

# The vertices of the components of the permutation module induced from parabolic groups

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## Abstract

We consider the permutation module  $k_P \uparrow^{\mathrm{GL}_n(p^f)}$ , where  $P$  is a parabolic group in the general linear group  $\mathrm{GL}_n(p^f)$  and  $k$  is an algebraically closed field of prime characteristic  $p$ . The vertices of the components of these modules have been calculated in [9] by Tinberg, who studied these modules for all groups with split BN-pairs in characteristic  $p$ . In this paper we show that the idea of suitability is strong enough to find all  $p$ -groups that are vertex of some component of  $k_P \uparrow^{\mathrm{GL}_n(p^f)}$ . Furthermore using a result of Burry and Carlson we show that all components have a different vertex.

## 1 Introduction

Let  $G$  be a finite group with subgroup  $H$ . We are interested in the  $p$ -groups of  $G$  that are vertices of components of the permutation-module  $k_H \uparrow^G$ , where  $k$  is an algebraically closed field of characteristic  $p$ . In [7] the author introduces the idea of  $H$ -suitability and in an example shows how suitability could be applied to detect all  $p$ -groups that are vertices of some component of  $k_H \uparrow^G$ . Furthermore in that example suitability is at its most restrictive as every  $H$ -suitable group turns out to be the vertex of a component of  $k_H \uparrow^G$ . In the present paper we present another example where this concept of suitability unfolds its full potential by giving us all  $p$ -groups that are vertices of some component of the given permutation module  $k_H \uparrow^G$ .

Next we describe the notion of  $H$ -suitability. A  $p$ -subgroup  $V$  of  $G$  is called  *$H$ -suitable* (with respect to  $G$  and  $p$ ), if for every  $S \in \mathrm{Syl}_p(G)$  with  $V \leq S$  there exists some  $g \in G$  so that  $V = S \cap H^g$ . Here  $H^g$  denotes the conjugate  $g^{-1}Hg$  of  $H$ . In [7] we show that  $H$ -suitability is a necessary condition for a  $p$ -group to be the vertex of a component of  $k_H \uparrow^G$  and present an example in which it is also sufficient. However in general an  $H$ -suitable group is not necessarily the vertex of some component of  $k_H \uparrow^G$ . For instance with respect to the symmetric group  $S_5$  and  $p = 2$  the trivial group is  $D$ -suitable, for the dihedral group  $D := \langle (1234), (13) \rangle \leq S_5$ , but  $k_D \uparrow^{S_5}$  has no projective summand. Nevertheless one can use the concept of suitability to produce a list of potential candidates for a vertex by finding all  $H$ -suitable groups. If this list is short enough one may then deal with each candidate individually.

In [7] we see that if  $V$  is  $H$ -suitable then so is every  $G$ -conjugate of  $V$ . Hence given a Sylow- $p$ -group  $S$  of  $G$  the set  $\mathcal{A} := \{S \cap H^g : g \in G\}$  contains a representative of each conjugacy class of  $H$ -suitable groups. The following

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statement presents a method to shorten the list given by  $\mathcal{A}$  by finding groups that are not  $H$ -suitable. A more general version of the statement with proof can be found in [7, Theorem 2.6.].

**Lemma 1.1** *Let  $S \in \text{Syl}_p(G)$  and  $K \leq G$ . Also let  $T \subseteq G$  such that*

- (a) *For every  $g \in G$  there is  $\alpha \in T$  so that  $S \cap H^g =_K S \cap H^{\alpha^{-1}}$ .*
- (b) *For all  $\alpha \in T$  we have  $S^\alpha \cap H \leq S$*
- (c) *If  $S^\alpha \cap H =_K S^{\alpha'} \cap H$ , for  $\alpha, \alpha' \in T$ , then  $\alpha = \alpha'$ .*

*Furthermore let  $V \leq S$  be  $H$ -suitable. Then there is some  $\alpha \in T$  such that  $V =_K S^\alpha \cap H$ .*

Observe that Lemma 1.1 implies that every  $H$ -suitable group  $V$  that is contained in  $S$  is a  $K$ -conjugate of both  $S \cap H^{\alpha^{-1}}$  and  $S^\beta \cap H$ , for some  $\alpha, \beta \in T$ . In particular any group in the set  $\{S \cap H^{\alpha^{-1}} : \alpha \in T\}$  which has no  $K$ -conjugate in the set  $\{S^\alpha \cap H : \alpha \in T\}$  fails to be  $H$ -suitable.

Let us now turn to our example for which we need the following notation. For the remainder of this paper let  $n, f \geq 1$  and set  $q = p^f$ , for some prime number  $p$ . Furthermore let  $G := \text{GL}_n(q)$  be the general linear group, that is, the group of invertible  $n \times n$ -matrices over the finite field  $\mathbb{F}_q$  with  $q$  elements. Finally let  $k$  be an algebraically closed field of characteristic  $p$ .

For any integer  $j \geq 1$  we define  $\text{GL}_j := \text{GL}_j(q)$ . By  $B_j$  we denote the group of upper-triangular matrices in  $\text{GL}_j$ . Those elements in  $B_j$  with 1's on the main diagonal form the group  $U_j$ . Recall that  $U_j$  is a Sylow- $p$ -group of  $\text{GL}_j$  and  $B_j$  is the normalizer of  $U_j$  in  $\text{GL}_j$ .

Furthermore  $\mathcal{W}_j$  denotes the group of permutation matrices in  $\text{GL}_j$ . There is a bijection between the symmetric group  $\text{Sym}(j)$  and  $\mathcal{W}_j$ , in the sense that the permutation  $\omega \in \text{Sym}(j)$  corresponds to the permutation matrix  $(\delta_{r, \omega(s)})_{r,s}$ , where  $\delta$  denotes the Kronecker-symbol.

Let  $\lambda = (\lambda_1, \dots, \lambda_r)$  be a *composition* of  $n$ , that is,  $\lambda_1, \dots, \lambda_r$  are positive integers such that  $\lambda_1 + \dots + \lambda_r = n$ . Then for  $X_t \in \text{GL}_{\lambda_t}$ , let  $D_n(X_1, X_2, \dots, X_r)$  be the matrix in  $G$ , which has  $X_1, \dots, X_r$  along the main diagonal and zeros otherwise. Similarly  $D_n(X_1 \bullet, X_2 \bullet, \dots, X_r)$  denotes any matrix in  $G$  which has  $X_1, \dots, X_r$  along the main diagonal, arbitrary elements in  $\mathbb{F}_q$  above and zeros below that diagonal. Finally we define

$$P_\lambda = \{D_n(X_1 \bullet, X_2 \bullet, \dots, X_r) : X_t \in \text{GL}_{\lambda_t}\}.$$

Then  $P_\lambda$  is a *parabolic subgroup* of  $G$ . For instance if  $\lambda = (1^n)$  then  $P_\lambda = B_n$ .

The permutation module  $k_{P_\lambda} \uparrow^G$ , where  $\lambda$  is a composition of  $n$ , has been studied before and in more generality. Tinberg [9] studied these modules for all groups with split BN-pairs in characteristic  $p$ . Further work was done by Canbanes [3] and Sin [8]. In particular the vertices of all components are known as well as their Green correspondents. In this paper we revisit  $k_{P_\lambda} \uparrow^G$  and show that  $P_\lambda$ -suitability is a strong enough tool to find all  $p$ -groups that are vertices

of components of  $k_{P_\lambda} \uparrow^G$ . In section 2 we find all  $P_\lambda$ -suitable groups up to  $G$ -conjugation and in section 3 we show that all  $P_\lambda$ -suitable groups are vertex of exactly one component of  $k_{P_\lambda} \uparrow^G$ .

Next let  $1 \leq k, l \leq n$  such that  $k \neq l$ . By  $F_{k,l}$  we denote the subgroup of  $G$  that consists of exactly those matrices that have ones on the main diagonal and zeros everywhere else except in the  $(k, l)$ -entry, which is arbitrary in  $\mathbb{F}_q$ . One checks easily that for  $\omega \in \mathcal{W}_n$  and  $k \neq l$  we have  $F_{k,l} = (F_{\omega(k), \omega(l)})^\omega$ . Finally by [5] a group  $V \leq U_n$  is called a *pattern group* if  $X_{k,l} \neq 0$ , for some  $X \in V$  and  $1 \leq k < l \leq n$ , implies that  $F_{k,l} \leq V$ . Note that a pattern group which contains both  $F_{k,l}$  and  $F_{l,m}$ , for  $k < l < m$ , also contains  $F_{k,m}$ .

**Lemma 1.2** *Let  $V, W \leq U_n$  be pattern groups that are  $B_n$ -conjugate. Then  $V = W$ .*

Proof: It is enough to show that  $F_{k,l} \leq V$  implies  $F_{k,l} \leq W$ . So let  $F_{k,l} \leq V$ , for some  $k < l$ , and let  $g \in F_{k,l}$  be non-trivial. By assumption there is  $h \in B_n$  so that  $g' := g^h \in W$ . As  $g'_{k,l} \neq 0$  it now follows that  $F_{k,l} \leq W$ .

□

## 2 $P_\lambda$ -suitable groups

In this section let  $\lambda = (\lambda_1, \dots, \lambda_r)$  be a composition of  $n$ . Also set  $s_i := \sum_{k=1}^{i-1} \lambda_k$ , for  $i = 1, \dots, r$ , and  $s_{r+1} := n$ .

### 2.1 Good Permutations

We call  $\omega \in \mathcal{W}_n$  *good* with respect to  $\lambda$ , if  $\omega(s_i + 1) < \dots < \omega(s_i + \lambda_i)$ , for all  $i = 1, \dots, r$ .

**Lemma 2.1** *Every  $(B_n, P_\lambda)$ -double coset in  $G$  contains a permutation matrix in  $\mathcal{W}_n$  that is good with respect to  $\lambda$ .*

Proof: Let  $g \in G$ . Then by the Bruhat decomposition, (see [1]), and since  $B_n \leq P_\lambda$ , there exists  $\mu \in \mathcal{W}_n$  such that  $B_n \cdot g \cdot P_\lambda = B_n \cdot \mu \cdot P_\lambda$ . Next let

$$\mu(\{s_i + 1, \dots, s_i + \lambda_i\}) = \{k_{s_i+1}, \dots, k_{s_i+\lambda_i}\}, \text{ where } k_{s_i+1} < \dots < k_{s_i+\lambda_i},$$

for all  $i = 1, \dots, r$ . Set  $\gamma(j) := \mu^{-1}(k_j)$ , for all  $j = 1, \dots, n$ . Then  $\gamma \in \mathcal{W}_n$ . Also observe that  $\gamma$  acts on the sets  $\{s_i + 1, \dots, s_i + \lambda_i\}$ . Hence  $\gamma \in P_\lambda$ .

Next set  $\omega := \mu \cdot \gamma \in \mathcal{W}_n$ . Since  $\omega(j) = \mu(\mu^{-1}(k_j)) = k_j$ , for all  $j = 1, \dots, n$ , it follows that  $\omega$  is good. As now  $B_n \cdot g \cdot P_\lambda = B_n \cdot \omega \cdot P_\lambda$ , the proof is complete.

□

For any  $g \in G$  we define  $V(g) := U_n^g \cap P_\lambda$ . The following lemma is then an easy exercise.

**Lemma 2.2** *Let  $\omega \in \mathcal{W}_n$  be good with respect to  $\lambda$ . Then*

$$V(\omega) = \{X = D_n(A_1\bullet, A_2\bullet, \dots, A_r) : A_i \in U_{\lambda_i}, \text{ and} \\ X_{k,l} = 0, \text{ for all } k, l = 1, \dots, n \text{ so that } \omega(k) > \omega(l)\}. \quad (1)$$

*In particular  $V(\omega) \leq U_n$ , and  $V(\omega)$  is a pattern group.*

Observe that Lemma 2.1 implies the following

**Corollary 2.3** *Every  $P_\lambda$ -suitable group is  $G$ -conjugate to some  $V(\omega)$ , where  $\omega \in \mathcal{W}_n$  is good with respect to  $\lambda$ .*

**Lemma 2.4** *The set  $T$  of all  $\omega \in \mathcal{W}_n$  that are good with respect to  $\lambda$  satisfies the properties (a)-(c) in Lemma 1.1, where  $H = P_\lambda$ ,  $S = U_n$  and  $K = B_n$ .*

Proof: Property (a) is a consequence of Lemma 2.1, property (b) follows since  $V(\omega) \leq U_n$  and property (c) is a consequence of Lemma 1.2 and the structure of  $V(\omega)$ , as given in (1). □

**Proposition 2.5** *Let  $V \leq U_n$  be a pattern group. If  $V$  is  $P_\lambda$ -suitable, then  $V = V(\omega)$ , where  $\omega \in \mathcal{W}_n$  is good with respect to  $\lambda$ .*

Proof: By Lemma 2.4 and Lemma 1.1 we have  $V =_{B_n} V(\omega)$ , where  $\omega \in \mathcal{W}_n$  is good with respect to  $\lambda$ . Now  $V = V(\omega)$  follows from Lemma 1.2. □

## 2.2 $\lambda$ -permutations

Recall that  $I_j$  denotes the identity matrix in  $GL_j$ . By  $\overline{I_j}$  we mean the permutation matrix in  $GL_j$  that has all its ones on the anti-diagonal. Also let  $\omega = D_n(K_1, \dots, K_u) \in \mathcal{W}_n$ , where  $K_i \in \{I_{n_i}, \overline{I_{n_i}}\}$ , and  $n_1 + \dots + n_u = n$ . We call such  $\omega$  a  $\lambda$ -permutation if for all  $t = 1, \dots, r$  so that  $\lambda_t \geq 2$  we have that  $\omega$  acts on each of the following sets

$$\{1, \dots, s_t + 1\}, \{s_t + 2\}, \dots, \{s_{t+1} - 1\}, \{s_{t+1}, \dots, n\}.$$

Observe that every  $\lambda$ -permutation is good with respect to  $\lambda$ . In the following we show that if  $\omega \in \mathcal{W}_n$  is a  $\lambda$ -permutation, then  $V(\omega)$  is  $P_\lambda$ -suitable.

**Lemma 2.6** *Let  $\omega \in \mathcal{W}_n$  be a  $\lambda$ -permutation. Then*

$$V(\omega) = \{D_n(A_1\bullet, A_2\bullet, \dots, A_u) : \text{where } A_i \in U_{n_i}, \text{ if } K_i = I_{n_i}, \\ \text{and } A_i = I_{n_i}, \text{ if } K_i = \overline{I_{n_i}}.\}$$

Proof: Let  $S$  be the set on the right hand side. Then  $S = U_n^\omega \cap U_n \leq P_\lambda$ . Since  $V(\omega) \leq U_n$  we get  $S \leq U_n^\omega \cap P_\lambda = V(\omega) \leq U_n^\omega \cap U_n = S$ . Thus  $V(\omega) = S$ .

□

Lemma 2.6 implies the following

**Corollary 2.7** *Let  $\omega \in \mathcal{W}_n$  be a  $\lambda$ -permutation. Then*

- (1)  $B_n \leq N_G(V(\omega))$
- (2) *If  $\mu \in \mathcal{W}_n$  such that  $V(\omega)^\mu \leq U_n$ , then  $\mu = D_n(\mu_1, \dots, \mu_u)$  where  $\mu_i = I_{n_i}$ , if  $K_i = I_{n_i}$ , and  $\mu_i \in \mathcal{W}_{n_i}$ , if  $K_i = \overline{I_{n_i}}$ .*

**Lemma 2.8** *Let  $\omega \in \mathcal{W}_n$  be a  $\lambda$ -permutation. Also let  $N$  be the set of all matrices of the form  $D_n(A_1 \bullet, A_2 \bullet, \dots, A_u)$  where  $A_i \in B_{n_i}$ , if  $K_i = I_{n_i}$ , and  $A_i \in \text{GL}_{n_i}$ , if  $K_i = \overline{I_{n_i}}$ . Moreover let  $g \in G$  such that  $V(\omega)^g \leq U_n$ . Then  $g \in N$ . In particular  $N_G(V(\omega)) = N$ .*

Proof: By the Bruhat decomposition there are  $A, B \in B_n$  and  $\mu \in \mathcal{W}_n$  such that  $g = A\mu B$ . As  $B_n \subseteq N_G(V(\omega))$  we have  $V(\omega)^\mu \leq U_n$ , and thus  $\mu \in N$ , by Corollary 2.7. Since  $B_n \subseteq N$ , we get  $g \in N$ . Now  $N_G(V(\omega)) \subseteq N$  follows. As one checks easily that  $N$  normalizes  $V(\omega)$ , the proof is complete.

□

**Corollary 2.9** *Let  $\omega_1, \omega_2 \in \mathcal{W}_n$  be  $\lambda$ -permutations so that  $V(\omega_1) =_G V(\omega_2)$ . Then  $\omega_1 = \omega_2$ .*

Proof: As  $V(\omega_1)^g = V(\omega_2) \leq U_n$ , for some  $g \in G$ , we have  $g \in N_G(V(\omega_1))$ , by Lemma 2.8. Thus  $V(\omega_1) = V(\omega_2)$ . Now  $\omega_1 = \omega_2$  follows from (1).

□

**Proposition 2.10** *Let  $\omega \in \mathcal{W}_n$  be a  $\lambda$ -permutation. Then  $V(\omega)$  is  $P_\lambda$ -suitable.*

Proof: Since  $V(\omega)^{\omega^{-1}} \leq U_n$ , it follows  $\omega^{-1} \in N_G(V(\omega))$ , by Lemma 2.8. Thus  $V(\omega) = U_n \cap P_\lambda^{\omega^{-1}}$ . Note that  $U_n \in \text{Syl}_p(N_G(V(\omega)))$ . So the statement follows from [7, Lemma 2.2.(3)].

□

### 2.3 The $P_\lambda$ -suitable $V(\omega)$

In the following let  $\omega \in \mathcal{W}_n$  be good with respect to  $\lambda$  so that  $V(\omega)$  is  $P_\lambda$ -suitable. We aim to show that  $\omega$  is a  $\lambda$ -permutation.

**Lemma 2.11** *Let  $F_{k,l} \leq V(\omega)$ , that is,  $\omega(k) < \omega(l)$ , for some  $k < l$ . Then*

- (1)  $F_{t,l} \leq V(\omega)$ , that is,  $\omega(t) < \omega(l)$ , for all  $t = 1, \dots, k$
- (2)  $F_{k,t} \leq V(\omega)$ , that is,  $\omega(k) < \omega(t)$ , for all  $t = l, \dots, n$

Proof: We prove part (1) by contradiction. Without loss of generality we may assume that  $F_{k-1,l} \not\leq V(\omega)$ . In particular  $F_{k-1,k} \not\leq V(\omega)$ . Next let  $\mu \in \mathcal{W}_n$  correspond to the permutation  $(k-1, k) \in \text{Sym}(n)$ , and set  $V := V(\omega)^\mu$ . Then  $V \leq U_n$  is a  $P_\lambda$ -suitable pattern group, and so, by Proposition 2.5, there is  $\alpha \in \mathcal{W}_n$  that is good with respect to  $\lambda$  such that  $V = V(\alpha)$ .

As  $F_{k,k-1} \not\leq V(\omega)$ ,  $F_{k-1,l} \not\leq V(\omega)$  and  $F_{k,l} \leq V(\omega)$ , we have  $F_{k-1,k} \not\leq V(\alpha)$ ,  $F_{k,l} \not\leq V(\alpha)$  and  $F_{k-1,l} \leq V(\alpha)$ . That forces  $\alpha(k) < \alpha(k-1)$ ,  $\alpha(l) < \alpha(k)$  and  $\alpha(k-1) < \alpha(l)$ , respectively. This contradiction proves part (1). A similar argument proves part (2). □

**Corollary 2.12** *If  $\omega(k) < \omega(k+1)$ , then  $\omega$  acts on the sets  $\{1, \dots, k\}$  and  $\{k+1, \dots, n\}$ .*

Proof: Let  $r \in \{1, \dots, k\}$ . Then  $\omega(r) < \omega(k+1)$ , by Lemma 2.11 (1). Hence  $\omega(r) < \omega(t)$ , for all  $t = k+1, \dots, n$ , by Lemma 2.11 (2). In particular  $\omega(\{1, \dots, k\}) \subseteq \{1, \dots, k\}$ , and the statement follows. □

**Proposition 2.13** *Let  $\omega \in \mathcal{W}_n$  be good with respect to  $\lambda$  such that  $V(\omega)$  is  $P_\lambda$ -suitable. Then  $\omega$  is a  $\lambda$ -permutation.*

Proof: First we show that  $\omega = D_n(K_1, \dots, K_u)$ , where  $K_i \in \{I_{n_i}, \overline{I_{n_i}}\}$ , and  $n_1 + \dots + n_u = n$ . Clearly there is some  $k \in \{0, 1, \dots, n\}$  such that  $\omega = D_n(K_1, \dots, K_u, \omega')$ , where  $K_i \in \{I_{n_i}, \overline{I_{n_i}}\}$ ,  $n_1 + \dots + n_u = k$  and  $\omega' \in \mathcal{W}_{n-k}$ . Let us choose  $k$  maximal with this property and suppose  $k < n$ . Observe that  $\omega$  acts on the sets  $\{1, \dots, k\}$  and  $\{k+1, \dots, n\}$ . Also the maximality of  $k$  implies that  $\omega(k+1) \neq k+1$ . Next let  $l \geq k+1$  be maximal such that  $\omega(k+1) > \omega(k+2) \dots > \omega(l)$ . Since  $l = n$  or  $\omega(l) < \omega(l+1)$ , it follows from Corollary 2.12 that  $\omega$  acts on the set  $\{k+1, \dots, l\}$  and it does so like the permutation matrix  $\overline{I_{l-k}}$ . But this contradicts the maximality of  $k$ , and thus  $k = n$ .

Now let  $i \in \{1, \dots, r\}$  so that  $\lambda_i \geq 2$ . As  $\omega$  is good with respect to  $\lambda$  we have  $\omega(s_i+1) < \dots < \omega(s_{i+1})$ . Now Corollary 2.12 implies that  $\omega$  acts on the sets  $\{1, \dots, s_i+1\}, \{s_i+2\}, \dots, \{s_{i+1}-1\}, \{s_{i+1}, \dots, n\}$ . In particular  $\omega$  is a  $\lambda$ -permutation. □

**Theorem 2.14** *The set  $\mathcal{C} := \{V(\omega) : \omega \in \mathcal{W}_n \text{ is a } \lambda\text{-permutation}\}$  provides a full set of representatives for the  $G$ -conjugacy classes of  $P_\lambda$ -suitable groups without repetitions.*

Proof: In Proposition 2.10 we have established that  $V(\omega)$ , where  $\omega \in \mathcal{W}_n$  is a  $\lambda$ -permutation, is  $P_\lambda$ -suitable. Furthermore any two different such groups lie in different conjugacy classes, by Corollary 2.9.

Now let  $V$  be  $P_\lambda$ -suitable. Then  $V$  is  $G$ -conjugate to some  $V' \leq U_n$ . Hence  $V' = U_n \cap P_\lambda^g$ , for some  $g \in G$ . By Lemma 2.1 it follows that  $V'$  is  $G$ -conjugate to  $V(\omega)$ , where  $\omega \in \mathcal{W}_n$  is good with respect to  $\lambda$ . In particular  $\omega$  is a  $\lambda$ -permutation, by Proposition 2.13.

□

### 3 The number of components of $k_{P_\lambda} \uparrow^{\text{GL}_n}$ and their vertices

Let  $\lambda = (\lambda_1, \dots, \lambda_r)$  be a composition of  $n$ . In the previous section we found all  $P_\lambda$ -suitable groups up to  $G$ -conjugation. In fact we have shown that  $V$  is  $P_\lambda$ -suitable if and only if  $V =_G V(\omega)$ , where  $\omega$  is a  $\lambda$ -permutation. Hence the set  $\mathcal{C}$  from Theorem 2.14 contains a vertex for each component of  $k_{P_\lambda} \uparrow^{\text{GL}_n}$ . In this section we want to determine how many components have each of the groups in  $\mathcal{C}$  as a vertex. So for the remainder of this paper let  $\omega \in \mathcal{W}_n$  be a  $\lambda$ -permutation and set  $V := V(\omega)$ .

By a result of Burry and Carlson [2] the  $kG$ -module  $k_{P_\lambda} \uparrow^G$  and the  $kN_G(V)$ -module  $\bigoplus_{g \in A} k_{P_\lambda^g \cap N_G(V)} \uparrow^{N_G(V)}$  have the same number of components with vertex

$V$ , where  $A$  is a set of representatives  $g$  for those  $(P_\lambda, N_G(V))$ -double cosets of  $G$  with  $V \leq P_\lambda^g$ .

Next suppose that  $V \leq P_\lambda^g$ , for some  $g \in G$ . Since  $U_n$  is a Sylow- $p$ -group of  $P_\lambda$  it follows that  $V^{g^{-1}h} \leq U_n$ , for some  $h \in P_\lambda$ . Now by Lemma 2.8 we have  $g^{-1}h \in N_G(V)$ , and thus  $g \in P_\lambda \cdot N_G(V)$ . Hence by the above paragraph we obtain that  $k_{P_\lambda} \uparrow^G$  has the same number of components with vertex  $V$  as  $k_{P_\lambda \cap N_G(V)} \uparrow^{N_G(V)}$ .

**Lemma 3.1** *We have  $P_\lambda \cap N_G(V(\omega)) = B_n$ .*

Proof: Clearly  $B_n \subseteq P_\lambda \cap N_G(V)$ . Next let  $X \in P_\lambda \cap N_G(V)$ , such that  $X \notin B_n$ . Hence  $X_{l,k} \neq 0$ , for some  $k < l$ . Since  $X \in N_G(V)$ , Lemma 2.8 implies that there is some  $j \in \{1, \dots, u\}$  such that  $K_j = \overline{I_{n_j}}$  and  $n_1 + \dots + n_{j-1} < k < l \leq n_1 + \dots + n_j$ . In particular  $\omega(k) > \omega(l)$ .

On the other hand as  $X \in P_\lambda$ , then  $s_i + 1 \leq k < l \leq s_{i+1}$ , for some  $i \in \{1, \dots, r\}$ . But then  $\omega(k) < \omega(l)$ , as  $\omega$  is good with respect to  $\lambda$ . This contradiction completes the proof.

□

**Lemma 3.2** *Let  $\omega \in \mathcal{W}_n$  be a  $\lambda$ -permutation. Then there is exactly one component in  $k_{P_\lambda} \uparrow^{\text{GL}_n}$  with vertex  $V(\omega)$ .*

Proof: Let  $V = V(\omega)$ . By the introduction of this section and Lemma 3.1 we know that the number of components of  $k_{P_\lambda} \uparrow^{\text{GL}_n}$  with vertex  $V$  coincides

with the number of components of  $k_{B_n} \uparrow^{\text{N}_G(V)}$  with vertex  $V$ . This number in turn equals the number of projective components of  $k_{B_n/V} \uparrow^{\text{N}_G(V)/V}$ . We have

$$B_n/V \cong \{D_n(A_1, \dots, A_u) : \text{where } A_i \in D_{n_i}, \text{ if } K_i = I_{n_i}, \\ \text{and } A_i \in B_{n_i}, \text{ if } K_i = \overline{I_{n_i}}\}$$

and

$$\text{N}_G(V)/V \cong \{D_n(A_1, \dots, A_u) : \text{where } A_i \in D_{n_i}, \text{ if } K_i = I_{n_i}, \\ \text{and } A_i \in \text{GL}_{n_i}, \text{ if } K_i = \overline{I_{n_i}}\}$$

where  $D_{n_i}$  is the group of diagonal matrices in  $\text{GL}_{n_i}$ . Hence

$$k_{B_n/V} \uparrow^{\text{N}_G(V)/V} \cong \bigotimes_{i: K_i = \overline{I_{n_i}}} k_{B_{n_i}} \uparrow^{\text{GL}_{n_i}}.$$

But every  $k_{B_j} \uparrow^{\text{GL}_j}$  contains exactly one projective component, known as the Steinberg-module, (see [4] for more details). Now the statement follows by [6, Proposition 1.2]. □

Finally we can state our main result.

**Theorem 3.3** *The number of components of  $k_{P_\lambda} \uparrow^{\text{GL}_n}$  coincides with the number of  $\lambda$ -permutations in  $\mathcal{W}_n$ , and  $\{V(\omega) : \omega \in \mathcal{W}_n \text{ is a } \lambda\text{-permutation}\}$  gives a full set of the different vertices of the components of  $k_{P_\lambda} \uparrow^{\text{GL}_n}$ .*

Proof: We have seen that every component of  $k_{P_\lambda} \uparrow^{\text{GL}_n}$  has a vertex of the form  $V(\omega)$ , where  $\omega \in \mathcal{W}_n$  is a  $\lambda$ -permutation. By Corollary 2.9 we know that two such groups are not  $G$ -conjugate. The result of Lemma 3.2 completes the proof. □

We conclude our paper with a specific example. Let  $\lambda = (3, 1, 2)$  be a composition of  $n = 6$ . Then  $s_1 = 0$ ,  $s_2 = 3$ ,  $s_3 = 4$  and  $s_4 = 6$ . Next observe that a  $\lambda$ -permutation  $\omega$  acts on the sets  $\{1\}$ ,  $\{2\}$ ,  $\{3, 4, 5\}$  and  $\{6\}$ . Hence there are exactly four  $\lambda$ -permutations and they are  $\omega_1 = I_6$ ,  $\omega_2 = D_6(I_2, \overline{I_3}, I_1)$ ,  $\omega_3 = D_6(I_2, \overline{I_2}, I_2)$  and  $\omega_4 = D_6(I_3, \overline{I_2}, I_1)$ . In particular  $k_{P_\lambda} \uparrow^{\text{GL}_6}$  has exactly four components with the respective vertices

$$\begin{aligned} V(\omega_1) &= U_6 \\ V(\omega_2) &= \{X \in U_6 : X_{3,4} = X_{3,5} = X_{4,5} = 0\}, \\ V(\omega_3) &= \{X \in U_6 : X_{3,4} = 0\}, \\ V(\omega_4) &= \{X \in U_6 : X_{4,5} = 0\}. \end{aligned}$$



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