the function $f(u)$ is a real, smooth function of $u \in [0, \infty)$ such that $f(0) \geq 0$ and

$$f(u) = u - \mu u^\alpha \quad \text{for sufficiently large } u \geq 0;$$

and the function $g(u)$ is given by

$$g(u) = u(1 + u)^{\beta-1} \quad \text{for } u \geq 0,$$

where the exponents α and β satisfy the relations

$$\alpha > 1 \quad \text{and} \quad 0 < \beta \leq 2, \tag{1}$$

and μ is a positive constant. From the results quoted above, we find that in the n -dimensional domain ($n \geq 2$), a blow-up can occur when $\alpha = 1$ and $\beta = 1$ with a special choice of $\mu = 1$, and the blow-up of solutions is prevented and the global existence of solutions is assured when $\alpha = 2$ and $\beta = 1$. We can then conjecture that the critical degradation order α_{cr} is in the interval $1 \leq \alpha_{\text{cr}} \leq 2$ under linear production $\beta = 1$; however, it has not been determined for the parabolic-parabolic chemotaxis-growth system (E). Recently, Xiang[29] showed global existence of solutions under $\beta = 1$ when $\alpha > 19/9$ if $n = 3$ and when $\alpha > n - 1$ if $n > 3$.

In the two- and three-dimensional cases, the authors [16, 17] introduced sublinear production order $\beta < 1$, and showed a sufficient condition $2(n+4)/(n+6) < \alpha \leq 2$ and $0 < \beta < (n+6)(\alpha-1)/[2(n+2)]$ for the existence of global and bounded solutions to (E) in a Hilbert space $H_2^{(n/2)-1}(\Omega) \times H_2^{(n/2)+\varepsilon}(\Omega) \subset L_n(\Omega) \times \mathcal{C}(\bar{\Omega})$ (their results would include Xiang's results [29] when $n = 3$ if the existence of local solutions were assured for $\alpha > 2$). The authors have also shown in the previous paper [18] the global existence of solutions in L_p -space of arbitrary space dimension n with $p > n$, where (α, β) is merely allowed for $0 < \beta < (\alpha - 1)/2$.

In this paper, we revise the results obtained in [18] considerably by combining the semigroup method and the energy estimates and by applying the technique

of trace operator [9, 13] (see Step 2 of Proof of Lemma 9). The main theorem of this paper is as follows:

Theorem 1. *Assume that the exponents α and β satisfy the relations (1) and*

$$\beta \leq \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha-1). \quad (2)$$

Let p be an arbitrarily fixed exponent with

$$\max\{2, n, (\alpha-2)n\} < p < \infty. \quad (3)$$

Then, for each pair of nonnegative initial functions $(u_0, v_0) \in L_p(\Omega) \times H_p^1(\Omega) \subset L_n(\Omega) \times \mathcal{C}(\overline{\Omega})$, the system (E) admits a unique global solution (u, v) in the function space

$$\begin{cases} 0 \leq u \in \mathcal{C}([0, \infty); L_p(\Omega)) \cap \mathcal{C}((0, \infty); H_{p,N}^2(\Omega)) \cap \mathcal{C}^1((0, \infty); L_p(\Omega)), \\ 0 \leq v \in \mathcal{C}([0, \infty); H_p^1(\Omega)) \cap \mathcal{C}((0, \infty); H_{p,N}^3(\Omega)) \cap \mathcal{C}^1((0, \infty); H_p^1(\Omega)). \end{cases} \quad (4)$$

Moreover, the solution satisfies the estimate

$$\|u(t)\|_{L_p} + \|v(t)\|_{H_p^1} \leq \psi \left(\|u_0\|_{L_p} + \|v_0\|_{H_p^1} \right), \quad t \geq 0, \quad (5)$$

with some increasing function $\psi(\cdot)$.

The definition and notation of function spaces will be given below and in Section 2. Theorem 1 above does not yet cover the case $(\alpha, \beta) = (2, 1)$ for $n \geq 2$ shown by Winkler [27], but the theorem requires no assumption on the largeness of μ nor the convexity of Ω considered in [27]. Our new results also contain the uniform boundedness of solutions with respect to the size of initial data.

We conclude this introduction by referring the results on the parabolic-elliptic chemotaxis systems. The parabolic-elliptic simplifications correspond to the situation where the chemical substance diffuses very quickly, which implies that the time scale τ tends to 0 in (E). For the n -dimensional parabolic-elliptic system with α -th order growth and linear secretion, that is, in the case of $\tau = 0$ and $\beta = 1$ in (E), the problem on the global existence and blow-up of solutions has largely been solved by Winkler [25, 28]: global existence and boundedness are assured when $\alpha > \max\{n/2, 2 - (1/n)\}$ [25]; also, there exists a blow-up solution when $1 < \alpha < 3/2 + 1/(2n-2)$ with $n \geq 5$ [28].

This paper is organized as follows. We provide preliminary results that we utilize in subsequent sections. In Section 3 we show the local existence of solutions by using a semigroup method (Theorem 5). In the final section we construct several a priori energy estimates by combining semigroup and energy methods. After obtaining the a priori estimates, we give the proof of the main theorem.

Notations. Let Ω be a smooth bounded domain in \mathbb{R}^n . For $1 \leq p \leq \infty$, the space of complex-valued L_p functions in Ω is denoted by $L_p(\Omega)$ with the usual norm $\|\cdot\|_{L_p}$. The complex Sobolev space in Ω of order k , $k = 0, 1, 2, \dots$, and exponent p , $1 \leq p \leq \infty$, is denoted by $H_p^k(\Omega)$ with norm $\|\cdot\|_{H_p^k}$. More generally, the Sobolev space of fractional order $s > 0$ and exponent $1 \leq p \leq \infty$ is denoted

by $H_p^s(\Omega)$ with norm $\|\cdot\|_{H_p^s}$. The space of complex-valued continuous functions on $\overline{\Omega}$ is denoted by $\mathcal{C}(\overline{\Omega})$ with norm $\|\cdot\|_{\mathcal{C}}$. Let X be a Banach space and I an interval of \mathbb{R} . $\mathcal{C}(I; X)$ and $\mathcal{C}^1(I; X)$ denote the space of X -valued continuous functions and of X -valued continuously differentiable functions, respectively. $\mathcal{B}(I; X)$ denotes the space of X -valued bounded functions. For simplicity, we will use a universal notation C to denote various constants that are determined for each occurrence by Ω in a specific way. In a situation where C also depends on some parameter, say η , it will be denoted by C_η . In addition, by a universal notation $\psi(\cdot)$ we will denote continuous increasing functions, which may change depending on the context.

2 Preliminaries

In this section we shall list some well-known results in the theories of function spaces and linear operators [19, 22, 24, 30].

Interpolation of Sobolev spaces. For $0 \leq s_0 < s < s_1 < \infty$ and $1 < p < \infty$, $H_p^s(\Omega)$ is the interpolation space $[H_p^{s_0}(\Omega), H_p^{s_1}(\Omega)]_\theta$ between $H_p^{s_0}(\Omega)$ and $H_p^{s_1}(\Omega)$, where $s = (1 - \theta)s_0 + \theta s_1$, with the estimate

$$\|w\|_{H_p^s} \leq C \|w\|_{H_p^{s_0}}^{1-\theta} \|w\|_{H_p^{s_1}}^\theta \quad \text{for } w \in H_p^{s_1}(\Omega). \quad (6)$$

See [30, Theorem 1.35].

Embedding theorem of Sobolev spaces. Let $1 < p < \infty$.

If $0 \leq s < n/p$, then $H_p^s(\Omega) \subset L_r(\Omega)$ for any $p \leq r \leq pn/(n - ps) = [(1/p) - (s/n)]^{-1}$ with continuous embedding

$$\|w\|_{L_r} \leq C_{s,p} \|w\|_{H_p^s} \quad \text{for } w \in H_p^s(\Omega). \quad (7)$$

If $s = n/p$, then $H_p^s(\Omega) \subset L_r(\Omega)$ for any finite $p \leq r < \infty$ with continuous embedding

$$\|w\|_{L_r} \leq C_{s,p} \|w\|_{H_p^s} \quad \text{for } w \in H_p^s(\Omega). \quad (8)$$

If $n/p < s < \infty$, then $H_p^s(\Omega) \subset \mathcal{C}(\overline{\Omega})$ with continuous embedding

$$\|w\|_{\mathcal{C}} \leq C_{s,p} \|w\|_{H_p^s} \quad \text{for } w \in H_p^s(\Omega). \quad (9)$$

See [30, Theorem 1.36].

If $1 \leq r \leq p < \infty$, then $L_r(\Omega)$ is embedded in $(H_{p'}^s(\Omega))'$, the dual space of $H_{p'}^s(\Omega)$ with respect to L_2 -inner product, for $(n/r) - (n/p) \leq s < \infty$ and $p' = p/(p - 1)$ with continuous embedding

$$\|w\|_{(H_{p'}^s)'} \leq C_r \|w\|_{L_r} \quad \text{for } w \in L_r(\Omega). \quad (10)$$

Gagliardo-Nirenberg's inequality. Let $1 \leq q \leq p \leq \infty$. Then the embedding $H_p^1(\Omega) \cap L_q(\Omega) \subset L_r(\Omega)$ holds for

$$\begin{cases} q \leq r \leq pn/(n - p) & \text{if } 1 \leq p < n; \\ q \leq r < \infty & \text{if } p = n; \\ q \leq r \leq \infty & \text{if } n < p \leq \infty, \end{cases} \quad (11)$$

with the estimate

$$\|w\|_{L_r} \leq C_{p,q,r} \|w\|_{H_p^1}^a \|w\|_{L_q}^{1-a} \quad \text{for } w \in H_p^1(\Omega), \quad (12)$$

where a is given by

$$\frac{1}{r} = a \left(\frac{1}{p} - \frac{1}{n} \right) + \frac{1-a}{q}. \quad (13)$$

See [30, Theorem 1.37].

Norms of a product of two functions. For $1 < p < \infty$ and $s > n/p$, from (9),

$$\|uv\|_{L_p} \leq C_p \|u\|_{L_p} \|v\|_{L_\infty} \leq C_{p,s} \|u\|_{L_p} \|v\|_{H_p^s} \quad \text{for } u \in L_p(\Omega), v \in H_p^s(\Omega). \quad (14)$$

As a corollary,

$$\begin{aligned} \|\nabla \cdot (u \nabla v)\|_{L_p} &\leq \|\nabla u \cdot \nabla v\|_{L_p} + \|u \Delta v\|_{L_p} \leq \|\nabla u\|_{L_p} \|\nabla v\|_{L_\infty} + \|u\|_{L_\infty} \|\Delta v\|_{L_p} \\ &\leq C_{p,s} (\|u\|_{H_p^1} \|v\|_{H_p^{1+s}} + \|u\|_{H_p^s} \|v\|_{H_p^2}) \\ &\quad \text{for } u \in H_p^1(\Omega) \cap H_p^s(\Omega), v \in H_p^2(\Omega) \cap H_p^{1+s}(\Omega). \end{aligned} \quad (15)$$

When $n < p < \infty$, since $H_p^1(\Omega) \subset L_\infty(\Omega)$ by (9), it holds that

$$\|\nabla \cdot (u \nabla v)\|_{L_p} \leq C_p \|u\|_{H_p^1} \|v\|_{H_p^2} \quad \text{for } u \in H_p^1(\Omega), v \in H_p^2(\Omega). \quad (16)$$

Domains of fractional powers of Laplace operators in L_p -spaces. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and $A_0 = -\Delta + 1$, Δ being the Laplace operator with Neumann boundary condition. Then, for each $1 < p < \infty$, A_0 is considered as a closed operator in $L_p(\Omega)$, the domain of which is $H_{p,N}^2(\Omega)$ (see [2, Theorem 2.4.1.3], [24, Theorem 5.3.4] or [30, Theorem 2.15]). Let us denote $A_p = A_0|_{L_p}$; then $\mathcal{D}(A_p) = H_{p,N}^2(\Omega)$. Moreover, by the shift property (see [2, Theorem 2.5.1.1] or [24, Theorems 5.3.4 and 5.4.1]) it holds that $\mathcal{D}(A_p|_{H_p^1}) = H_{p,N}^3(\Omega)$ with norm equivalence.

The domains of fractional powers of A_p are characterized by

$$\mathcal{D}(A_p^\theta) = \begin{cases} H_p^{2\theta}(\Omega) & \text{for } 0 \leq \theta < \frac{1}{2} + \frac{1}{2p} \\ H_{p,N}^{2\theta}(\Omega) & \text{for } \frac{1}{2} + \frac{1}{2p} < \theta \leq \frac{3}{2} \end{cases} \quad (17)$$

with norm equivalence. Here, $H_{p,N}^s(\Omega)$ for $s > 1 + (1/p)$ denotes a closed subspace of $H_p^s(\Omega)$ such that

$$H_{p,N}^s(\Omega) = \left\{ w \in H_p^s(\Omega); \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega \right\} \quad \text{for } s > 1 + \frac{1}{p}.$$

Indeed, we can see that A_p has a bounded H_∞ functional calculus (see Yagi [30, Sec.16.1.2]) in $L_p(\Omega)$ and $H_p^1(\Omega)$, and by Yagi [30, Theorem 16.5], that the interpolation $\mathcal{D}(A_p^\theta) = [L_p(\Omega), H_{p,N}^2(\Omega)]_\theta$ and $\mathcal{D}((A_p|_{H_p^1})^\theta) = [H_p^1(\Omega), H_{p,N}^3(\Omega)]_\theta$ hold for $0 < \theta < 1$ with norm equivalence. Then, carefully following the proof of [30, Theorem 16.11], we can verify the rest part of (17). For the detail see Appendix.

Analytic semigroups generated by Laplace operators in L_p -spaces.

For each $1 < p < \infty$, A_0 defined above generates in L_p -space an analytic semigroup e^{-tA_0} (it is independent of p in the sense that $e^{-tA_p}w = e^{-tA_0}w$ for $w \in L_p(\Omega) \cap L_2(\Omega)$). For $\gamma \geq 0$ it satisfies the estimate

$$\|A_0^\gamma e^{-tA_0}w\|_{L_p} \leq Ct^{-\gamma} e^{-\delta_0 t} \|w\|_{L_p}, \quad t > 0, w \in L_p(\Omega), \quad (18)$$

with some fixed constant $\delta_0 > 0$. See [8, Sec. 2] (see also [26, Lemma 1.3], [30, Theorems 2.19 and 2.27] and [22, Sec. 13.7]).

A differential geometric property of functions with Neumann boundary condition.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. If the function $w \in C^2(\overline{\Omega})$ satisfies $\partial w / \partial \nu = 0$ on $\partial\Omega$, then it holds that

$$\frac{\partial |\nabla w|^2}{\partial \nu} \leq 2\kappa_\Omega |\nabla w|^2 \quad \text{on } \partial\Omega, \quad (19)$$

where κ_Ω is an upper bound for the curvatures of $\partial\Omega$; $\kappa_\Omega = 0$ when Ω is convex. See [13, Lemma 4.2]. See also [9].

Boundedness of trace operators. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Let $1 < p < \infty$ and $s > 1/p$. Then, the trace $T : f \mapsto f|_{\partial\Omega}$ is a bounded linear operator from $H_p^s(\Omega)$ to $L_p(\partial\Omega)$. Hence, we have

$$\|w\|_{L_p(\partial\Omega)} \leq C_{s,p} \|w\|_{H_p^s(\Omega)}, \quad w \in H_p^s(\Omega). \quad (20)$$

See [30, Theorem 1.39] or [24, Theorem 4.7.1].

3 Local solutions

By similar arguments to those in [17, 18, 19] or [30, Chap. 12], we can show the existence of local solutions to (E). We first review the existence theorem by Yagi [30, Chap. 4] (see also [20]) for local solutions to an abstract equation in a Banach space. Let X be a Banach space with norm $\|\cdot\|_X$. We consider the following Cauchy problem for a semilinear abstract evolution equation in X :

$$\begin{cases} \frac{dU}{dt} + AU = F(U), & t > 0, \\ U(0) = U_0. \end{cases} \quad (21)$$

Here, A is a sectorial operator of X satisfying that its spectral set is contained in a sectorial domain $\Sigma = \{\lambda \in \mathbb{C}; |\arg \lambda| \leq \phi\}$ with some $0 \leq \phi < \pi/2$, and $\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq M/(|\lambda| + 1)$, $\lambda \notin \Sigma$ with constant M . The nonlinear operator F is a mapping from $\mathcal{D}(A^\eta)$ to X , where $0 < \eta < 1$, and it also satisfies a Lipschitz condition:

$$\begin{aligned} \|F(U) - F(\tilde{U})\|_X &\leq \varphi \left(\|A^\gamma U\|_X + \|A^\gamma \tilde{U}\|_X \right) \\ &\times \left[\|A^\eta(U - \tilde{U})\|_X + \left(\|A^\eta U\|_X + \|A^\eta \tilde{U}\|_X \right) \|A^\gamma(U - \tilde{U})\|_X \right], \\ &U, \tilde{U} \in \mathcal{D}(A^\eta), \end{aligned} \quad (22)$$

where γ is an exponent such that $0 < \gamma \leq \eta < 1$, and $\varphi(\cdot)$ is some increasing continuous function. The initial value U_0 is taken in $\mathcal{D}(A^\gamma)$. Then, from [30, Theorem 4.1] (or [20, Theorem 3.1]) we have the existence theorem of the local solutions to (21):

Theorem 2 ([30, Theorem 4.1]). *Under the above assumptions, for any $U_0 \in \mathcal{D}(A^\gamma)$, (21) possesses a unique local solution U in the function space:*

$$\begin{cases} U \in \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A)) \cap \mathcal{C}([0, T_{U_0}]; \mathcal{D}(A^\gamma)) \cap \mathcal{C}^1((0, T_{U_0}]; X), \\ t^{1-\gamma}U \in \mathcal{B}((0, T_{U_0}]; \mathcal{D}(A)) \end{cases}$$

with the estimate

$$t^{1-\gamma}\|AU(t)\|_X + \|A^\gamma U(t)\|_X \leq C_{U_0}, \quad 0 < t \leq T_{U_0},$$

where T_{U_0} and C_{U_0} are positive constants depending only on the norm $\|A^\gamma U_0\|_X$.

By applying Theorem 2, we can show the existence of the local solutions to (E). The following proposition has been proved in [18].

Proposition 3 ([18, Proposition 3]). *Let $n \in \mathbb{N}$, assume the relation (1) for α and β , and let p be an exponent satisfying*

$$\max\{n, (\alpha - 2)n\} < p < \infty. \quad (23)$$

Then, for each pair of initial functions $(u_0, v_0) \in L_p(\Omega) \times H_p^1(\Omega) \subset L_n(\Omega) \times \mathcal{C}(\overline{\Omega})$, the problem (E) admits a unique local solution (u, v) in the function space

$$\begin{cases} u \in \mathcal{C}((0, T]; H_p^1(\Omega)) \cap \mathcal{C}([0, T]; L_p(\Omega)) \cap \mathcal{C}^1((0, T]; (H_{p'}^1(\Omega))'), \\ v \in \mathcal{C}((0, T]; H_{p,N}^2(\Omega)) \cap \mathcal{C}([0, T]; H_p^1(\Omega)) \cap \mathcal{C}^1((0, T]; L_p(\Omega)) \end{cases} \quad (24)$$

with the estimate

$$t^{\frac{1}{2}} \left\{ \|u(t)\|_{H_p^1} + \|v(t)\|_{H_p^2} \right\} + \left\{ \|u(t)\|_{L_p} + \|v(t)\|_{H_p^1} \right\} \leq C, \quad 0 < t \leq T,$$

where $p' = p/(p-1)$, and T and C are positive constants depending only on the norm $\|u_0\|_{L_p} + \|v_0\|_{H_p^1}$.

By a solution (u, v) to (E) in the function space (24) we mean that the pair of functions (u, v) contained in (24) satisfies

$$\begin{cases} \frac{d}{dt} \langle u, w \rangle_{L_2} = -\langle \nabla u, \nabla w \rangle_{L_2} + \chi \langle u \nabla v, \nabla w \rangle_{L_2} + \langle f(u), w \rangle_{L_2} \\ \qquad \qquad \qquad \text{for any } w \in H_{p'}^1(\Omega) \text{ and } 0 < t < \infty, \\ \tau \frac{\partial v}{\partial t} = \Delta v - v + g(u) \quad \text{in } \Omega \times (0, \infty). \end{cases}$$

Next, we will show the local existence of solutions in the second function space:

Proposition 4. *Let $n \in \mathbb{N}$, assume the relation (1) for α and β , and let p be an exponent satisfying $n < p < \infty$. Then, for each pair of initial functions*

$(u_0, v_0) \in H_p^1(\Omega) \times H_{p,N}^2(\Omega)$, the problem (E) admits a unique local solution (u, v) in the function space

$$\begin{cases} u \in \mathcal{C}((0, T]; H_{p,N}^2(\Omega)) \cap \mathcal{C}([0, T]; H_p^1(\Omega)) \cap \mathcal{C}^1((0, T]; L_p(\Omega)), \\ v \in \mathcal{C}((0, T]; H_{p,N}^3(\Omega)) \cap \mathcal{C}([0, T]; H_{p,N}^2(\Omega)) \cap \mathcal{C}^1((0, T]; H_p^1(\Omega)) \end{cases}$$

with the estimate

$$t^{\frac{1}{2}} \left\{ \|u(t)\|_{H_p^2} + \|v(t)\|_{H_p^3} \right\} + \left\{ \|u(t)\|_{H_p^1} + \|v(t)\|_{H_p^2} \right\} \leq C, \quad 0 < t \leq T,$$

where T and C are positive constants depending only on the norm $\|u_0\|_{H_p^1} + \|v_0\|_{H_p^2}$.

Proof. The system (E) can be expressed as a semilinear parabolic equation

$$\begin{cases} \frac{dU}{dt} + AU = F(U), & t > 0, \\ U(0) = U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \end{cases} \quad (25)$$

in a product Banach space $X = L_p(\Omega) \times H_p^1(\Omega)$. Here, we define the linear operator A by

$$A = \begin{bmatrix} -\Delta + 1 & 0 \\ 0 & \tau^{-1}(-\Delta + 1) \end{bmatrix}, \quad \mathcal{D}(A) = H_{p,N}^2(\Omega) \times H_{p,N}^3(\Omega).$$

The nonlinear operator F is defined by

$$F(U) = \begin{bmatrix} -\chi \nabla \cdot (u \nabla v) + \bar{f}(u) + u \\ \bar{g}(u) \end{bmatrix}, \quad U = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{D}(A^\eta) = H_p^1(\Omega) \times H_{p,N}^2(\Omega)$$

with $\eta = 1/2$. Here, $\bar{f}(u)$ and $\bar{g}(u)$ denote some smooth extensions of $f(u)$ and $g(u)$ for the variable $u \in \mathbb{C}$ satisfying $f(u) \geq 0$ for $u < 0$ and $g(u) = 0$ for $u < -1$, respectively. The initial value U_0 is taken in the function space $\mathcal{D}(A^\gamma) = \mathcal{D}(A^\eta)$, that is $\gamma = \eta$. Under this setting, we need to verify only the Lipschitz condition (22). For $U = \begin{bmatrix} u \\ v \end{bmatrix}, \tilde{U} = \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \in \mathcal{D}(A^\eta)$,

$$\begin{aligned} \|F(U) - F(\tilde{U})\|_X &\leq \chi \|\nabla \cdot (u \nabla v - \tilde{u} \nabla \tilde{v})\|_{L_p} \\ &\quad + \|u - \tilde{u}\|_{L_p} + \|\bar{f}(u) - \bar{f}(\tilde{u})\|_{L_p} + \|\bar{g}(u) - \bar{g}(\tilde{u})\|_{H_p^1}. \end{aligned}$$

For the first term, applying (16), we see

$$\begin{aligned} \|\nabla \cdot (u \nabla v) - \nabla \cdot (\tilde{u} \nabla \tilde{v})\|_{L_p} &\leq \|\nabla \cdot ((u - \tilde{u}) \nabla v)\|_{L_p} + \|\nabla \cdot (\tilde{u} \nabla (v - \tilde{v}))\|_{L_p} \\ &\leq C_{p,s} (\|u - \tilde{u}\|_{H_p^1} \|v\|_{H_p^2} + \|\tilde{u}\|_{H_p^1} \|v - \tilde{v}\|_{H_p^2}) \\ &\leq C (\|A^\gamma U\|_X + \|A^\gamma \tilde{U}\|_X) \|A^\gamma (U - \tilde{U})\|_X. \end{aligned}$$

For the third and fourth terms, using (1) and $H_p^s(\Omega) \subset L_\infty(\Omega)$ by (9), we can easily see that

$$\begin{aligned} \|\bar{f}(u) - \bar{f}(\tilde{u})\|_{L_p} &\leq C(1 + \|u\|_{L_\infty} + \|\tilde{u}\|_{L_\infty})^{\alpha-1} \|u - \tilde{u}\|_{L_p} \\ &\leq C(1 + \|u\|_{H_p^1} + \|\tilde{u}\|_{H_p^1})^{\alpha-1} \|u - \tilde{u}\|_{L_p}, \end{aligned}$$

$$\begin{aligned}
& \|\bar{g}(u) - \bar{g}(\tilde{u})\|_{H_p^1} \leq \|\bar{g}(u) - \bar{g}(\tilde{u})\|_{L_p} + \|\bar{g}'(u)\nabla(u - \tilde{u})\|_{L_p} + \|\{\bar{g}'(u) - \bar{g}'(\tilde{u})\}\nabla\tilde{u}\|_{L_p} \\
& \leq C(1 + \|u\|_{L_\infty} + \|\tilde{u}\|_{L_\infty})(\|u - \tilde{u}\|_{L_p} + \|\nabla(u - \tilde{u})\|_{L_p}) + C\|u - \tilde{u}\|_{L_\infty}\|\nabla\tilde{u}\|_{L_p} \\
& \leq C(1 + \|u\|_{H_p^1} + \|\tilde{u}\|_{H_p^1})\|u - \tilde{u}\|_{H_p^1}.
\end{aligned}$$

Thus $F(U)$ satisfies the Lipschitz condition (22). We complete the proof. \square

Now we can state our main theorem of this section:

Theorem 5. *Let $n \in \mathbb{N}$, assume the relation (1) for α and β , and let p be an arbitrarily fixed exponent satisfying (23). Then, for each pair of nonnegative initial functions $(u_0, v_0) \in L_p(\Omega) \times H_p^1(\Omega) \subset L_n(\Omega) \times \mathcal{C}(\bar{\Omega})$, the problem (E) admits a unique local solution (u, v) in the function space*

$$\begin{cases} 0 \leq u \in \mathcal{C}([0, T]; L_p(\Omega)) \cap \mathcal{C}((0, T]; H_{p,N}^2(\Omega)) \cap \mathcal{C}^1((0, T]; L_p(\Omega)), \\ 0 \leq v \in \mathcal{C}([0, T]; H_p^1(\Omega)) \cap \mathcal{C}((0, T]; H_{p,N}^3(\Omega)) \cap \mathcal{C}^1((0, T]; H_p^1(\Omega)) \end{cases} \quad (26)$$

with the estimate

$$\|u(t)\|_{L_p} + \|v(t)\|_{H_p^1} \leq C, \quad 0 < t \leq T, \quad (27)$$

where T and C are positive constants depending only on the norm $\|u_0\|_{L_p} + \|v_0\|_{H_p^1}$.

Proof. It is clear that the local solutions belong to the function space (26) from Propositions 3 and 4. The nonnegativity of solutions has been proved in [18, Theorem 4] with the aid of the truncation method [30, Section 12.1.3]. Hence, we conclude the proof. \square

4 A priori estimates and global solutions

In this section we will construct several a priori estimates. The a priori estimates hold with each of the inequalities of α and β in the lemmas. Throughout this section, except for in the global existence theorem, we assume that $0 \leq u_0 \in H_{p,N}^2(\Omega) \subset H_\infty^1(\Omega)$ and $0 \leq v_0 \in H_{p,N}^3(\Omega) \subset H_\infty^2(\Omega)$ with $n < p < \infty$. In this case, applying [30, Theorem 4.2], we can verify that $0 \leq u \in \mathcal{C}([0, T]; H_{p,N}^2(\Omega))$ and $0 \leq v \in \mathcal{C}([0, T]; H_{p,N}^3(\Omega))$ with the estimate $\|u(t)\|_{H_p^2} + \|v(t)\|_{H_p^3} \leq C_{U_0}$ for $0 \leq t \leq T$, where C_{U_0} is some positive constant. For a local solution (u, v) to (E) and exponents $z > 0$ and $\omega > 0$, we define

$$I_\omega^z(t) = \int_0^t \omega e^{-\omega(t-s)} \int_\Omega u^z dx ds.$$

The following lemma will be used frequently in this section.

Lemma 6 (Gronwall's inequality). *Assume that a smooth real function $h(t)$ satisfies the differential inequality*

$$h'(t) + ah(t) \leq K(t), \quad t_0 \leq t \leq T,$$

with a positive constant a and an integrable real function $K(t)$. Then, $h(t)$ is estimated by

$$h(t) \leq h(t_0)e^{-a(t-t_0)} + \int_{t_0}^t e^{-a(t-s)}K(s)ds, \quad t_0 \leq t \leq T.$$

Lemma 7. Let (u, v) be a local solution to (E), and assume that

$$\alpha > 1.$$

Then, it holds that

$$\|u\|_{L_1} = \int_{\Omega} u \, dx \leq e^{-t} \|u_0\|_{L_1} + a_1 |\Omega| \quad (28)$$

with a constant $a_1 = \max\{f(u) + u; u \geq 0\}$. In addition, for an arbitrary constant $\omega > 0$,

$$I_{\omega}^{\alpha}(t) \leq \frac{2}{\mu} \{(a + a_1\omega) |\Omega| + \omega \|u_0\|_{L_1}\} \equiv \bar{I}_{\omega}^{\alpha} \quad (29)$$

holds with a constant $a = \max\{f(u) + \mu u^{\alpha}/2; u \geq 0\}$.

Proof. (Just the same as [17, Lemmas 4.1 and 4.2] or the first half part of [18, Lemma 5].) Integrating the first equation of (E) over Ω , we have

$$\frac{d}{dt} \int_{\Omega} u \, dx = \int_{\Omega} f(u) \, dx \leq \int_{\Omega} (a_1 - u) \, dx.$$

Then, by Lemma 6, we obtain (28). From these inequalities, we see that

$$\begin{aligned} \frac{\mu}{2} I_{\omega}^{\alpha}(t) &\leq \int_0^t \omega e^{-\omega(t-s)} \int_{\Omega} \{a - f(u)\} \, dx \, ds \\ &= \int_0^t \omega e^{-\omega(t-s)} \left\{ a |\Omega| - \frac{d}{ds} \|u\|_{L_1} \right\} \, ds \\ &\leq a |\Omega| (1 - e^{-\omega t}) + \omega e^{-\omega t} \|u_0\|_{L_1} + \int_0^t \omega^2 e^{-\omega(t-s)} \|u\|_{L_1} \, ds \\ &\leq (a + a_1\omega) |\Omega| + \omega \|u_0\|_{L_1}, \end{aligned}$$

which yields (29). \square

Lemma 8. Let (u, v) be a local solution to (E), and assume that

$$\alpha > 1 \quad \text{and} \quad 0 < \beta \leq \frac{\alpha}{2}.$$

Then, for any exponent $2 \leq q \leq \alpha/\beta$,

$$\|v\|_{H_q^1}^q \leq C_q e^{-\delta_q t} \|v_0\|_{H_q^1}^q + C_q (|\Omega| + \|u_0\|_{L_1}) \quad (30)$$

holds with some positive constants C_q and δ_q .

Proof. When $q = 2$ (see [17, Proposition 4.4]), multiplying the second equation of (E) by $-\Delta v + v$ and integrating it over Ω , we see that

$$\begin{aligned} \frac{\tau}{2} \frac{d}{dt} \int_{\Omega} (|\nabla v|^2 + v^2) \, dx &\leq -\frac{1}{2} \int_{\Omega} (\Delta v)^2 \, dx - 2 \int_{\Omega} |\nabla v|^2 \, dx \\ &\quad - \frac{1}{2} \int_{\Omega} v^2 \, dx + \int_{\Omega} (1 + u)^{2\beta} \, dx, \end{aligned}$$

that is,

$$\begin{aligned} \tau \frac{d}{dt} \int_{\Omega} (|\nabla v|^2 + v^2) dx + \int_{\Omega} (|\nabla v|^2 + v^2) dx \\ + \int_{\Omega} (\Delta v)^2 dx + 3 \int_{\Omega} |\nabla v|^2 dx \leq 2 \int_{\Omega} (1 + u)^{2\beta} dx. \end{aligned} \quad (31)$$

Thus, by Lemma 6 again, we verify

$$\|v\|_{H_2^1}^2 \leq e^{-t/\tau} \|v_0\|_{H_2^1}^2 + 2 \int_0^t e^{-(t-s)/\tau} \int_{\Omega} (1 + u)^{2\beta} dx ds.$$

When $q > 2$ (just in the same way as the second half part of [18, Lemma 5]), we utilize the semigroup $e^{-tA_0/\tau}$ of $A_0 = -\Delta + 1$, Δ be the Laplace operator with Neumann boundary condition. Then, the second equation of (E) gives

$$v(t) = e^{-tA_0/\tau} v_0 + \frac{1}{\tau} \int_0^t e^{-(t-s)A_0/\tau} g(u(s)) ds. \quad (32)$$

Operating $A_0^{1/2}$ to this equality and applying (17) and (18), we have

$$\begin{aligned} \|v\|_{H_q^1} &\leq C_q \|A_0^{1/2} v\|_{L_q} \\ &\leq C_q \|A_0^{1/2} e^{-tA_0/\tau} v_0\|_{L_q} + \frac{1}{\tau} \int_0^t C_q \|A_0^{1/2} e^{-(t-s)A_0/\tau} g(u)\|_{L_q} ds \\ &\leq C_q e^{-\delta_0 t/\tau} \|A_0^{1/2} v_0\|_{L_q} + \int_0^t C_q (t-s)^{-1/2} e^{-\delta_0(t-s)/\tau} \|g(u)\|_{L_q} ds \\ &\leq C_q e^{-\delta_0 t/\tau} \|v_0\|_{H_q^1} + \int_0^t C_q (t-s)^{-1/2} e^{-\delta_0(t-s)/\tau} \|(1+u)^\beta\|_{L_q} ds. \end{aligned}$$

The last term can be estimated as

$$\begin{aligned} \int_0^t (t-s)^{-1/2} e^{-\delta_0(t-s)/\tau} \|(1+u)^\beta\|_{L_q} ds \\ = \int_0^t (t-s)^{-1/2} e^{-\delta_0(t-s)/\tau} \|1+u\|_{L_{q\beta}^\beta} ds \\ \leq \left(\int_0^t (t-s)^{-q'/2} e^{-\delta_0(t-s)/\tau} ds \right)^{1/q'} \left(\int_0^t e^{-\delta_0(t-s)/\tau} \|1+u\|_{L_{q\beta}^\beta}^{q\beta} ds \right)^{1/q}, \end{aligned}$$

where $q' = q/(q-1)$. Here we notice that $q'/2 < 1$ and the singular integral converges. Hence, we have

$$\|v\|_{H_q^1} \leq C_q e^{-\delta_0 t/\tau} \|v_0\|_{H_q^1} + C_q \left(\int_0^t e^{-\delta_0(t-s)/\tau} \int_{\Omega} (1+u)^{q\beta} dx ds \right)^{1/q}.$$

Combining both cases when $q = 2$ and when $q > 2$, we have

$$\|v\|_{H_q^1}^q \leq C_q e^{-\delta_q t/\tau} \|v_0\|_{H_q^1}^q + C_q \int_0^t e^{-\delta_q(t-s)/\tau} \int_{\Omega} (1+u)^{q\beta} dx ds \quad (33)$$

for $q \geq 2$ with some positive constants C_q and δ_q . Applying (29), we prove (30) for $2 \leq q \leq \alpha/\beta$. \square

Lemma 9. Let (u, v) be a local solution to (E), and assume that

$$\alpha > 1, \quad 0 < \beta \leq \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha-1).$$

Then, for any $2 \leq q \leq \alpha/\beta$ satisfying $q > 2n\alpha/[(n+2)(\alpha-1)]$, and for any exponent $1 < \theta \leq \{q(n+2)/(2n) - 1\}(\alpha-1)$,

$$\|1+u\|_{L_\theta}^\theta \leq e^{-qt/(2\tau)} \|1+u_0\|_{L_\theta}^\theta + \psi_{\theta,q} \left(\|1+u_0\|_{L_1} + \|v_0\|_{H_q^1} \right) \quad (34)$$

holds with some increasing function $\psi_{\theta,q}(\cdot)$. In addition, for an arbitrary constant $\omega > 0$,

$$\begin{aligned} I_\omega^{\alpha+\theta-1}(t) &\leq \frac{4}{\mu} \left\{ \left(1 + \frac{\omega}{\theta}\right) \psi_{\theta,q} \left(\|u_0\|_{L_1} + \|v_0\|_{H_q^1} \right) \right. \\ &\quad \left. + \omega \left(\frac{1}{\theta} \|1+u_0\|_{L_\theta}^\theta + \zeta \frac{2\tau}{q} \|v_0\|_{H_q^1}^q \right) \right\} \equiv \bar{I}_\omega^{\alpha+\theta-1} \quad (35) \end{aligned}$$

holds with some constant $\zeta > 0$.

Proof. We describe the proof in several steps.

Step 1. Multiplying the first equation of (E) by $(1+u)^{\theta-1}$ and integrating it over Ω , we see that

$$\begin{aligned} \frac{1}{\theta} \frac{d}{dt} \int_\Omega (1+u)^\theta dx &= -(\theta-1) \int_\Omega (1+u)^{\theta-2} |\nabla u|^2 dx \\ &\quad + \chi(\theta-1) \int_\Omega u(1+u)^{\theta-2} \nabla u \cdot \nabla v dx + \int_\Omega (1+u)^{\theta-1} f(u) dx \\ &\leq -\frac{\theta-1}{2} \int_\Omega (1+u)^{\theta-2} |\nabla u|^2 dx + \frac{\chi^2(\theta-1)}{2} \int_\Omega (1+u)^\theta |\nabla v|^2 dx \\ &\quad + \int_\Omega (1+u)^{\theta-1} f(u) dx. \end{aligned}$$

For the second term on the right-hand side, using (12), we note that

$$\begin{aligned} \frac{\chi^2(\theta-1)}{2} \int_\Omega (1+u)^\theta |\nabla v|^2 dx &\leq \frac{\chi^2(\theta-1)}{2} \|(1+u)^\theta\|_{L_{\kappa/(\kappa-1)}} \|\nabla v\|_{L_\kappa}^2 \\ &= \frac{\chi^2(\theta-1)}{2} \|(1+u)^\theta\|_{L_{\kappa/(\kappa-1)}} \|\nabla v\|_{L_{4\kappa/q}}^{4/q} \\ &\leq \frac{\chi^2(\theta-1)}{2} \|(1+u)^\theta\|_{L_{\kappa/(\kappa-1)}} \cdot C_q \|\nabla v\|_{H_2^1}^{2/\kappa} \|\nabla v\|_{L_2}^{(4/q)-(2/\kappa)} \\ &\leq C_q \eta^{-\kappa+1} \chi^{2\kappa} (\theta-1)^\kappa \|\nabla v\|_{H_2^1}^2 + \eta \|\nabla v\|_{L_q}^{(2\kappa-q)/(\kappa-1)} \int_\Omega (1+u)^{\theta\kappa/(\kappa-1)} dx \end{aligned}$$

with $\kappa = q(n+2)/(2n)$ and an arbitrary $\eta > 0$. Hence we have

$$\begin{aligned} \frac{1}{\theta} \frac{d}{dt} \int_\Omega (1+u)^\theta dx &\leq -\frac{\theta-1}{2} \int_\Omega (1+u)^{\theta-2} |\nabla u|^2 dx + C_q \eta^{-\kappa+1} \chi^{2\kappa} (\theta-1)^\kappa \|\nabla v\|_{H_2^1}^2 \\ &\quad + \int_\Omega \left[\eta \|\nabla v\|_{L_q}^{(2\kappa-q)/(\kappa-1)} (1+u)^{\theta\kappa/(\kappa-1)} + (1+u)^{\theta-1} f(u) \right] dx. \quad (36) \end{aligned}$$

Step 2. We present the differential inequality on $\|v\|_{H_q^1}^q$ for $q \geq 2$. For the present assume $q > 2$. Firstly, multiplying the second equation of (E) by v^{q-1} and integrating it over Ω , we see that

$$\begin{aligned} \frac{\tau}{q} \frac{d}{dt} \int_{\Omega} v^q dx &= -(q-1) \int_{\Omega} v^{q-2} |\nabla v|^2 dx - \int_{\Omega} v^q dx + \int_{\Omega} v^{q-1} g(u) \\ &\leq -(q-1) \int_{\Omega} v^{q-2} |\nabla v|^2 dx - \frac{1}{2} \int_{\Omega} v^q dx + \frac{C'_q}{2} \int_{\Omega} (1+u)^{q\beta} dx. \end{aligned} \quad (37)$$

Next, differentiating the second equation of (E), we have

$$\tau \frac{\partial}{\partial t} |\nabla v|^2 = 2\tau \nabla v \cdot \nabla v_t = 2\nabla v \cdot \nabla \Delta v - 2|\nabla v|^2 + 2\nabla v \cdot \nabla g(u).$$

Noting that $\Delta |\nabla v|^2 = 2|D^2 v|^2 + 2\nabla v \cdot \nabla \Delta v$ and $(\Delta v)^2 \leq n|D^2 v|^2$, where $|D^2 v|^2 = \sum_{i,j} |D_i D_j v|^2$, we see

$$\tau \frac{\partial}{\partial t} |\nabla v|^2 \leq \Delta |\nabla v|^2 - \frac{2}{n} (\Delta v)^2 - 2|\nabla v|^2 + 2\nabla v \cdot \nabla g(u).$$

Multiplying this inequality by $|\nabla v|^{q-2}$, integrating it over Ω and applying (19), we obtain

$$\begin{aligned} &\frac{2\tau}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^q dx \\ &\leq \int_{\Omega} |\nabla v|^{q-2} \left\{ \Delta |\nabla v|^2 - \frac{2}{n} (\Delta v)^2 - 2|\nabla v|^2 + 2\nabla v \cdot \nabla g(u) \right\} dx \\ &= \int_{\partial\Omega} |\nabla v|^{q-2} \frac{\partial |\nabla v|^2}{\partial \nu} dx - \int_{\Omega} \nabla |\nabla v|^{q-2} \cdot \nabla |\nabla v|^2 dx \\ &\quad - \int_{\Omega} |\nabla v|^{q-2} \left\{ \frac{2}{n} (\Delta v)^2 + 2|\nabla v|^2 \right\} dx + \int_{\Omega} 2|\nabla v|^{q-2} \nabla v \cdot \nabla g(u) dx \\ &\leq 2\kappa_{\Omega} \int_{\partial\Omega} |\nabla v|^q dx - \int_{\Omega} \frac{q-2}{2} |\nabla v|^{q-4} |\nabla |\nabla v|^2|^2 dx - \int_{\Omega} \frac{2}{n} |\nabla v|^{q-2} (\Delta v)^2 dx \\ &\quad - \int_{\Omega} 2|\nabla v|^q dx + \int_{\Omega} 2|\nabla v|^{q-2} \nabla v \cdot \nabla g(u) dx. \end{aligned}$$

For the first term on the right-hand side, applying (20) and (6) with any $1/2 < s < 1$ and $\varepsilon > 0$, we see that

$$\begin{aligned} 2\kappa_{\Omega} \int_{\partial\Omega} |\nabla v|^q dx &= 2\kappa_{\Omega} \left\| |\nabla v|^{q/2} \right\|_{L_2(\partial\Omega)}^2 \leq C \left\| |\nabla v|^{q/2} \right\|_{H_2^s(\Omega)}^2 \\ &\leq C \left\| |\nabla v|^{q/2} \right\|_{H_2^s}^{2s} \left\| |\nabla v|^{q/2} \right\|_{L_2}^{2(1-s)} \leq \varepsilon \left\| \nabla \left(|\nabla v|^{q/2} \right) \right\|_{L_2}^2 + C_{\varepsilon} \left\| |\nabla v|^{q/2} \right\|_{L_2}^2. \end{aligned}$$

For the last term on the right-hand side, we see

$$\begin{aligned}
& \int_{\Omega} 2|\nabla v|^{q-2} \nabla v \cdot \nabla g(u) dx \\
&= - \int_{\Omega} \left\{ (q-2)|\nabla v|^{q-4} \nabla |\nabla v|^2 \cdot \nabla v + 2|\nabla v|^{q-2} \Delta v \right\} g(u) dx \\
&\leq \int_{\Omega} (q-2)|\nabla v|^{q-3} |\nabla |\nabla v|^2| (1+u)^{\beta} dx + \int_{\Omega} 2|\nabla v|^{q-2} |\Delta v| (1+u)^{\beta} dx \\
&\leq \frac{q-2}{4} \int_{\Omega} |\nabla v|^{q-4} |\nabla |\nabla v|^2|^2 dx + \frac{1}{n} \int_{\Omega} |\nabla v|^{q-2} (\Delta v)^2 dx + \int_{\Omega} |\nabla v|^q dx \\
&\quad + C_q'' (n+q-2)^{q/2} \int_{\Omega} (1+u)^{q\beta} dx.
\end{aligned}$$

Hence, noting that $|\nabla (|\nabla v|^{q/2})|^2 = (q^2/16)|\nabla v|^{q-4} |\nabla |\nabla v|^2|^2$, we have

$$\begin{aligned}
& \frac{2\tau}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^q dx + \frac{4(q-2)}{q^2} \int_{\Omega} \left| \nabla (|\nabla v|^{q/2}) \right|^2 dx \\
&\quad + \frac{1}{n} \int_{\Omega} |\nabla v|^{q-2} (\Delta v)^2 dx + \int_{\Omega} |\nabla v|^q dx \\
&\leq \varepsilon \int_{\Omega} \left| \nabla (|\nabla v|^{q/2}) \right|^2 dx + C_{\varepsilon} \int_{\Omega} |\nabla v|^q dx \\
&\quad + C_q'' (n+q-2)^{q/2} \int_{\Omega} (1+u)^{q\beta} dx. \quad (38)
\end{aligned}$$

Adding (38) to (37) and taking $\varepsilon = 2(q-2)/q^2$, we see

$$\begin{aligned}
& \frac{2\tau}{q} \frac{d}{dt} \int_{\Omega} (|\nabla v|^q + v^q) dx + \int_{\Omega} (|\nabla v|^q + v^q) dx \\
&\quad + \frac{2(q-2)}{q^2} \int_{\Omega} \left| \nabla (|\nabla v|^{q/2}) \right|^2 dx + \frac{1}{n} \int_{\Omega} |\nabla v|^{q-2} (\Delta v)^2 dx \\
&\quad + \frac{8(q-1)}{q^2} \int_{\Omega} \left| \nabla (v^{q/2}) \right|^2 dx \leq C \int_{\Omega} |\nabla v|^q dx + C_q \int_{\Omega} (1+u)^{q\beta} dx. \quad (39)
\end{aligned}$$

This inequality holds also for $q = 2$ (see (31)). The right-hand side is bounded in terms of $\|u_0\|_{L^1}$ and $\|v_0\|_{H_q^1}$ in view of Lemmas 7 and 8 since $q\beta \leq \alpha$.

Step 3. Adding (39) multiplied by some weight $\zeta > 0$ to (36), we see

$$\begin{aligned}
& \frac{1}{\theta} \frac{d}{dt} \int_{\Omega} (1+u)^{\theta} dx + \zeta \left(\frac{2\tau}{q} \frac{d}{dt} \|v\|_{H_q^1}^q + \|v\|_{H_q^1}^q + \frac{2(q-2)}{q^2} \left\| |\nabla v|^{q/2} \right\|_{H_q^1}^2 \right) \\
&\quad \leq C_q \eta^{-\kappa+1} \chi^{2\kappa} (\theta-1)^{\kappa} \left\| |\nabla v|^{q/2} \right\|_{H_q^1}^2 + C \|\nabla v\|_{L_q}^q \\
&\quad + \int_{\Omega} \left[\eta \|\nabla v\|_{L_q}^{(2\kappa-q)/(\kappa-1)} (1+u)^{\theta\kappa/(\kappa-1)} + (1+u)^{\theta-1} f(u) \right. \\
&\quad \quad \left. + \zeta C_q (1+u)^{q\beta} \right] dx. \quad (40)
\end{aligned}$$

Since $q\beta \leq \alpha < \alpha + \theta - 1$ and $\theta\kappa/(\kappa - 1) \leq \alpha + \theta - 1$ from the assumptions, suitable choice of η and ζ yields

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{\theta} \|1 + u\|_{L_\theta}^\theta + \zeta \frac{2\tau}{q} \|v\|_{H_q^1}^q \right) + \frac{q}{2\tau} \left(\frac{1}{\theta} \|1 + u\|_{L_\theta}^\theta + \zeta \frac{2\tau}{q} \|v\|_{H_q^1}^q \right) \\ \leq \psi_{\theta,q} \left(\|v\|_{H_q^1} \right) - \frac{\mu}{4} \int_{\Omega} u^{\alpha+\theta-1} dx, \quad (41) \end{aligned}$$

and hence, by Lemma 6,

$$\begin{aligned} \frac{1}{\theta} \|1 + u\|_{L_\theta}^\theta + \zeta \frac{2\tau}{q} \|v\|_{H_q^1}^q \leq e^{-qt/(2\tau)} \left(\frac{1}{\theta} \|1 + u_0\|_{L_\theta}^\theta + \zeta \frac{2\tau}{q} \|v_0\|_{H_q^1}^q \right) \\ + \int_0^t e^{-q(t-s)/(2\tau)} \psi_{\theta,q} \left(\|v\|_{H_q^1} \right) ds. \end{aligned}$$

Application of (30) to the right-hand side of this inequality leads to (34).

Step 4. The proof of (35) is very similar to that of (29), as follows: using (41),

$$\begin{aligned} \frac{\mu}{4} I_\omega^{\alpha+\theta-1}(t) \\ \leq \int_0^t \omega e^{-\omega(t-s)} \left\{ \psi_{\theta,q} \left(\|v\|_{H_q^1} \right) - \frac{d}{dt} \left(\frac{1}{\theta} \|1 + u\|_{L_\theta}^\theta + \zeta \frac{2\tau}{q} \|v\|_{H_q^1}^q \right) \right\} ds \\ \leq \psi_{\theta,q} \left(\sup_{t \geq 0} \|v\|_{H_q^1} \right) + \omega e^{-\omega t} \left(\frac{1}{\theta} \|1 + u_0\|_{L_\theta}^\theta + \zeta \frac{2\tau}{q} \|v_0\|_{H_q^1}^q \right) \\ + \int_0^t \omega^2 e^{-\omega(t-s)} \left(\frac{1}{\theta} \|1 + u\|_{L_\theta}^\theta + \zeta \frac{2\tau}{q} \|v\|_{H_q^1}^q \right) ds \\ \leq \left(1 + \frac{\omega}{\theta} \right) \psi_{\theta,q} \left(\sup_{t \geq 0} \|v\|_{H_q^1} \right) + \omega \left(\frac{1}{\theta} \|1 + u_0\|_{L_\theta}^\theta + \zeta \frac{2\tau}{q} \|v_0\|_{H_q^1}^q \right). \end{aligned}$$

Thus we complete the proof of the lemma. \square

Lemma 10. *Let (u, v) be a local solution to (E), and assume that*

$$\alpha > 1, \quad 0 < \beta \leq \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha - 1).$$

Suppose that for some exponent $\sigma > 1$ and $r > 2n\alpha/[(n+2)(\alpha-1)]$ the integral $I_\omega^{\alpha+\sigma-1}(t)$ is bounded by

$$I_\omega^{\alpha+\sigma-1}(t) \leq (1 + \omega) \psi_{\sigma,r} (\|1 + u_0\|_{L_\sigma} + \|v_0\|_{H_r^1}) \equiv \bar{I}_\omega^{\alpha+\sigma-1} \quad (42)$$

for an arbitrary constant $\omega > 0$. Then, for any exponent $q \geq 2$ satisfying $r \leq q \leq (\alpha + \sigma - 1)/\beta$,

$$\|v\|_{H_q^1}^q \leq C_q e^{-\delta_q t} \|v_0\|_{H_q^1}^q + C_q \psi_{\sigma,r} (\|1 + u_0\|_{L_\sigma} + \|v_0\|_{H_r^1}) \quad (43)$$

holds with some positive constants C_q and δ_q . Moreover, for any exponent $\sigma \leq \theta \leq \{q(n+2)/(2n) - 1\}(\alpha - 1)$,

$$\|1 + u\|_{L_\theta}^\theta \leq e^{-qt/(2\tau)} \|1 + u_0\|_{L_\theta}^\theta + \psi_{\theta,q} \left(\|1 + u_0\|_{L_\sigma} + \|v_0\|_{H_q^1} \right) \quad (44)$$

holds with some increasing function $\psi_{\theta,q}(\cdot)$. In addition, for an arbitrary constant $\omega > 0$, it holds that

$$I_\omega^{\alpha+\theta-1}(t) \leq (1+\omega) \psi_{\theta,q} \left(\|1+u_0\|_{L_\theta} + \|v_0\|_{H_q^1} \right) \equiv \bar{I}_\omega^{\alpha+\theta-1}. \quad (45)$$

Proof. We can prove the lemma in the similar argument as in Lemmas 8 and 9.

Firstly, the inequality (33) holds also in this case. Since $q\beta \leq \alpha + \sigma - 1$, by Lemma 6 again, we verify

$$\|v\|_{H_q^1}^q \leq C_q e^{-\delta_q t} \|v_0\|_{H_q^1}^q + C_q b_{q,\alpha+\sigma-1} \left(|\Omega| + I_{q/2\tau}^{\alpha+\sigma-1}(t) \right).$$

By (42), we obtain (43).

The estimate (44) is verified from the inequality (41) together with (43), since $q\beta \leq \alpha + \sigma - 1 \leq \alpha + \theta - 1$ and $\theta\kappa/(\kappa-1) \leq \alpha + \theta - 1$ with $\kappa = q(n+2)/(2n)$. The proof of (45) is just the same as that of (35).

Thus we complete the proof of the lemma. \square

For obtaining the final a priori estimate, we apply Lemma 10 iteratively. We then show the following a priori estimate.

Proposition 11. *Let (u, v) be a local solution to (E), and assume that*

$$\alpha > 1, \quad 0 < \beta \leq \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha-1).$$

Then, for any exponent $p > 2$, it holds that

$$\begin{aligned} \|1+u\|_{L_p}^p + \|v\|_{H_p^1}^p &\leq C e^{-pt/(2\tau)} \left(\|1+u_0\|_{L_p}^p + \|v_0\|_{H_p^1}^p \right) \\ &\quad + \psi_p \left(\|1+u_0\|_{L_\sigma} + \|v_0\|_{H_\tau^1} \right) \end{aligned} \quad (46)$$

with some exponents $1 < \sigma < p$, $\alpha/\beta < r < p$ and some increasing function $\psi_p(\cdot)$.

Proof. The proof is given by induction. Firstly we have estimates (28) on $\|u\|_{L_1}$. Let

$$\theta_0 = 1.$$

Secondly we have (30) on $\|v\|_{H_q^1}$ for $2 \leq q \leq \alpha/\beta$ by Lemma 8 and (34) on $\|1+u\|_{L_\theta}$ for $1 < \theta \leq \{q(n+2)/(2n) - 1\}(\alpha-1)$ by Lemma 9. Let

$$q_1 = \frac{\theta_0 + \alpha - 1}{\beta} = \frac{\alpha}{\beta}, \quad \theta_1 = \left(\frac{n+2}{2n} q_1 - 1 \right) (\alpha - 1).$$

For each integer k and given θ_k , we can obtain by Lemma 10 the estimates (43) on $\|v\|_{H_q^1}$ for $2 < q \leq (\theta_k + \alpha - 1)/\beta$ and (44) on $\|1+u\|_{L_\theta}$ for $1 < \theta \leq \{q(n+2)/(2n) - 1\}(\alpha-1)$. Define

$$q_{k+1} = \frac{\theta_k + \alpha - 1}{\beta} = \frac{(n+2)(\alpha-1)}{2n\beta} q_k, \quad \theta_{k+1} = \left(\frac{n+2}{2n} q_{k+1} - 1 \right) (\alpha - 1).$$

Since $(n+2)(\alpha-1)/(2n\beta) > 1$ by assumption, we can easily see that

$$q_k \rightarrow \infty \quad \text{and} \quad \theta_k \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty.$$

Hence, for any $p > 1$, there exists a finite integer k_0 such that $q_{k_0} > p$ and $\theta_{k_0} > p$, and the desired estimates are obtained. \square

By using the a priori estimates shown above, we prove the main theorem for the global existence of the solutions.

Proof of Theorem 1. From Theorem 5 for each pair of nonnegative initial functions (u_0, v_0) there exists a unique nonnegative local solution (u, v) on the interval $[0, T]$ with the estimate (27), and the existence time T depends only on the norm $\|u_0\|_{L_p} + \|v_0\|_{H_p^1}$. In addition, from Proposition 11, the norm $\|u(t)\|_{L_p} + \|v(t)\|_{H_p^1}$, $0 \leq t \leq T$, is estimated from above by a uniform constant C_{U_0} also depending only on the norm $\|u_0\|_{L_p} + \|v_0\|_{H_p^1}$. Hence, the interval can be extended to $[0, T + \tilde{T}]$, where the extended time \tilde{T} and the norm $\|u(t)\|_{L_p} + \|v(t)\|_{H_p^1}$, $0 \leq t \leq T + \tilde{T}$, are estimated by the same constant C_{U_0} . The existence interval can be again extended, to $[0, T + 2\tilde{T}]$. Repeating this procedure proves the global existence theorem with the estimate (5). \square

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Appendix. On the domains of fractional powers of Laplace operators in L_p -spaces

Here we discuss the characterization of the domains of definition of fractional powers of Laplace operator $A_0 = -\Delta + 1$ with Neumann boundary condition on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, as a closed operator in $L_p(\Omega)$ for each $1 < p < \infty$.

We have already known the following facts.

Theorem A.1 ([2, Theorem 2.4.1.3], [24, Theorem 5.3.4], [30, Theorem 2.15]). *For each $1 < p < \infty$, A_0 is considered as a closed operator in $L_p(\Omega)$, the domain of which is $H_{p,N}^2(\Omega)$. If we denote $A_p = A_0|_{L_p}$, then it holds that $\mathcal{D}(A_p) = H_{p,N}^2(\Omega)$ with norm equivalence.*

Theorem A.2 ([2, Theorem 2.5.1.1], [24, Theorems 5.3.4 and 5.4.1]). *Let k be a positive integer and $1 < p < \infty$. Then $u \in H_p^{k+2}(\Omega) \cap H_{p,N}^2(\Omega)$ yields $A_0 u \in H_p^k(\Omega)$. Moreover, if $u \in H_{p,N}^2(\Omega)$ satisfies $A_0 u \in H_p^k(\Omega)$, then $u \in H_p^{k+2}(\Omega)$. That means, as the first example, if we denote $\mathfrak{A}_p = A_0|_{H_p^1}$, that the identity $\mathcal{D}(\mathfrak{A}_p) = H_{p,N}^3(\Omega)$ holds with norm equivalence.*

To interpolate these results between $k = 0$ and $k = 2$, we apply the theory of bounded H_∞ functional calculus in $L_p(\Omega)$ given by Yagi [30, Sec.16.1.2].

Theorem A.3. *For the operator $A_p = A_0|_{L_p}$, the identity*

$$\mathcal{D}(A_p^\theta) = [L_p(\Omega), H_{p,N}^2(\Omega)]_\theta = \begin{cases} H_p^{2\theta}(\Omega) & \text{for } 0 \leq \theta < \frac{1}{2} + \frac{1}{2p} \\ H_{p,N}^{2\theta}(\Omega) & \text{for } \frac{1}{2} + \frac{1}{2p} < \theta \leq 1 \end{cases} \quad (\text{A.1})$$

holds with norm equivalence.

Proof. Firstly, it is obvious that $L_p(\Omega)$ is a reflexive Banach space and A_p is a sectorial operator in $L_p(\Omega)$ with angle $\omega_A = 0$. Hence, we can directly verify the following condition given in [30, Theorem 16.5] with $A = A_p$, $X = L_p(\Omega)$, $X^* = L_{p'}(\Omega)$ and $\langle \cdot, \cdot \rangle$ their duality product:

(H) For every angle $\omega_A < \omega < \pi$ and every exponent $0 < \theta < 1$, the integrable condition along the V-shaped contour $\Gamma_\omega : \lambda = \rho e^{\pm i\omega}$ ($0 \leq \rho < \infty$)

$$I_{\omega, \theta} = \int_{\Gamma_\omega} |\lambda|^{2\theta-1} \left| \langle A^{2(1-\theta)} (\lambda - A)^{-2} F, G \rangle \right| |d\lambda| \leq C_{\omega, \theta} \|F\| \|G\|_*,$$

$$F \in X, G \in X^*, \quad (\text{A.2})$$

holds with some constant $C_{\omega, \theta} > 0$.

We omit the detail here. Then, by [30, Theorem 16.5] it is verified that A_p has a bounded H_∞ functional calculus in $L_p(\Omega)$. Again by [30, Theorem 16.5], we have the first identity of (A.1). The rest part of the theorem has been already shown in [30, Theorem 16.11]. \square

The next theorem shows the interpolation result between $k = 1$ and $k = 3$.

Theorem A.4. *For the operator $\mathfrak{A}_p = A_0|_{H_p^1}$, the identity*

$$\mathcal{D}(\mathfrak{A}_p^\theta) = [H_p^1(\Omega), H_{p,N}^3(\Omega)]_\theta = \begin{cases} H_p^{1+2\theta}(\Omega) & \text{for } 0 \leq \theta < \frac{1}{2p} \\ H_{p,N}^{1+2\theta}(\Omega) & \text{for } \frac{1}{2p} < \theta \leq 1 \end{cases} \quad (\text{A.3})$$

holds with norm equivalence.

Proof. It is also obvious that $H_p^1(\Omega)$ is a reflexive Banach space with duality product $\langle \cdot, \cdot \rangle$ of $H_p^1(\Omega) \times H_{p'}^1(\Omega)$, and that \mathfrak{A}_p is a sectorial operator in $H_p^1(\Omega)$ with angle $\omega_A = 0$. Hence, again we can prove the condition (H) above by direct calculation to see that the Laplace operator $\mathfrak{A}_p = A_0|_{H_p^1}$ has a bounded H_∞ functional calculus in $H_p^1(\Omega)$. Then, we have the first identity of (A.3) again by [30, Theorem 16.5]. For the proof of the rest part of the theorem, we must follow carefully the proof of [30, Theorem 16.11].

Step 1: $\mathcal{D}(\mathfrak{A}_p^\theta) \subset H_{p,(N)}^{1+2\theta}(\Omega)$. We can easily see that $[H_p^1(\Omega), H_{p,N}^3(\Omega)]_\theta \subset [H_p^1(\Omega), H_p^3(\Omega)]_\theta = H_p^{1+2\theta}(\Omega)$. To see the boundary condition for $1/(2p) < \theta < 1$, take a sequence $\{u_k\} \subset \mathcal{D}(\mathfrak{A}_p) = H_{p,N}^3(\Omega)$ converging to u in $\mathcal{D}(\mathfrak{A}_p^\theta)$. This implies that every $u \in \mathcal{D}(\mathfrak{A}_p^\theta)$ satisfies the Neumann boundary condition on $\partial\Omega$.

Step 2: $\mathcal{D}(\mathfrak{A}_p^\theta) \supset H_{p,(N)}^{1+2\theta}(\Omega)$. We divide this step into three parts.

(i) $\frac{1}{2} \leq \theta \leq 1$. Let $u \in H_{p,N}^{1+2\theta}(\Omega)$. Then, for any $v \in H_{p',N}^3(\Omega) = \mathcal{D}(\mathfrak{A}_p^*)$, similarly in the proof of [30, Theorem 16.11], we see

$$\begin{aligned} |\langle \langle u, (\mathfrak{A}_p^*)^\theta v \rangle \rangle| &= \left| \frac{1}{2\pi i} \int_\Gamma \lambda^\theta \langle \langle u, (\lambda - \mathfrak{A}_p^*)^{-1} v \rangle \rangle d\lambda \right| \\ &\leq \|A_0 u\|_{H_p^{2\theta-1}} \left\| \frac{1}{2\pi i} \int_\Gamma \lambda^\theta (\lambda - A_0)^{-1} v d\lambda \right\|_{(H_{p'}^{2\theta-1})} \\ &\leq C \|A_0 u\|_{H_p^{2\theta-1}} \left\| A_0^{\theta-\frac{1}{2}} v \right\|_{H_{p'}^{2-2\theta}} \leq C \|u\|_{H_p^{1+2\theta}} \|v\|_{H_{p'}^1}. \end{aligned}$$

Here we utilize [30, Theorem 1.43] for the boundedness of $A_0 = 1 - \sum_k D_k^2 : H_p^{2\theta+1}(\Omega) \rightarrow H_p^{2\theta-1}(\Omega)$ and Theorem 3 for the boundedness of the fractional powers of $A_0|_{L_p} = A_p$. This inequality yields that, for each fixed $u \in H_{p,N}^{1+2\theta}(\Omega)$, the linear form $\langle\langle u, (\mathfrak{A}_p^*)^\theta v \rangle\rangle$ is a bounded linear functional of $v \in H_{p'}^1(\Omega)$, that is, there exists $w \in H_p^1(\Omega)$ such that $\langle\langle u, (\mathfrak{A}_p^*)^\theta v \rangle\rangle = \langle\langle w, v \rangle\rangle$ for any $v \in H_{p'}^1(\Omega)$. Hence, $w = \mathfrak{A}_p^\theta u$ and $u \in \mathcal{D}(\mathfrak{A}_p^\theta)$.

(ii) $\frac{1}{2p} < \theta < \frac{1}{2}$. Let $u \in H_{p,N}^{1+2\theta}(\Omega)$. Then, for any $v \in H_{p',N}^3(\Omega) = \mathcal{D}(\mathfrak{A}_p^*)$, by the quite similar argument as (i), we see

$$\begin{aligned} |\langle\langle u, (\mathfrak{A}_p^*)^\theta v \rangle\rangle| &\leq C \|u\|_{H_p^{2\theta+1}} \left\| \frac{1}{2\pi i} \int_\Gamma \lambda^\theta (\lambda - A_0)^{-1} v \, d\lambda \right\|_{H_{p'}^{1-2\theta}} \\ &\leq C \|u\|_{H_p^{2\theta+1}} \|v\|_{H_{p'}^1}. \end{aligned}$$

Here we utilize again [30, Theorem 1.43] for the boundedness of $D_k : H_{p'}^{1-2\theta}(\Omega) \rightarrow (H_p^{2\theta}(\Omega))'$. Thus, for each fixed $u \in H_{p,N}^{1+2\theta}(\Omega)$, there exists $w \in H_p^1(\Omega)$ such that $\langle\langle u, (\mathfrak{A}_p^*)^\theta v \rangle\rangle = \langle\langle w, v \rangle\rangle$ for any $v \in H_{p'}^1(\Omega)$. Hence, $w = \mathfrak{A}_p^\theta u$ and $u \in \mathcal{D}(\mathfrak{A}_p^\theta)$.

(iii) $0 < \theta < \frac{1}{2p}$. We can verify that $u \in H_p^{1+2\theta}(\Omega)$ is contained in $\mathcal{D}(\mathfrak{A}_p^\theta)$ by the same argument as (ii) except for the boundary conditions.

Hence we complete the proof. \square

As the consequence, we conclude (17) in Section 2.

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