Initial-boundary value problem for the degenerate hyperbolic equation of a hanging string

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Abstract

We consider an initial-boundary value problem for the degenerate linear hyperbolic equation as a model of the motion of an inextensible string fixed at one end in the gravity field. We shall show the existence and the uniqueness of the solution and study the regularity of the solution.

1 Introduction

We are concerned with the motion of an inextensible string of finite length with uniform density having one end fixed and another end free and acted on solely by forces of the gravity and the tension. Let L be the length of the string and s ($\in [0, L]$) be the arc length measured from the free end of the string. Suppose that the string is described as a curve

$$u(s,t) = (u_1(s,t), u_2(s,t), u_3(s,t)), \qquad s \in [0, L]$$

at time t and that the fixed end of the string is at the origin in $\mathbf{R}^3$ (see Figure 1).

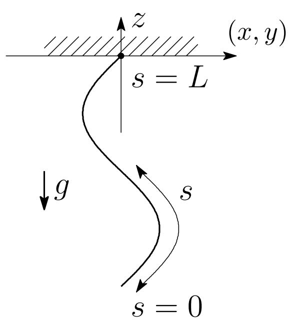

Figure 1: Hanging string

The motion of the string is dominated by the force of gravity and the tension of the string. Let ρ be the density of the string, $g = (0, 0, -1)$ the acceleration of gravity vector,

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and $\tau = \tau(s,t)$ the (scalar) tension of the string. Then the equation of the motion of the string has the form

$$\rho u_{tt} - (\tau u_s)_s = \rho g \quad \text{in} \quad (0,L)\times(0,T) \tag{1.1}$$

(for instance, see [9]). Inextensibility of the string is mathematically enforced by requiring that

$$|u_s| \equiv 1 \quad \text{in} \quad (0,L)\times(0,T). \tag{1.2}$$

As for the boundary conditions, we impose that

$$u|_{s=L} = 0, \quad \tau|_{s=0} = 0 \quad \text{on} \quad (0,T). \tag{1.3}$$

The first condition of (1.3) means that the fixed end ($s = L$) of the string is at the origin in $\mathbf{R}^3$ and the second one means that the tension vanishes at the free end ($s = 0$). As for the initial conditions, we impose that

$$(u, u_t)|_{t=0} = (u_0(s), v_0(s)) \qquad \text{in} \quad (0,L). \tag{1.4}$$

There are few results about the existence and the uniqueness of the solution of this initial-boundary value problem. Reeken [6, 7] considered the motion of an inextensible string of *infinite* length having one end fixed at the point $(0,0,+\infty)$ in a gravity field. For technical reasons he assumed that the acceleration of gravity vector g is not a constant. In precise, he assumed that $g = g(s) \in C^\infty$ is constant for $s \in [0,l]$ and grows linearly beyond $s = l$ for some l. Under this non-physical condition, he proved the existence locally in time and the uniqueness of the solution provided the initial data are sufficiently close to a trivial stationary solution. Preston [3] considered the motion of an inextensible string of finite length in *the absence of gravity*, that is $g = 0$. He proved the existence locally in time and the uniqueness of the solution for arbitrary initial data.

Remark. The stationary solution $(\bar{u}, \bar{\tau})$ of the boundary value problem (1.1)–(1.3) is given by

$$\bar{u}(s) = (0,0,s-L), \quad \bar{\tau}(s) = \rho s.$$

This implies that the stationary tension $\bar{\tau}(s)$ is positive except at the free end ($s = 0$) and degenerates linearly at this end.

In what follows we assume, for simplicity, that $\rho = 1$ and $L = 1$ and set $I = (0,1)$. In this paper, assuming that the function $\tau(s,t)$ is given, we discuss on the solution u of the initial-boundary value problem (1.1), (1.3), and (1.4) neglecting (1.2). More precisely, for given functions $\tau(s,t)$ and $f(s,t)$, we consider the following initial-boundary value problem:

$$u_{tt} - (\tau(s,t)u_s)_s = f(s,t) \qquad \text{in} \quad I\times(0,T), \tag{1.5}$$

$$u|_{s=1} = 0 \qquad \text{on} \quad (0,T), \tag{1.6}$$

$$(u, u_t)|_{t=0} = (u_0(s), v_0(s)) \qquad \text{in} \quad I, \tag{1.7}$$

where $u(s,t)$ is a scalar unknown function. We note that the relationship between the position vector u and the tension τ can be determined by the inextensibility constraint (1.2). If $\tau(s,t)$ is strictly positive, then (1.5) is a wave equation with a non-degenerate coefficient, and hence the existence and the uniqueness of the solution of the initial-boundary value problem (1.5)–(1.7) are well-known. However, from the boundary condition (1.3) it is natural to assume that $\tau(s,t)$ degenerates at $s=0$.

Koshlyakov, Gliner, and Smilnov [1] considered the case $\tau(s,t)=s$, which is the stationary tension of the boundary value problem (1.1)–(1.3). By using eigenfunction expansion method they proved the existence of the solution of the initial-boundary value problem (1.5)–(1.7) with $f\equiv 0$. Yamaguchi [8] also considered the case $\tau(s,t)=s$. However, instead of a linear equation (1.5), he considered a semi-linear equation

$$u_{tt}-(\tau(s,t)u_s)_s=f(s,u) \quad \text{in} \quad I\times(0,T). \tag{1.8}$$

Under some technical conditions of $f(s,u)$ he proved the existence of a time global solution of the initial-boundary value problem (1.8), (1.6), and (1.7) provided the initial data are sufficiently small.

Remark. If (u,τ) is a smooth solution of the initial-boundary value problem (1.1)–(1.4), then the tension τ satisfies

$$\tau(0,t)=0, \quad \tau_s(0,t)>0, \quad \text{and} \quad \tau(s,t)>0 \quad \text{for} \quad (s,t)\in(0,L]\times[0,T]$$

under a physically natural condition. In other words, the tension $\tau(s,t)$ is positive except at the free end ($s=0$) and degenerates linearly at this end.

Taking into account of Remark above, for the given function $\tau(s,t)$ we assume that

$$\tau(s,t)=sa(s,t), \quad \text{where } a\in C^\infty(\bar I\times[0,T]) \text{ is strictly positive on } \bar I\times[0,T]. \tag{1.9}$$

In this paper, we shall show the existence and the uniqueness of the solution of the initial-boundary value problem (1.5)–(1.7) and study the regularity of the solution under this assumption (1.9).

This paper is organized as follows. In Section 2 we define function spaces $X^m(\mathbf{R}_+)$ which are essentially the same as in [6]. We also state our main theorem. In Section 3 we introduce maps $\sharp$ and $\flat$ which play crucial roles to prove main theorem. We also give the proof of main theorem admitting that some propositions hold. Section 4 is devoted to functions with weights. In Section 4.1 we prove some inequalities. In Section 4.2 we prove a function in $X^m(\mathbf{R}_+)$ can be approximated bysmooth functions. In Section 5 we give the proofs of propositions used in Section 3 admitting that Proposition 5.1 holds. In Sections 6 we give the proof of Proposition 5.1.

2 Notation and main theorem

Throughout this paper, for two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ we will write $\|v\|_1\lesssim\|v\|_2$ to denote the fact that $\|v\|_1\le C\|v\|_2$ for a certain constant C which is independent of v. We will also write $\|v\|_1 \simeq \|v\|_2$ to denote the fact that $\|v\|_1 \lesssim \|v\|_2$ and $\|v\|_2 \lesssim \|v\|_1$. In other words, $\|v\|_1 \simeq \|v\|_2$ means that the two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Let $\mathbf{Z}_+$ denote the set of non-negative integers and $\mathbf{R}_+$ denote the set of positive real numbers, namely, $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$ and $\mathbf{R}_+ = (0, +\infty)$. We use the following notation for the function spaces: For $m \in \mathbf{Z}_+$ we define

$$X^m(\mathbf{R}_+) = \{u \in L^2(\mathbf{R}_+);\ \|u\|_{X^m(\mathbf{R}_+)} < +\infty\} \tag{2.1}$$

equipped with the norm

$$\|u\|_{X^m(\mathbf{R}_+)} = \begin{cases} \displaystyle\sum_{j=0}^{k} \|\partial_s^j u\|_{L^2(\mathbf{R}_+)} + \sum_{j=0}^{k} \|s^j \partial_s^{k+j} u\|_{L^2(\mathbf{R}_+)}, & m = 2k,\ k \in \mathbf{Z}_+, \\ \displaystyle\sum_{j=0}^{k} \|\partial_s^j u\|_{L^2(\mathbf{R}_+)} + \sum_{j=0}^{k} \|s^{j+\frac{1}{2}} \partial_s^{k+1+j} u\|_{L^2(\mathbf{R}_+)}, & m = 2k+1,\ k \in \mathbf{Z}_+, \end{cases} \tag{2.2}$$

where $\partial_s u = \frac{d}{ds}u(s)$. Similarly, we define the function space $X^m(I)$ and the norm $\|\cdot\|_{X^m(I)}$ replaced $\mathbf{R}_+$ by I in the definitions (2.1) and (2.2) above (these function spaces $X^m(I)$ are essentially the same as those in Section 2 in [6]). We note that a function $u \in X^m(I)$ is expressed as $u = \tilde{u}|_I$ for some $\tilde{u} \in X^m(\mathbf{R}_+)$. Furthermore, we define

$$\Lambda_X^m = \bigcap_{j=0}^{m} C^j([0, T]; X^{m-j}(I))$$

equipped with the norm

$$\|u\|_{\Lambda_X^m} = \sum_{j=0}^{m} \sup_{t \in [0,T]} \|\partial_t^j u(\cdot, t)\|_{X^{m-j}(I)}.$$

Remark. Let $m \geq 2$ be an integer and suppose that

$$f \in \Lambda_X^{m-1} \quad \text{and} \quad (u_0, v_0) \in X^m(I) \times X^{m-1}(I).$$

If u is a solution of the initial-boundary value problem (1.5)–(1.7) in the class $u \in \Lambda_X^m$, then we have $\partial_t^j u(0, s) = U^{(j)}(s)$ $(j = 0, 1, \ldots, m)$, where $U^{(j)}(s)$ is determined from the initial data (u_0, v_0) inductively by

$$U^{(0)}(s) = u_0(s), \qquad U^{(1)}(s) = v_0(s),$$

and

$$U^{(j)}(s) = \sum_{i=0}^{j-2} \binom{j-2}{i} \partial_s\big(\partial_t^{j-2-i}\tau(0, s)\partial_s U^{(i)}(s)\big) + \partial_t^{j-2} f(0, s), \qquad j = 2, 3, \ldots, m.$$

Since $U^{(j)} \in H^{m-j}(\delta, 1)$ $(j = 0, 1, \ldots, m-1)$ for any $\delta \in (0, 1)$, we can define the trace of the function $U^{(j)}$ $(j = 0, 1, \ldots, m-1)$ at $s = 1$. Therefore, the compatibility conditions which are necessary to insure that the solution u is in Λ_X^m are given by

$$U^{(j)}|_{s=1} = 0, \qquad j = 0, 1, \ldots, m-1. \tag{2.3}$$

Our main theorem is as follows:

Theorem 2.1 *Let $m \geq 2$ be an integer and $T > 0$. Suppose that*

$$f \in \Lambda_X^{m-1} \quad \text{and} \quad (u_0, v_0) \in X^m(I) \times X^{m-1}(I)$$

satisfy the compatibility conditions (2.3). *Then the initial-boundary value problem* (1.5)–(1.7) *has a unique solution $u \in \Lambda_X^m$, which satisfies the estimate*

$$\|u\|_{\Lambda_X^m} \lesssim \|u_0\|_{X^m(I)} + \|v_0\|_{X^{m-1}(I)} + \|f\|_{\Lambda_X^{m-1}}. \tag{2.4}$$

3 Proof of the main theorem

We use the following notation for the function spaces: For $m \in \mathbf{Z}_+$ we define

$$H^m_{\mathrm{rad}}(\mathbf{R}^2) = \{w \in H^m(\mathbf{R}^2);\ w \text{ is a radially symmetric function}\} \tag{3.1}$$

and denote $H^0_{\mathrm{rad}}(\mathbf{R}^2)$ by $L^2_{\mathrm{rad}}(\mathbf{R}^2)$. Similarly, we set $\Omega = \{(x, y) \in \mathbf{R}^2;\ x^2 + y^2 < 1\}$ and define the function space $H^m_{\mathrm{rad}}(\Omega)$ replaced $\mathbf{R}^2$ by Ω in the definition (3.1) above. We note that a function $w \in H^m_{\mathrm{rad}}(\Omega)$ is expressed as $w = \tilde{w}|_\Omega$ for some $\tilde{w} \in H^m_{\mathrm{rad}}(\mathbf{R}^2)$.

For the proof of Theorem 2.1, we introduce the following notation: For $u \in L^2(\mathbf{R}_+)$ we define the function $u^\sharp : \mathbf{R}^2 \to \mathbf{R}$ as

$$u^\sharp(x, y) = u(x^2 + y^2), \qquad (x, y) \in \mathbf{R}^2.$$

Then it is easily checked that the map $\sharp :\ L^2(\mathbf{R}_+) \ni u \mapsto u^\sharp \in L^2_{\mathrm{rad}}(\mathbf{R}^2)$ is bijective and norm-preserving in the sense that

$$\|u^\sharp\|_{L^2(\mathbf{R}^2)} = \sqrt{\pi}\|u\|_{L^2(\mathbf{R}_+)} \quad \text{for} \quad u \in L^2(\mathbf{R}_+).$$

This map $\sharp$ gives the following relationship between $X^m(\mathbf{R}_+)$ and $H^m_{\mathrm{rad}}(\mathbf{R}^2)$ and between $X^m(I)$ and $H^m_{\mathrm{rad}}(\Omega)$.

Proposition 3.1 *Let $m \in \mathbf{Z}_+$. Then the map $\sharp :\ X^m(\mathbf{R}_+) \ni u \mapsto u^\sharp \in H^m_{\mathrm{rad}}(\mathbf{R}^2)$ is bijective and for $u \in X^m(\mathbf{R}_+)$ it holds that*

$$\|u^\sharp\|_{H^m(\mathbf{R}^2)} \simeq \|u\|_{X^m(\mathbf{R}_+)}.$$

Proposition 3.2 *Let $m \in \mathbf{Z}_+$. Then the map $\sharp :\ X^m(I) \ni u \mapsto u^\sharp \in H^m_{\mathrm{rad}}(\Omega)$ is bijective and for $u \in X^m(I)$ it holds that*

$$\|u^\sharp\|_{H^m(\Omega)} \simeq \|u\|_{X^m(I)}.$$

The proofs of Propositions 3.1 and 3.2 are given in Section 5.

For convenience, we also construct the inverse map of $\sharp:\ L^2(\mathbf{R}_+)\to L^2_{\rm rad}(\mathbf{R}^2)$. For given $w\in L^2_{\rm rad}(\mathbf{R}^2)$, since w is radially symmetric, there exists a function $W:\ \mathbf{R}_+\to\mathbf{R}$ such that $w(x,y)=W(r)$, where $r=\sqrt{x^2+y^2}$. Using this function W, we define the function $w^\flat:\ \mathbf{R}_+\to\mathbf{R}$ as

$$w^\flat(s)=W(s^{\frac{1}{2}}),\qquad s\in\mathbf{R}_+.$$

Then it is easily checked that the map $\flat:\ L^2_{\rm rad}(\mathbf{R}^2)\ni w\mapsto w^\flat\in L^2(\mathbf{R}_+)$ is also bijective and norm-preserving in the sense that

$$\|w^\flat\|_{L^2(\mathbf{R}_+)}=\frac{1}{\sqrt{\pi}}\|w\|_{L^2(\mathbf{R}^2)}\quad\text{for}\quad w\in\ L^2_{\rm rad}(\mathbf{R}^2).$$

This map $\flat: L^2_{\rm rad}(\mathbf{R}^2)\to L^2(\mathbf{R}_+)$ is the inverse map of $\sharp: L^2(\mathbf{R}_+)\to L^2_{\rm rad}(\mathbf{R}^2)$. Therefore, from Propositions 3.1 and 3.2 we immediately have

Corollary 3.3 *Let $m\in\mathbf{Z}_+$. Then the map $\flat:\ H^m_{\rm rad}(\mathbf{R}^2)\ni w\mapsto w^\flat\in X^m(\mathbf{R}_+)$ is bijective and for $w\in H^m_{\rm rad}(\mathbf{R}^2)$ it holds that*

$$\|w^\flat\|_{X^m(\mathbf{R}_+)}\simeq\|w\|_{H^m(\mathbf{R}^2)}.$$

Corollary 3.4 *Let $m\in\mathbf{Z}_+$. Then the map $\flat:\ H^m_{\rm rad}(\Omega)\ni w\mapsto w^\flat\in X^m(I)$ is bijective and for $w\in H^m_{\rm rad}(\Omega)$ it holds that*

$$\|w^\flat\|_{X^m(I)}\simeq\|w\|_{H^m(\Omega)}.$$

We also introduce the following differential operators: For $m\in\mathbf{Z}_+$ we define A_m as

$$A_0u=u,\qquad A_1u=s^{\frac{1}{2}}\partial_su,\qquad A_2u=\partial_s(s\partial_su),$$

and

$$A_mu=\begin{cases}A_2^ku, & m=2k,\ k\in\mathbf{Z}_+,\\ A_1A_2^ku, & m=2k+1,\ k\in\mathbf{Z}_+\end{cases}$$

(these operators A_m are essentially the same as those in Section 2 in [6]). Then we have

Proposition 3.5 *Let $w\in X^2(\mathbf{R}_+)$ and $u,v\in X^1(\mathbf{R}_+)$. Then it holds that*

$$(A_2w)^\sharp=\frac{1}{4}\Delta w^\sharp,\qquad (A_1u)^\sharp(A_1v)^\sharp=\frac{1}{4}(\nabla u^\sharp\cdot\nabla v^\sharp),\tag{3.2}$$

where Δ and ∇ are the Laplacian and the gradient in $\mathbf{R}^2$, respectively.

Proposition 3.6 *Let $w\in H^2_{\rm rad}(\mathbf{R}^2)$ and $u,v\in H^1_{\rm rad}(\mathbf{R}^2)$. Then it holds that*

$$A_2w^\flat=\frac{1}{4}(\Delta w)^\flat,\qquad (A_1u^\flat)(A_1v^\flat)=\frac{1}{4}(\nabla u\cdot\nabla v)^\flat.\tag{3.3}$$

The proofs of Propositions 3.5 and 3.6 are given in Section 5. In what follows, admitting that Propositions 3.2, 3.5, and 3.6 hold, we shall give the proof of Theorem 2.1.

3.1 Uniqueness of the solution

Let $u \in \Lambda_X^m$ be a solution of the initial-boundary value problem (1.5)–(1.7) and set $w = u^\sharp$. Then Proposition 3.2 yields

$$w \in \Lambda_{H,\,\mathrm{rad}}^m := \bigcap_{j=0}^{m} C^j([0,T]; H_{\mathrm{rad}}^{m-j}(\Omega)).$$

Furthermore, from the assumption (1.9) we have

$$(\tau u_s)_s = (sau_s)_s = a(su_s)_s + sa_s u_s = aA_2 u + (A_1 a)(A_1 u).$$

Thus Proposition 3.5 shows that w is a solution of the following initial-boundary value problem:

$$w_{tt} - \frac{1}{4}(a^\sharp \Delta w + \nabla a^\sharp \cdot \nabla w) = f^\sharp \qquad \text{in} \quad \Omega \times (0,T), \tag{3.4}$$

$$w = 0 \qquad \text{on} \quad \partial\Omega \times (0,T), \tag{3.5}$$

$$(w, w_t)|_{t=0} = (u_0^\sharp, v_0^\sharp) \qquad \text{in} \quad \Omega. \tag{3.6}$$

From the assumption (1.9), the function $a^\sharp(s,t)$ is strictly positive on $\overline{\Omega} \times [0,T]$, and hence (3.4) is a wave equation with a non-degenerate coefficient. Thus, by standard arguments of initial-boundary value problem for a linear hyperbolic equation (for instance, see Proposition 2.1 in [5]), the solution w of the initial-boundary value problem (3.4)–(3.6) is unique. Therefore the solution u of the initial-boundary value problem (1.5)–(1.7) is also unique. □

3.2 Existence and estimate of the solution

Let $m \geq 2$ be an integer and suppose that

$$f \in \Lambda_X^{m-1} \qquad \text{and} \qquad (u_0, v_0) \in X^m(I) \times X^{m-1}(I)$$

satisfy the compatibility conditions (2.3). We now consider the initial-boundary value problem (3.4)–(3.6). Proposition 3.2 yields

$$f^\sharp \in \Lambda_{H,\,\mathrm{rad}}^{m-1} \qquad \text{and} \qquad (u_0^\sharp, v_0^\sharp) \in H_{\mathrm{rad}}^m(\Omega) \times H_{\mathrm{rad}}^{m-1}(\Omega).$$

Moreover, $f^\sharp$ and $(u_0^\sharp, v_0^\sharp)$ also satisfy the compatibility conditions of the initial-boundary value problem (3.4)–(3.6). Thus, by standard arguments (for instance, see Theorem 3.1 in [5]), we obtain a unique solution

$$w \in \Lambda_H^m := \bigcap_{j=0}^{m} C^j([0,T]; H^{m-j}(\Omega))$$

of the initial-boundary value problem (3.4)–(3.6), which satisfies

$$\|w\|_{\Lambda_H^m} \lesssim \|u_0^\sharp\|_{H^m(\Omega)} + \|v_0^\sharp\|_{H^{m-1}(\Omega)} + \|f^\sharp\|_{\Lambda_H^{m-1}}. \tag{3.7}$$

Since the solution of the initial-boundary value problem (3.4)–(3.6) is unique and functions $a^\sharp, f^\sharp, u_0^\sharp, v_0^\sharp$ are radially symmetric, the solution w is also radially symmetric, namely, $w \in \Lambda^m_{H,\mathrm{rad}}$. Now let us set $u = w^\flat$. From Corollary 3.4 we have $u \in \Lambda^m_X$. Moreover, Proposition 3.6 shows that u is a solution of the initial-boundary value problem (1.5)–(1.7). The estimate (2.4) follows from Proposition 3.2 and (3.7). $\square$

4 Functions with weights

4.1 Auxiliary inequalities

We start with the following lemma.

Lemma 4.1 *Let $w \in L^2(0,1)$ and set*

$$W(s) = s^{-\frac{1}{2}} \int_0^s w(\sigma)\, d\sigma, \qquad s \in (0,1).$$

Then we have $W \in L^2(0,1)$ and it holds that

$$\|W\|_{L^2(0,1)} \le \frac{4}{e} \|w\|_{L^2(0,1)}.$$

Proof. For $0 < \delta < 1$ we set

$$E_\delta = \Big(\int_\delta^1 W(s)^2\, ds \Big)^{\frac{1}{2}}.$$

Using integration by parts we have

$$\begin{aligned}
E_\delta^2 &= \int_\delta^1 \frac{1}{s} \Big(\int_0^s w(\sigma)\, d\sigma \Big)^2 ds \\
&= (\log s) \Big(\int_0^s w(\sigma)\, d\sigma \Big)^2 \Big|_{s=\delta}^{s=1} - 2 \int_\delta^1 (\log s) \Big(\int_0^s w(\sigma)\, d\sigma \Big) w(s)\, ds \\
&= |\log \delta| \Big(\int_0^\delta w(\sigma)\, d\sigma \Big)^2 - 2 \int_\delta^1 s^{\frac{1}{2}} (\log s)\, W(s) w(s)\, ds \\
&\le |\log \delta| \Big(\int_0^\delta w(\sigma)\, d\sigma \Big)^2 + \frac{4}{e} \|w\|_{L^2(0,1)} E_\delta,
\end{aligned}$$

where we used $0 \le s^{\frac{1}{2}} |\log s| \le \frac{2}{e}$ for $s \in (0,1)$. In short, we obtain

$$E_\delta^2 \le A_\delta + B E_\delta, \quad \text{where} \quad A_\delta = |\log \delta| \Big(\int_0^\delta w(\sigma)\, d\sigma \Big)^2, \quad B = \frac{4}{e} \|w\|_{L^2(0,1)}. \tag{4.1}$$

Solving the quadratic inequality (4.1), we have

$$E_\delta \le \frac{1}{2} \big(B + \sqrt{B^2 + 4A_\delta}\, \big). \tag{4.2}$$

On the other hand, we note that

$$|A_\delta| \le |\log\delta|\Big(\int_0^\delta 1\,d\sigma\Big)\Big(\int_0^\delta w(\sigma)^2\,d\sigma\Big) = \delta|\log\delta|\Big(\int_0^\delta w(\sigma)^2\,d\sigma\Big) \to 0 \quad \text{as} \quad \delta\to +0.$$

Therefore, passing to the limit as $\delta \to +0$ in (4.2), we conclude the proof. $\square$

As an immediate corollary to Lemma 4.1 we have

Corollary 4.2 *Let $u \in L^2(\mathbf{R}_+)$ and suppose that $A_1u, A_2u \in L^2(\mathbf{R}_+)$. Then we have*

$$\partial_s u(s) = \frac{1}{s}\int_0^s w(\sigma)\,d\sigma, \quad \textit{where} \quad w = A_2u.$$

Proof. Since $\partial_s(s\partial_s u) = w$, there exists a constant $c \in \mathbf{R}$ such that

$$s\partial_s u(s) = c + \int_0^s w(\sigma)\,d\sigma.$$

This implies

$$cs^{-\frac{1}{2}} = A_1u - s^{-\frac{1}{2}}\int_0^s w(\sigma)\,d\sigma. \tag{4.3}$$

From Lemma 4.1 and the fact that $A_1u, w \in L^2(0,1)$, the right-hand side of (4.3) belongs to $L^2(0,1)$. On the other hand, we note that $s^{-\frac{1}{2}} \notin L^2(0,1)$. This implies $c = 0$, which concludes the proof. $\square$

Lemma 4.3 *Let $v \in L^2(0,1)$ and suppose that $\partial_s(sv) \in L^2(0,1)$. Then it holds that*

$$\sup_{s\in(0,1)} |s^{\frac{1}{2}}v(s)| \le \|\partial_s(sv)\|_{L^2(0,1)}.$$

Proof. Let us set $w = \partial_s(sv)$. Then, by using the same argument as in the proof of Corollary 4.2 we have

$$sv(s) = \int_0^s w(\sigma)\,d\sigma, \qquad s \in (0,1),$$

and hence

$$|sv(s)| = \Big|\int_0^s w(\sigma)\,d\sigma\Big| \le \Big(\int_0^s 1\,d\sigma\Big)^{\frac{1}{2}}\Big(\int_0^s w(\sigma)^2\,d\sigma\Big)^{\frac{1}{2}} \le s^{\frac{1}{2}}\|w\|_{L^2(0,1)}, \qquad s\in(0,1).$$

This concludes the proof. $\square$

Lemma 4.4 *Let $\alpha > -\frac{1}{2}$ and $v \in L^2(\mathbf{R}_+)$, and set*

$$V(s) = \frac{1}{s^{\alpha+1}}\int_0^s \sigma^\alpha v(\sigma)\,d\sigma, \qquad s \in \mathbf{R}_+.$$

Then we have $V \in L^2(\mathbf{R}_+)$ and it holds that

$$\|V\|_{L^2(\mathbf{R}_+)} \le \frac{2}{1+2\alpha}\|v\|_{L^2(\mathbf{R}_+)}.$$

By using an argument similar to that in the proof of Lemma 4.1, we can also prove Lemma 4.4 (see also Lemma 1 in [7]). Thus we omit the proof.

Lemma 4.5 *Let $\alpha, \beta \in \mathbf{R}$ and suppose that $\alpha > -\frac{1}{2}$ and $\alpha + \beta \geq -1$. In addition, let $w \in L^2(\mathbf{R}_+)$ and suppose that* $\operatorname{supp} w \subset [0, R]$ *for some $R > 0$. Moreover, set*

$$W(s) = s^\alpha \int_s^{+\infty} \sigma^\beta w(\sigma)\, d\sigma, \qquad s \in \mathbf{R}_+.$$

Then we have $W \in L^2(\mathbf{R}_+)$ and it holds that

$$\|W\|_{L^2(\mathbf{R}_+)} \leq \frac{2}{1+2\alpha} \|s^{\alpha+\beta+1} w\|_{L^2(\mathbf{R}_+)}.$$

Proof. For $0 < \delta < 1$ we set

$$E_\delta = \Big(\int_\delta^{+\infty} W(s)^2\, ds \Big)^{\frac{1}{2}}.$$

By using an argument similar to that in the proof of Lemma 4.1, we have

$$\begin{aligned} E_\delta^2 = {} & \frac{1}{1+2\alpha}\, s^{1+2\alpha} \Big(\int_s^{+\infty} \sigma^\beta w(\sigma)\, d\sigma \Big)^2 \Big|_{s=\delta}^{s=+\infty} \\ & + \frac{2}{1+2\alpha} \int_\delta^{+\infty} s^{1+2\alpha} \Big(\int_s^{+\infty} \sigma^\beta w(\sigma)\, d\sigma \Big) s^\beta u(s)\, ds. \end{aligned}$$

Since $\operatorname{supp} w \subset [0, R]$, we have $s^{1+2\alpha}(\int_s^{+\infty} \sigma^\beta w(\sigma)\, d\sigma)^2|_{s=+\infty} = 0$. Thus we obtain

$$\begin{aligned} E_\delta^2 & \leq \frac{2}{1+2\alpha} \int_\delta^{+\infty} s^{1+2\alpha} \Big(\int_s^{+\infty} \sigma^\beta w(\sigma)\, d\sigma \Big) s^\beta u(s)\, ds \\ & = \frac{2}{1+2\alpha} \int_\delta^{+\infty} W(s) s^{\alpha+\beta+1} u(s)\, ds \leq \frac{2}{1+2\alpha} \|s^{\alpha+\beta+1} u\|_{L^2(\mathbf{R}_+)} E_\delta. \end{aligned}$$

This implies

$$E_\delta \leq \frac{2}{1+2\alpha} \|s^{\alpha+\beta+1} u\|_{L^2(\mathbf{R}_+)}. \tag{4.4}$$

Therefore, passing to the limit as $\delta \to +0$ in (4.4), we conclude the proof. □

4.2 Approximating functions

We shall show that a function in $X^m(\mathbf{R}_+)$ can be approximated by smooth functions. Let $\overline{\mathbf{R}_+}$ denote the set of non-negative real numbers, namely, $\overline{\mathbf{R}_+} = [0, +\infty)$. We define

$$C_0^\infty(\overline{\mathbf{R}_+}) = \{u :\ \overline{\mathbf{R}_+} \to \mathbf{R};\ u = \tilde{u}|_{\overline{\mathbf{R}_+}} \text{ for some } \tilde{u} \in C_0^\infty(\mathbf{R})\}.$$

Then we have

Lemma 4.6 *Let $m \in \mathbf{Z}_+$. Then $C_0^\infty(\overline{\mathbf{R}_+})$ is dense in $X^m(\mathbf{R}_+)$.*

Proof. We divide the proof into three steps.

FIRST STEP: Suppose that $u \in X^m(\mathbf{R}_+)$. Let us take $\rho \in C_0^\infty(\mathbf{R})$ with $\rho(s) \geq 0$, $\int_{\mathbf{R}} \rho(s)ds = 1$ and set

$$u_\epsilon(s) = J_\epsilon u(s) \equiv \int_{\mathbf{R}} u(se^{-\epsilon\sigma})\rho(\sigma)\, d\sigma, \qquad s \in \mathbf{R}_+$$

for $0 < \epsilon \leq 1$ (this operator J_ϵ is the same as that in Section 2 in [4]). We note the following:

Lemma 4.7 *Let $\alpha \in \mathbf{R}$ and take $\rho \in C_0^\infty(\mathbf{R})$ with $\rho(s) \geq 0$, $\int_{\mathbf{R}} \rho(s)ds = 1$. In addition, let $w \in L^2(\mathbf{R}_+)$ and set*

$$W_\epsilon(s) = \int_{\mathbf{R}} w(se^{-\epsilon\sigma})e^{\epsilon\alpha\sigma}\rho(\sigma)\, d\sigma, \qquad s \in \mathbf{R}_+$$

for $0 < \epsilon \leq 1$. Then we have $W_\epsilon \in L^2(\mathbf{R}_+)$ for $0 < \epsilon \leq 1$ and it holds that

$$W_\epsilon \to w \quad in \quad L^2(\mathbf{R}_+) \quad as \quad \epsilon \to +0.$$

By using arguments similar to those in Section 2 in [4] and in Section 3 in [2] we can prove Lemma 4.7. Thus we omit the proof. We continue the proof of Lemma 4.6. From Lemma 4.7 it is easily checked that $u_\epsilon \to u$ in $X^m(\mathbf{R}_+)$ as $\epsilon \to +0$. Moreover, we have $u_\epsilon \in X^m(\mathbf{R}_+) \cap C^\infty(\mathbf{R}_+)$ for $0 < \epsilon \leq 1$ (see Section 2 in [4], Section 3 in [2]). Therefore, replacing u by u_ϵ, we may suppose that $u \in X^m(\mathbf{R}_+) \cap C^\infty(\mathbf{R}_+)$ without loss of generality.

SECOND STEP: Suppose that $u \in X^m(\mathbf{R}_+) \cap C^\infty(\mathbf{R}_+)$. Let us take $\psi \in C_0^\infty(\mathbf{R})$ with $\operatorname{supp}\psi \subset (-1, 1)$ such that $\psi(s) \equiv 1$ near $s = 0$ and set

$$u_\epsilon(s) = \psi(\epsilon s)u(s), \qquad s \in \mathbf{R}_+$$

for $0 < \epsilon \leq 1$. Then it is easily checked that $u_\epsilon \to u$ in $X^m(\mathbf{R}_+)$ as $\epsilon \to +0$. Moreover, we have $u_\epsilon \in X^m(\mathbf{R}_+) \cap C^\infty(\mathbf{R}_+)$ with $\operatorname{supp} u_\epsilon \subset [0, \frac{1}{\epsilon}]$ for $0 < \epsilon \leq 1$. Therefore, replacing u by u_ϵ, we may suppose that $u \in X^m(\mathbf{R}_+) \cap C^\infty(\mathbf{R}_+)$ with $\operatorname{supp} u \subset [0, R]$ for some $R > 0$ without loss of generality.

THIRD STEP: Suppose that $u \in X^m(\mathbf{R}_+) \cap C^\infty(\mathbf{R}_+)$ with $\operatorname{supp} u \subset [0, R]$ for some $R > 0$. Let us take $\chi \in C_0^\infty(\mathbf{R})$ with $\operatorname{supp}\chi \subset (-1, 1)$ such that $\chi(s) \equiv 1$ near $s = 0$ and set $\chi_\epsilon(s) = \chi(\frac{s}{\epsilon})$ for $0 < \epsilon \leq 1$. Moreover, we set

$$u_\epsilon(s) = \frac{1}{(m-1)!} \int_s^{+\infty} (s - \sigma)^{m-1}(\chi_\epsilon(\sigma) - 1)(\partial_s^m u)(\sigma)\, d\sigma, \qquad s \in \mathbf{R}_+$$

for $0 < \epsilon \leq 1$. Since $(\chi_\epsilon - 1)\partial_s^m u \in C_0^\infty(\mathbf{R}_+)$, we have $u_\epsilon \in C_0^\infty(\overline{\mathbf{R}_+})$ for $0 < \epsilon \leq 1$. In what follows, we shall show that $u_\epsilon \to u$ in $X^m(\mathbf{R}_+)$ as $\epsilon \to +0$.

Repeating integration by parts we have

$$u(s) = -\frac{1}{(m-1)!}\int_s^{+\infty} (s-\sigma)^{m-1}(\partial_s^m u)(\sigma)\,d\sigma,$$

and hence u_ϵ is written as $u_\epsilon = u + v_\epsilon$, where

$$v_\epsilon(s) = \frac{1}{(m-1)!}\int_s^{+\infty} (s-\sigma)^{m-1}\chi_\epsilon(\sigma)(\partial_s^m u)(\sigma)\,d\sigma.$$

Thus, in order to prove that $u_\epsilon \to u$ in $X^m(\mathbf{R}_+)$ as $\epsilon \to +0$, it suffices to show that

$$v_\epsilon \to 0 \quad \text{in} \quad X^m(\mathbf{R}_+) \quad \text{as} \quad \epsilon \to +0. \tag{4.5}$$

We consider two cases depending on whether m is even or odd. Since the proofs are the same in the two cases, we only give the proof of (4.5) in the case where m is odd ($m = 2k+1$) and omit it in the case where m is even. In what follows, we shall show that

$$\|v_\epsilon\|_{X^{2k+1}(\mathbf{R}_+)} = \sum_{j=0}^{k} \|\partial_s^j v_\epsilon\|_{L^2(\mathbf{R}_+)} + \sum_{j=0}^{k} \|s^{j+\frac{1}{2}}\partial_s^{k+1+j} v_\epsilon\|_{L^2(\mathbf{R}_+)} \to 0 \quad \text{as} \quad \epsilon \to +0.$$

Let us set $w_\epsilon = \chi_\epsilon\, s^{k+\frac{1}{2}}\partial_s^{2k+1}u$. Since $s^{k+\frac{1}{2}}\partial_s^{2k+1}u \in L^2(\mathbf{R}_+)$, it is easily checked that $w_\epsilon \to 0$ in $L^2(\mathbf{R}_+)$ as $\epsilon \to +0$. We note that v_ϵ is written as

$$v_\epsilon(s) = \frac{1}{(2k)!}\int_s^{+\infty} (s-\sigma)^{2k}\sigma^{-k-\frac{1}{2}} w_\epsilon(\sigma)\,d\sigma. \tag{4.6}$$

Thus, for $j = 0, 1, \ldots, k$, taking the j-th order derivative of (4.6), we have

$$\partial_s^j v_\epsilon(s) = \frac{1}{(2k-j)!}\int_s^{+\infty} (s-\sigma)^{2k-j}\sigma^{-k-\frac{1}{2}} w_\epsilon(\sigma)\,d\sigma.$$

Therefore, using Lemma 4.5 and noting $\operatorname{supp} w_\epsilon \subset [0, R]$ we obtain

$$\|\partial_s^j v_\epsilon\|_{L^2(\mathbf{R}_+)} \lesssim \|s^{k+\frac{1}{2}-j} w_\epsilon\|_{L^2(\mathbf{R}_+)} \lesssim \|w_\epsilon\|_{L^2(\mathbf{R}_+)} \to 0 \quad \text{as} \quad \epsilon \to +0.$$

Similarly, for $j = 0, 1, \ldots, k$, taking the $(k+1+j)$-th order derivative of (4.6), we have

$$\begin{aligned} &s^{j+\frac{1}{2}}\partial_s^{k+1+j} v_\epsilon(s) \\ &= \begin{cases} \dfrac{s^{j+\frac{1}{2}}}{(k-j-1)!}\displaystyle\int_s^{+\infty} (s-\sigma)^{k-j-1}\sigma^{-k-\frac{1}{2}} w_\epsilon(\sigma)\,d\sigma, & j = 0, 1, \ldots, k-1, \\ -w_\epsilon(s), & j = k. \end{cases} \end{aligned}$$

Thus Lemma 4.5 yields

$$\|s^{j+\frac{1}{2}}\partial_s^{k+1+j} v_\epsilon\|_{L^2(\mathbf{R}_+)} \lesssim \|w_\epsilon\|_{L^2(\mathbf{R}_+)} \to 0 \quad \text{as} \quad \epsilon \to +0.$$

Therefore, in the case where m is odd, we proved (4.5). Similarly, in the case where m is even, we can also prove (4.5). $\square$

5 Proofs of propositions

In this section, we shall give the proofs of propositions used in Section 3. The proofs of Propositions 3.1 and 3.2 are essentially the same. However, the proof of Proposition 3.1 is a little easier than that of Proposition 3.2, since I and Ω are bounded sets. Therefore, we only give the proof of Proposition 3.1 and omit the proof of Proposition 3.2. In what follows, admitting that Proposition 5.1 below holds, we shall prove Propositions 3.1, 3.5, and 3.6. The proof of Proposition 5.1 is given in Section 6.

Proposition 5.1 *Let $m \in \mathbf{Z}_+$. Then for $u \in L^2(\mathbf{R}_+)$ the following three conditions* (A), (B), *and* (C) *are equivalent;*

(A) $u \in X^m(\mathbf{R}_+)$,

(B) $s^{\frac{i}{2}}\partial_s^j u \in L^2(\mathbf{R}_+)$ *for* $i = 0, 1, \ldots, m$ *and* $j = i, i+1, \ldots, [\frac{m+i}{2}]$,

(C) $A_j u \in L^2(\mathbf{R}_+)$ *for* $j = 0, 1, \ldots, m$,

where $[\alpha]$ denotes the largest integer not greater than α. Moreover, for $u \in X^m(\mathbf{R}_+)$ it holds that

$$\|u\|_{X^m(\mathbf{R}_+)} \simeq \sum_{i=0}^{m} \sum_{j=i}^{[\frac{m+i}{2}]} \|s^{\frac{i}{2}}\partial_s^j u\|_{L^2(\mathbf{R}_+)} \simeq \sum_{j=0}^{m} \|A_j u\|_{L^2(\mathbf{R}_+)}.$$

Before the proofs of Propositions 3.1, 3.5, and 3.6 we consider higher order derivatives of $u^\sharp$. Let $u \in C^\infty(\overline{\mathbf{R}_+})$ and $p, q \in \mathbf{Z}_+$. Then, by direct calculation we have

$$\partial_x^p \partial_y^q u^\sharp = \sum_{i=0}^{[\frac{p}{2}]} \sum_{j=0}^{[\frac{q}{2}]} \frac{p!\,q!}{(p-2i)!\,i!\,(q-2j)!\,j!} \Big(\frac{2x}{r}\Big)^{p-2i} \Big(\frac{2y}{r}\Big)^{q-2j} (s^{\frac{p}{2}+\frac{q}{2}-i-j}\partial_s^{p+q-i-j}u)^\sharp, \tag{5.1}$$

where $r = \sqrt{x^2+y^2}$.

Lemma 5.2 *Let $m \in \mathbf{Z}_+$. Then for $u \in X^m(\mathbf{R}_+)$ and for $p, q \in \mathbf{Z}_+$ with $p+q \le m$,* (5.1) *also holds.*

Remark. For a general function $u \in C^\infty(\mathbf{R}_+)$, (5.1) does not necessarily hold. Indeed, suppose that (5.1) holds for $u(s) = \log s$ and we shall derive a contradiction as follows. From (5.1) we have

$$\begin{aligned} \Delta u^\sharp &= \sum_{i=0}^{1} \frac{2!}{(2-2i)!\,i!} \Big\{ \Big(\frac{2x}{r}\Big)^{2-2i} + \Big(\frac{2y}{r}\Big)^{2-2i} \Big\} (s^{1-i}\partial_s^{2-i}u)^\sharp \\ &= 4(s\partial_s^2 u)^\sharp + 4(\partial_s u)^\sharp = 4(A_2 u)^\sharp, \end{aligned} \tag{5.2}$$

where Δ is the Laplacian in $\mathbf{R}^2$. As is well-known, since $u^\sharp(x,y) = \log(x^2+y^2)$, we have $\Delta u^\sharp = 4\pi\delta$, where δ is the Dirac delta function. However, by direct calculation, we have $(A_2 u)^\sharp = 0$. This is a contradiction. Therefore Lemma 5.2 is not obvious.

Proof of Lemma 5.2. Let $u \in X^m(\mathbf{R}_+)$ and w denote the right-hand side of (5.1). We shall show that $\partial_x^p \partial_y^q u^\sharp = w$. From Lemma 4.6 we can choose $\{u_n\} \subset C_0^\infty(\overline{\mathbf{R}_+})$ such that $u_n \to u$ in $X^m(\mathbf{R}_+)$ as $n \to \infty$. Similarly, let w_n denote the right-hand side of (5.1) replaced u by u_n. Since $u_n \in C_0^\infty(\overline{\mathbf{R}_+})$, it follows from (5.1) that $\partial_x^p \partial_y^q u_n^\sharp = w_n$. Thus for arbitrary $\varphi \in C_0^\infty(\mathbf{R}^2)$ it holds that

$$(-1)^{p+q}(u_n^\sharp, \partial_x^p \partial_y^q \varphi)_{L^2(\mathbf{R}^2)} = (w_n, \varphi)_{L^2(\mathbf{R}^2)}. \tag{5.3}$$

We now note the following:

Lemma 5.3 *Let $m \in \mathbf{Z}_+$. Then for $u \in X^m(\mathbf{R}_+)$ and for $p, q \in \mathbf{Z}_+$ with $p + q \le m$ it holds that*

$$\sum_{i=0}^{[\frac{p}{2}]} \sum_{j=0}^{[\frac{q}{2}]} \| s^{\frac{p}{2}+\frac{q}{2}-i-j} \partial_s^{p+q-i-j} u \|_{L^2(\mathbf{R}_+)} \lesssim \|u\|_{X^m(\mathbf{R}_+)}.$$

We omit the proof of Lemma 5.3 because it is easily derived from Proposition 5.1. We continue the proof of Lemma 5.2. Since

$$\|u_n^\sharp - u^\sharp\|_{L^2(\mathbf{R}^2)} = \sqrt{\pi} \|u_n - u\|_{L^2(\mathbf{R}_+)} \lesssim \|u_n - u\|_{X^m(\mathbf{R}_+)},$$

we have $u_n^\sharp \to u^\sharp$ in $L^2(\mathbf{R}^2)$ as $n \to \infty$. Similarly, from Lemma 5.3 we obtain

$$\begin{aligned}
\|w_n - w\|_{L^2(\mathbf{R}^2)} &\lesssim \sum_{i=0}^{[\frac{p}{2}]} \sum_{j=0}^{[\frac{q}{2}]} \| (s^{\frac{p}{2}+\frac{q}{2}-i-j} \partial_s^{p+q-i-j} (u_n - u))^\sharp \|_{L^2(\mathbf{R}^2)} \\
&\simeq \sum_{i=0}^{[\frac{p}{2}]} \sum_{j=0}^{[\frac{q}{2}]} \| s^{\frac{p}{2}+\frac{q}{2}-i-j} \partial_s^{p+q-i-j} (u_n - u) \|_{L^2(\mathbf{R}_+)} \lesssim \|u_n - u\|_{X^m(\mathbf{R}_+)},
\end{aligned}$$

and hence $w_n \to w$ in $L^2(\mathbf{R}^2)$ as $n \to \infty$. Thus, passing to the limit as $n \to \infty$ in (5.3), we obtain

$$(-1)^{p+q}(u^\sharp, \partial_x^p \partial_y^q \varphi)_{L^2(\mathbf{R}^2)} = (w, \varphi)_{L^2(\mathbf{R}^2)},$$

which shows that $\partial_x^p \partial_y^q u^\sharp = w$. □

We now give the proofs of Propositions 3.5, 3.6, and 3.1.

Proof of Proposition 3.5. From Lemma 5.2 and (5.2) we obtain the first part of (3.2). Similarly, using Lemma 5.2 we can also prove the second part. □

Proof of Proposition 3.6. We shall show the first part of (3.3). Since w is radially symmetric, there exists a function W such that $w(x, y) = W(r)$, where $r = \sqrt{x^2 + y^2}$. In polar coordinates, it holds that $\Delta f = \partial_r^2 f + \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial_\theta^2 f$. Thus we have $\Delta w = W''(r) + \frac{1}{r}W'(r)$, and hence $(\Delta w)^\flat = W''(s^{\frac{1}{2}}) + s^{-\frac{1}{2}} W'(s^{\frac{1}{2}})$. On the other hand, by direct calculation we have $A_2 w^\flat = A_2(W(s^{\frac{1}{2}})) = \frac{1}{4}W''(s^{\frac{1}{2}}) + \frac{1}{4}s^{-\frac{1}{2}}W'(s^{\frac{1}{2}})$. This proves the first part of (3.3). Similarly, we can also prove the second part. □

Proof of Proposition 3.1. First we shall show that

$$\|u^\sharp\|_{H^m(\mathbf{R}^2)} \lesssim \|u\|_{X^m(\mathbf{R}_+)} \quad \text{for} \quad u \in X^m(\mathbf{R}_+). \tag{5.4}$$

If $p, q \in \mathbf{Z}_+$ satisfy $p+q \le m$, then from Lemmas 5.2 and 5.3 we have

$$\|\partial_x^p \partial_y^q u^\sharp\|_{L^2(\mathbf{R}^2)} \lesssim \sum_{i=0}^{[\frac{p}{2}]} \sum_{j=0}^{[\frac{q}{2}]} \|s^{\frac{p}{2}+\frac{q}{2}-i-j} \partial_s^{p+q-i-j} u\|_{L^2(\mathbf{R}_+)} \lesssim \|u\|_{X^m(\mathbf{R}_+)}.$$

This shows (5.4). Next we shall show that

$$\|w^\flat\|_{X^m(\mathbf{R}_+)} \lesssim \|w\|_{H^m(\mathbf{R}^2)} \quad \text{for} \quad w \in H^m_{\mathrm{rad}}(\mathbf{R}^2). \tag{5.5}$$

We note that $\|A_j w^\flat\|_{L^2(\mathbf{R}_+)} \lesssim \|w\|_{H^j(\mathbf{R}^2)}$ for $j = 0, 1, \dots, m$. Indeed, in the case where j is even ($j = 2k$), Proposition 3.6 yields

$$\|A_{2k} w^\flat\|_{L^2(\mathbf{R}_+)} = \frac{1}{4^k} \|(\Delta^k w)^\flat\|_{L^2(\mathbf{R}_+)} = \frac{1}{4^k \sqrt{\pi}} \|\Delta^k w\|_{L^2(\mathbf{R}^2)} \lesssim \|w\|_{H^{2k}(\mathbf{R}^2)}.$$

Similarly, in the case where j is odd ($j = 2k+1$), we have

$$\|A_{2k+1} w^\flat\|_{L^2(\mathbf{R}_+)} = \frac{1}{2^{2k+1}\sqrt{\pi}} \|\nabla(\Delta^k w)\|_{L^2(\mathbf{R}^2)} \lesssim \|w\|_{H^{2k+1}(\mathbf{R}^2)}.$$

Therefore Proposition 5.1 yields

$$\|w^\flat\|_{X^m(\mathbf{R}_+)} \simeq \sum_{j=0}^{m} \|A_j w^\flat\|_{L^2(\mathbf{R}_+)} \lesssim \sum_{j=0}^{m} \|w\|_{H^j(\mathbf{R}^2)} \lesssim \|w\|_{H^m(\mathbf{R}^2)}.$$

Since the map $\flat$ is the inverse map of $\sharp$, we complete the proof. □

6 Equivalent expressions

Proposition 5.1 follows directly from Propositions 6.1, 6.2, and 6.3 below, which shall be proved in this section.

Proposition 6.1 *Let $m \in \mathbf{Z}_+$ and $u \in X^m(\mathbf{R}_+)$. Then it holds that*

$$\sum_{i=0}^{m} \sum_{j=i}^{[\frac{m+i}{2}]} \|s^{\frac{i}{2}} \partial_s^j u\|_{L^2(\mathbf{R}_+)} \lesssim \|u\|_{X^m(\mathbf{R}_+)}.$$

Proposition 6.2 *Let $m \in \mathbf{Z}_+$ and $u \in L^2(\mathbf{R}_+)$ and suppose that $s^{\frac{i}{2}} \partial_s^j u \in L^2(\mathbf{R}_+)$ for $i = 0, 1, \dots, m$ and $j = i, i+1, \dots, [\frac{m+i}{2}]$. Then it holds that*

$$\sum_{j=0}^{m} \|A_j u\|_{L^2(\mathbf{R}_+)} \lesssim \sum_{i=0}^{m} \sum_{j=i}^{[\frac{m+i}{2}]} \|s^{\frac{i}{2}} \partial_s^j u\|_{L^2(\mathbf{R}_+)}.$$

Proposition 6.3 *Let $m \in \mathbf{Z}_+$ and $u \in L^2(\mathbf{R}_+)$ and suppose that $A_j u \in L^2(\mathbf{R}_+)$ for $j = 0, 1, \dots, m$. Then it holds that*

$$\|u\|_{X^m(\mathbf{R}_+)} \lesssim \sum_{j=0}^{m} \|A_j u\|_{L^2(\mathbf{R}_+)}.$$

6.1 Proof of Proposition 6.1

Proposition 6.1 follows from Lemmas 6.4 and 6.5 below.

Lemma 6.4 *Let $k \in \mathbf{Z}_+$ and $u \in X^{2k}(\mathbf{R}_+)$. Then for $l = 0, 1, \ldots, k$ it holds that*

$$\sum_{i=0}^{2l} \|s^{\frac{i}{2}} \partial_s^{k-l+i} u\|_{L^2(\mathbf{R}_+)} \lesssim \|u\|_{X^{2k}(\mathbf{R}_+)}. \tag{6.1}$$

Lemma 6.5 *Let $k \in \mathbf{Z}_+$ and $u \in X^{2k+1}(\mathbf{R}_+)$. Then for $l = 0, 1, \ldots, k$ it holds that*

$$\sum_{i=0}^{2l+1} \|s^{\frac{i}{2}} \partial_s^{k-l+i} u\|_{L^2(\mathbf{R}_+)} \lesssim \|u\|_{X^{2k+1}(\mathbf{R}_+)}.$$

Admitting for the moment that Lemmas 6.4 and 6.5 hold, we shall give the proof of Proposition 6.1.

Proof of Proposition 6.1. By rearranging the terms of the sum we note that

$$\sum_{i=0}^{m} \sum_{j=i}^{[\frac{m+i}{2}]} \|s^{\frac{i}{2}} \partial_s^{j} u\|_{L^2(\mathbf{R}_+)} = \begin{cases} \displaystyle\sum_{l=0}^{k} \sum_{i=0}^{2l} \|s^{\frac{i}{2}} \partial_s^{k-l+i} u\|_{L^2(\mathbf{R}_+)}, & m = 2k,\ k \in \mathbf{Z}_+, \\ \displaystyle\sum_{l=0}^{k} \sum_{i=0}^{2l+1} \|s^{\frac{i}{2}} \partial_s^{k-l+i} u\|_{L^2(\mathbf{R}_+)}, & m = 2k+1,\ k \in \mathbf{Z}_+. \end{cases}$$

Thus, from Lemmas 6.4 and 6.5 we conclude the proof of Proposition 6.1. □

Since the proofs of Lemmas 6.4 and 6.5 are the same, we only give the proof of Lemma 6.4 and omit the proof of Lemma 6.5.

Proof of Lemma 6.4. From Lemma 4.6 we may suppose that $u \in C_0^\infty(\overline{\mathbf{R}_+})$ without loss of generality. In this proof, for simplicity, we denote $L^2(\mathbf{R}_+)$ and $X^m(\mathbf{R}_+)$ by L^2 and X^m, respectively. We will use the same letter C to denote an inessential positive constant, which may vary from line to line. We prove the inequality (6.1) by induction on l. From the definition (2.2) the inequality (6.1) obviously holds for the case $l = 0$. Inductively assume that the inequality (6.1) holds for the case l. We consider the case $l + 1$. Let us set

$$x_i = \|s^{\frac{i}{2}} \partial_s^{k-(l+1)+i} u\|_{L^2}, \qquad i = 0, 1, \ldots, 2l + 2.$$

From the definition (2.2) we note that

$$x_0 = \|\partial_s^{k-l-1} u\|_{L^2} \lesssim \|u\|_{X^{2k}}, \qquad x_{2l+2} = \|s^{l+1} \partial_s^{k+l+1} u\|_{L^2} \lesssim \|u\|_{X^{2k}}.$$

Therefore, in order to prove the inequality (6.1) for the case $l + 1$, it suffices to show that

$$y := \sum_{i=1}^{2l+1} a_i x_i^2 \lesssim \|u\|_{X^{2k}}^2 \qquad \text{where} \qquad a_i = 2^{i^2}.$$

For $i = 1, 2, \dots, 2l+1$, using integration by parts we have

$$
\begin{aligned}
x_i^2 &= (\partial_s^{k-l-1+i}u, s^i\partial_s^{k-l-1+i}u)_{L^2} = -(\partial_s^{k-l-2+i}u, \partial_s(s^i\partial_s^{k-l-1+i}u))_{L^2} \\
&= -i\,(\partial_s^{k-l-2+i}u, s^{i-1}\partial_s^{k-l-1+i}u)_{L^2} - (\partial_s^{k-l-2+i}u, s^i\partial_s^{k-l+i}u)_{L^2} \\
&\le i\,\|s^{\frac{i-1}{2}}\partial_s^{k-l-2+i}u\|_{L^2}\|s^{\frac{i-1}{2}}\partial_s^{k-l-1+i}u\|_{L^2} + \|s^{\frac{i-1}{2}}\partial_s^{k-l-2+i}u\|_{L^2}\|s^{\frac{i+1}{2}}\partial_s^{k-l+i}u\|_{L^2} \\
&= i\,x_{i-1}\|s^{\frac{i-1}{2}}\partial_s^{k-l-1+i}u\|_{L^2} + x_{i-1}x_{i+1} \\
&\le \frac{a_{i-1}}{8a_i}x_{i-1}^2 + \frac{2a_i\,i^2}{a_{i-1}}\|s^{\frac{i-1}{2}}\partial_s^{k-l-1+i}u\|_{L^2}^2 + \frac{a_{i-1}}{2a_i}x_{i-1}^2 + \frac{a_i}{2a_{i-1}}x_{i+1}^2 \\
&= \frac{5a_{i-1}}{8a_i}x_{i-1}^2 + \frac{a_i}{2a_{i-1}}x_{i+1}^2 + \frac{2a_i\,i^2}{a_{i-1}}\|s^{\frac{i-1}{2}}\partial_s^{k-l+i-1}u\|_{L^2}^2.
\end{aligned}
$$

On the other hand, the inductive hypothesis implies

$$
\|s^{\frac{i-1}{2}}\partial_s^{k-l+i-1}u\|_{L^2} \le \sum_{i=0}^{2l}\|s^{\frac{i}{2}}\partial_s^{k-l+i}u\|_{L^2} \lesssim \|u\|_{X^{2k}},
$$

and hence

$$
x_i^2 \le \frac{5a_{i-1}}{8a_i}x_{i-1}^2 + \frac{a_i}{2a_{i-1}}x_{i+1}^2 + C\|u\|_{X^{2k}}^2.
$$

Moreover, since $2a_i^2 \le a_{i-1}a_{i+1}$, we have

$$
8a_ix_i^2 \le 5a_{i-1}x_{i-1}^2 + 2a_{i+1}x_{i+1}^2 + C\|u\|_{X^{2k}}^2. \tag{6.2}
$$

Taking the sum of the inequalities (6.2) over $i = 1, 2, \dots, 2l+1$, we obtain

$$
\begin{aligned}
8y = \sum_{i=1}^{2l+1} 8a_ix_i^2 &\le \sum_{i=1}^{2l+1}\big(5a_{i-1}x_{i-1}^2 + 2a_{i+1}x_{i+1}^2 + C\|u\|_{X^{2k}}^2\big) \\
&\le 5y + 5a_0x_0^2 + 2y + 2a_{2l+2}x_{2l+2}^2 + C\|u\|_{X^{2k}}^2,
\end{aligned}
$$

and hence

$$
y \le 5a_0x_0^2 + 2a_{2l+2}x_{2l+2}^2 + C\|u\|_{X^{2k}}^2 \lesssim \|u\|_{X^{2k}}^2,
$$

which concludes the proof. □

6.2 Proof of Proposition 6.2

For simplicity, we introduce the following notation: For $m \in \mathbf{Z}_+$ we define $|||\cdot|||_{X^m(\mathbf{R}_+)}$ as

$$
|||u|||_{X^m(\mathbf{R}_+)} = \sum_{i=0}^{m}\sum_{j=i}^{[\frac{m+i}{2}]}\|s^{\frac{i}{2}}\partial_s^j u\|_{L^2(\mathbf{R}_+)}.
$$

Then it is easily checked that $|||u|||_{X^m(\mathbf{R}_+)} \le |||u|||_{X^{m+1}(\mathbf{R}_+)}$. Therefore, in order to prove Proposition 6.2, it suffices to show the following lemma.

Lemma 6.6 *Let $k \in \mathbf{Z}_+$ and $u \in L^2(\mathbf{R}_+)$ and suppose that $|||u|||_{X^k(\mathbf{R}_+)} < +\infty$. Then it holds that*

$$\|A_k u\|_{L^2(\mathbf{R}_+)} \lesssim |||u|||_{X^k(\mathbf{R}_+)}. \tag{6.3}$$

Proof. We prove the inequality (6.3) by induction on k. Clearly we have

$$\|A_0 u\|_{L^2(\mathbf{R}_+)} = \|u\|_{L^2(\mathbf{R}_+)} = |||u|||_{X^0(\mathbf{R}_+)}, \quad \|A_1 u\|_{L^2(\mathbf{R}_+)} = \|s^{\frac{1}{2}}\partial_s u\|_{L^2(\mathbf{R}_+)} \le |||u|||_{X^1(\mathbf{R}_+)}.$$

Thus the inequality (6.3) for the case $k = 0, 1$ is trivial. Inductively assume that the inequality (6.3) holds for the case k. We consider the case $k+2$. Let $u \in L^2(\mathbf{R}_+)$ and suppose that $|||u|||_{X^{k+2}(\mathbf{R}_+)} < +\infty$. Set $w = A_2 u$. Then we have

$$\begin{aligned}
|||w|||_{X^k(\mathbf{R}_+)} = |||\partial_s(s\partial_s u)|||_{X^k(\mathbf{R}_+)} &= \sum_{i=0}^{k} \sum_{j=i}^{[\frac{k+i}{2}]} \|s^{\frac{i}{2}}\partial_s^{j+1}(s\partial_s u)\|_{L^2(\mathbf{R}_+)} \\
&\lesssim \sum_{i=0}^{k} \sum_{j=i}^{[\frac{k+i}{2}]} \left(\|s^{\frac{i}{2}}\partial_s^{j+1}u\|_{L^2(\mathbf{R}_+)} + \|s^{\frac{i+2}{2}}\partial_s^{j+2}u\|_{L^2(\mathbf{R}_+)}\right) \\
&\lesssim \sum_{i=0}^{k+2} \sum_{j=i}^{[\frac{k+i+2}{2}]} \|s^{\frac{i}{2}}\partial_s^{j}u\|_{L^2(\mathbf{R}_+)} = |||u|||_{X^{k+2}(\mathbf{R}_+)} < +\infty,
\end{aligned}$$

and hence the inductive hypothesis implies $\|A_k w\|_{L^2(\mathbf{R}_+)} \lesssim |||w|||_{X^k(\mathbf{R}_+)}$. Thus we obtain

$$\|A_{k+2}u\|_{L^2(\mathbf{R}_+)} = \|A_k w\|_{L^2(\mathbf{R}_+)} \lesssim |||w|||_{X^k(\mathbf{R}_+)} \lesssim |||u|||_{X^{k+2}(\mathbf{R}_+)},$$

which shows the inequality (6.3) for the case $k+2$. □

6.3 Proof of Proposition 6.3

For the proof of Proposition 6.3, we need Lemmas 6.7 and 6.8 below.

Lemma 6.7 *Let $k \in \mathbf{Z}_+$ and $u \in L^2(\mathbf{R}_+)$ and suppose that $A_1 u \in L^2(\mathbf{R}_+)$. In addition, set $w = A_2 u$ and suppose that $w \in X^{2k}(\mathbf{R}_+)$. Then for $j = 0, 1, \ldots, k$ it holds that*

$$\partial_s^{j+1}u(s) = \frac{1}{s^{j+1}} \int_0^s \sigma^j (\partial_s^j w)(\sigma)\, d\sigma. \tag{6.4}$$

Lemma 6.8 *Let $k \in \mathbf{Z}_+$ and $u \in L^2(\mathbf{R}_+)$ and suppose that $A_1 u \in L^2(\mathbf{R}_+)$. In addition, set $w = A_2 u$ and suppose that $w \in X^{2k+1}(\mathbf{R}_+)$. Then for $j = 0, 1, \ldots, k+1$, (6.4) also holds.*

Proof of Lemma 6.7. We prove (6.4) by induction on j. Corollary 4.2 shows that (6.4) holds for the case $j = 0$. Inductively assume that (6.4) holds for the case j. We consider the case $j+1$. Recalling the inductive hypothesis and using integration by parts we have

$$\begin{aligned}\partial_s^{j+2}u(s) &= \partial_s\Big(\frac{1}{s^{j+1}}\int_0^s \sigma^j(\partial_s^j w)(\sigma)\,d\sigma\Big) = \frac{1}{s}\partial_s^j w(s) - \frac{j+1}{s^{j+2}}\int_0^s \sigma^j(\partial_s^j w)(\sigma)\,d\sigma \\ &= \frac{1}{s}\partial_s^j w(s) - \frac{1}{s^{j+2}}\Big(\sigma^{j+1}(\partial_s^j w)(\sigma)\Big|_{\sigma=0}^{\sigma=s} - \int_0^s \sigma^{j+1}(\partial_s^{j+1}w)(\sigma)\,d\sigma\Big).\end{aligned}$$

On the other hand, since $w \in X^{2k}(\mathbf{R}_+)(\subset H^k(\mathbf{R}_+))$, we have $\partial_s^j w \in H^1(\mathbf{R}_+)$. Thus $\partial_s^j w$ is continuous on $\overline{\mathbf{R}_+}$. This yields $\sigma^{j+1}(\partial_s^j w)(\sigma)|_{\sigma=0}^{\sigma=s} = s^{j+1}\partial_s^j w(s)$. Therefore (6.4) also holds for the case $j+1$. $\square$

Proof of Lemma 6.8. In the same way as the proof of Lemma 6.7 we can prove (6.4) for the case $j = 0, 1, \dots, k$. It remains to show the case $j = k+1$. The same argument as in the proof of Lemma 6.7 implies

$$\partial_s^{k+2}u(s) = \frac{1}{s}\partial_s^k w(s) - \frac{1}{s^{k+2}}\Big(\sigma^{k+1}(\partial_s^k w)(\sigma)\Big|_{\sigma=0}^{\sigma=s} - \int_0^s \sigma^{k+1}(\partial_s^{k+1}w)(\sigma)\,d\sigma\Big).$$

Since $w \in X^{2k+1}(\mathbf{R}_+)$, we have $w \in H^{2k+1}(a,b)$ for arbitrary $a, b \in \mathbf{R}_+$ with $a < b$. Thus $\partial_s^k w$ is continuous in $\mathbf{R}_+$. This yields $\sigma^{k+1}(\partial_s^k w)(\sigma)|_{\sigma=s} = s^{k+1}\partial_s^k w(s)$.

Next we shall show that $\sigma^{k+1}(\partial_s^k w)(\sigma)|_{\sigma=0} = 0$. From Lemma 4.3 we have

$$\begin{aligned}\sup_{s\in(0,1)} |s^{\frac{1}{2}}\partial_s^k w(s)| &\lesssim \|\partial_s(s\partial_s^k w)\|_{L^2(0,1)} \lesssim \|\partial_s^k w\|_{L^2(0,1)} + \|s\partial_s^{k+1}w\|_{L^2(0,1)} \\ &\lesssim \|\partial_s^k w\|_{L^2(0,1)} + \|s^{\frac{1}{2}}\partial_s^{k+1}w\|_{L^2(0,1)} \lesssim \|w\|_{X^{2k+1}(\mathbf{R}_+)},\end{aligned}$$

and hence $\sigma^{k+1}(\partial_s^k w)(\sigma)|_{\sigma=0} = 0$. Thus (6.4) holds also for the case $k+1$. $\square$

We now give the proof of Proposition 6.3.

Proof of Proposition 6.3. We will give the proof of Proposition 6.3 only in the case where m is odd ($m = 2k+1$). More precisely, we will prove that if $u \in L^2(\mathbf{R}_+)$ and $A_j u \in L^2(\mathbf{R}_+)$ for $j = 0, 1, \dots, 2k+1$, then it holds that

$$\|u\|_{X^{2k+1}(\mathbf{R}_+)} \lesssim \sum_{j=0}^{2k+1} \|A_j u\|_{L^2(\mathbf{R}_+)}. \tag{6.5}$$

We prove (6.5) by induction on k. Clearly we have

$$\|u\|_{X^1(\mathbf{R}_+)} = \|u\|_{L^2(\mathbf{R}_+)} + \|s^{\frac{1}{2}}\partial_s u\|_{L^2(\mathbf{R}_+)} = \|A_0 u\|_{L^2(\mathbf{R}_+)} + \|A_1 u\|_{L^2(\mathbf{R}_+)}.$$

Thus (6.5) for the case $k = 0$ is trivial. Inductively assume that (6.5) holds for the case k. We consider the case $k+1$. Let $u \in L^2(\mathbf{R}_+)$ and suppose that $A_j u \in L^2(\mathbf{R}_+)$ for $j = 0, 1, \dots, 2k+3$. We shall prove that

$$\|u\|_{X^{2k+3}(\mathbf{R}_+)} \lesssim \|u\|_{X^{2k+1}(\mathbf{R}_+)} + \|w\|_{X^{2k+1}(\mathbf{R}_+)}, \tag{6.6}$$

where $w = A_2 u$. Indeed, the inequality (6.6) and the inductive hypothesis imply

$$\begin{aligned}\|u\|_{X^{2k+3}(\mathbf{R}_+)} &\lesssim \|u\|_{X^{2k+1}(\mathbf{R}_+)} + \|w\|_{X^{2k+1}(\mathbf{R}_+)} \lesssim \sum_{j=0}^{2k+1} \|A_j u\|_{L^2(\mathbf{R}_+)} + \sum_{j=0}^{2k+1} \|A_j w\|_{L^2(\mathbf{R}_+)} \\ &= \sum_{j=0}^{2k+1} \|A_j u\|_{L^2(\mathbf{R}_+)} + \sum_{j=0}^{2k+1} \|A_{j+2} u\|_{L^2(\mathbf{R}_+)} \lesssim \sum_{j=0}^{2k+3} \|A_j u\|_{L^2(\mathbf{R}_+)}.\end{aligned}$$

Therefore (6.5) also holds for the case $k+1$.

In what follows, we shall show the inequality (6.6), namely,

$$\sum_{j=0}^{k+1} \|\partial_s^j u\|_{L^2(\mathbf{R}_+)} + \sum_{j=0}^{k+1} \|s^{j+\frac{1}{2}} \partial_s^{k+2+j} u\|_{L^2(\mathbf{R}_+)} \lesssim \|u\|_{X^{2k+1}(\mathbf{R}_+)} + \|w\|_{X^{2k+1}(\mathbf{R}_+)}.$$

By the definition (2.2) the inequality $\|\partial_s^j u\|_{L^2(\mathbf{R}_+)} \leq \|u\|_{X^{2k+1}(\mathbf{R}_+)}$ for the case $j = 0, 1, \ldots, k$ is trivial. Next we shall show that $\|\partial_s^{k+1} u\|_{L^2(\mathbf{R}_+)} \lesssim \|w\|_{X^{2k+1}(\mathbf{R}_+)}$. From Lemma 6.8 we have

$$\partial_s^{k+1} u(s) = \frac{1}{s^{k+1}} \int_0^s \sigma^k (\partial_s^k w)(\sigma)\, d\sigma.$$

Thus, from Lemma 4.4 we obtain

$$\|\partial_s^{k+1} u\|_{L^2(\mathbf{R}_+)} \lesssim \|\partial_s^k w\|_{L^2(\mathbf{R}_+)} \lesssim \|w\|_{X^{2k+1}(\mathbf{R}_+)}.$$

Finally we shall show that

$$\|s^{j+\frac{1}{2}} \partial_s^{k+2+j} u\|_{L^2(\mathbf{R}_+)} \lesssim \|w\|_{X^{2k+1}(\mathbf{R}_+)} \tag{6.7}$$

for $j = 0, 1, \ldots, k+1$. We prove the inequality (6.7) by induction on j. We first consider the case $j = 0$. Lemma 6.8 yields

$$s^{\frac{1}{2}} \partial_s^{k+2} u(s) = \frac{1}{s^{k+\frac{3}{2}}} \int_0^s \sigma^{k+\frac{1}{2}} (s^{\frac{1}{2}} \partial_s^{k+1} w)(\sigma)\, d\sigma.$$

Furthermore, Lemma 4.4 implies

$$\|s^{\frac{1}{2}} \partial_s^{k+2} u\|_{L^2(\mathbf{R}_+)} \lesssim \|s^{\frac{1}{2}} \partial_s^{k+1} w\|_{L^2(\mathbf{R}_+)} \lesssim \|w\|_{X^{2k+1}(\mathbf{R}_+)}.$$

Thus the inequality (6.7) holds for the case $j = 0$. Inductively assume that the inequality (6.7) holds for the case j. We consider the case $j+1$. We note that

$$\partial_s^{k+1+j} w = \partial_s^{k+1+j} A_2 u = \partial_s^{k+2+j} (s \partial_s u) = s \partial_s^{k+3+j} u + (k+2+j) \partial_s^{k+2+j} u.$$

This yields

$$s^{j+\frac{3}{2}} \partial_s^{k+3+j} u = s^{j+\frac{1}{2}} \partial_s^{k+1+j} w - (k+2+j) s^{j+\frac{1}{2}} \partial_s^{k+2+j} u,$$

and hence

$$\|s^{j+1+\frac{1}{2}} \partial_s^{k+2+j+1} u\|_{L^2(\mathbf{R}_+)} \lesssim \|s^{j+\frac{1}{2}} \partial_s^{k+1+j} w\|_{L^2(\mathbf{R}_+)} + \|s^{j+\frac{1}{2}} \partial_s^{k+2+j} u\|_{L^2(\mathbf{R}_+)}.$$

On the other hand, from the definition (2.2) we have $\|s^{j+\frac{1}{2}} \partial_s^{k+1+j} w\|_{L^2(\mathbf{R}_+)} \leq \|w\|_{X^{2k+1}(\mathbf{R}_+)}$. Moreover, the inductive hypothesis implies $\|s^{j+\frac{1}{2}} \partial_s^{k+2+j} u\|_{L^2(\mathbf{R}_+)} \lesssim \|w\|_{X^{2k+1}(\mathbf{R}_+)}$. Thus the inequality (6.7) holds also for the case $j+1$. Therefore the inequality (6.6) is proved. □

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