

SELF-INTERSECTIONS OF CURVES ON A SURFACE AND BERNOULLI NUMBERS

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ABSTRACT. We study an operation which measures self-intersections of curves on an oriented surface. It turns out that a certain computation on this topological operation is related to the Bernoulli numbers B_m , and our study yields a family of explicit formulas for B_m . As a special case, this family contains the celebrated formula for B_m due to Kronecker.

1. INTRODUCTION

The Bernoulli numbers B_m ($m \geq 0$) are defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} x^m.$$

We have: $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_4 = -1/30, \dots$, and $B_m = 0$ for all odd $m \geq 3$. The appearance of the Bernoulli numbers is ubiquitous in mathematics, and a large number of identities involving the Bernoulli numbers has been known [3] [4] [9] [10].

In this article, we show that the Bernoulli numbers arise naturally from the topology of surfaces, i.e., 2-manifolds. In more detail, by studying self-intersections of curves on an oriented surface, we obtain the following family of explicit formulas for B_m :

Theorem 1. *Let $m \geq 2$. For any integers a and n satisfying $0 \leq a \leq m \leq n$, we have*

$$(1) \quad B_m = (-1)^a \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \sum_{i=1}^{k-1} i^a (k-i)^{m-a}.$$

Notice that the formula above has two parameters a and n . When $a = 0$ and $n = m$, the formula (1) reduces to the celebrated formula for B_m due

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to Kronecker ([7], see also [4] [5] [9] [10]): for $m \geq 2$,

$$(2) \quad B_m = \sum_{k=1}^{m+1} \frac{(-1)^{k+1}}{k} \binom{m+1}{k} \sum_{i=1}^{k-1} i^m.$$

In fact, using the classical formula for the sum of powers (known as Faulhaber's formula) and a property of binomial coefficients (see Lemma 2), one can derive the formula (1) from the Kronecker formula (2). However, our derivation of the formula (1) is self-contained and more direct.

Our proof of Theorem 1 is motivated by a topological consideration on an oriented surface. In §2, we consider an operation μ to a curve on the surface. This operation was introduced in [6] inspired by a construction of Turaev [11], and, among other things, it computes *self-intersections* of curves. The key is to compute $\mu(\log \gamma)$ for a simple loop γ and we find that it involves the Bernoulli numbers (Theorem 2). Here, we work with a suitable completion to be able to consider $\log \gamma$. In §3, we formalize the topological argument in §2 and prove the main results. In §4, we give another self-contained proof of Theorem 1 by introducing a certain generating function.

The Bernoulli numbers have already appeared in the study of intersections of *two curves* on an oriented surface [8]. Our formula provides yet another evidence for a close connection between the topology of surfaces and the Bernoulli numbers. This connection has been developed in [1] to an unexpected connection between the operation μ , or equivalently, the Turaev cobracket, and the Kashiwara-Vergne problem in the formulation by Alekseev-Torossian [2].

2. SELF-INTERSECTION MAP AND BERNOULLI NUMBERS

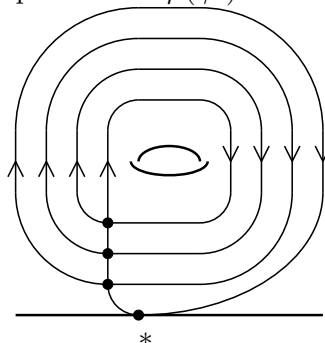
Let S be a compact connected oriented surface with $\partial S \neq \emptyset$. Fix a basepoint $* \in \partial S$ and set $\pi_1(S) := \pi_1(S, *)$. We denote by $\hat{\pi}(S)$ the set of free homotopy classes of oriented loops on S . For any $p \in S$, we denote by $|\cdot|: \pi_1(S, p) \rightarrow \hat{\pi}(S)$ the forgetful map of the basepoint.

We recall the operation $\mu: \mathbb{Q}\pi_1(S) \rightarrow \mathbb{Q}\pi_1(S) \otimes (\mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\mathbf{1})$, which has been introduced in [6] inspired by a construction of Turaev [11]. Here, $\mathbf{1}$ is the class of a constant loop. Let $\gamma: [0, 1] \rightarrow S$ be an immersed based loop. We arrange so that the pair of tangent vectors $(\dot{\gamma}(0), \dot{\gamma}(1))$ is a positive basis of the tangent space T_*S , and that the self-intersections of γ (except for the base point $*$) lie in the interior $\text{Int}(S)$ and consist of transverse double points. Let Γ be the set of such double points of γ . For $p \in \Gamma$ we denote $\gamma^{-1}(p) = \{t_1^p, t_2^p\}$, so that $0 < t_1^p < t_2^p < 1$. We define

$$\mu(\gamma) := - \sum_{p \in \Gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) (\gamma_{0t_1^p} \gamma_{t_2^p 1}) \otimes |\gamma_{t_1^p t_2^p}| \in \mathbb{Q}\pi_1(S) \otimes (\mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\mathbf{1}).$$

Here,

- the sign $\varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))$ is $+1$ if the pair $(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))$ is a positive basis of $T_p S$, and is -1 otherwise,

FIGURE 1. computation of $\mu(\gamma^k)$ for a simple γ ($k = 4$)


- the based loop $\gamma_{0t_1^p}\gamma_{t_2^p1}$ is the conjunction of the paths $\gamma|_{[0,t_1^p]}$ and $\gamma|_{[t_2^p,1]}$,
- the element $\gamma_{t_1^p t_2^p} \in \pi_1(S, p)$ is the restriction of γ to $[t_1^p, t_2^p]$ and we understand that $|\gamma_{t_1^p t_2^p}| = 0$ if the loop $\gamma_{t_1^p t_2^p}$ is homotopic to a constant loop.

Remark 1. The operation μ is essentially the same as Turaev's operation $\mu^T : \pi_1(S) \rightarrow \mathbb{Q}\pi_1(S)$ in [11]. In fact, we have $\mu^T(\gamma)\gamma = -(\text{id} \otimes \varepsilon)\mu(\gamma)$ for any $\gamma \in \pi_1(S)$, where $\varepsilon(\alpha) = 1$ for any $\alpha \in \hat{\pi}(S) \setminus \{\mathbf{1}\}$. Conversely, one can express μ in terms of μ^T . The alternating part of $(|\otimes 1)\mu(\gamma)$ is exactly the Turaev cobracket [12] of the free loop $|\gamma|$.

We observe that if γ is simple and the pair $(\dot{\gamma}(0), \dot{\gamma}(1))$ is a positive basis of T_*S , then for any integer $k \in \mathbb{Z}$,

$$(3) \quad \mu(\gamma^k) = \begin{cases} -\sum_{i=1}^{k-1} \gamma^i \otimes |\gamma^{k-i}| & (k > 0) \\ 0 & (k = 0) \\ \sum_{i=0}^{|k|-1} \gamma^{-i} \otimes |\gamma^{k+i}| & (k < 0). \end{cases}$$

See Figure 1.

In [6] §4, it was shown that the map μ extends to a map between completions $\mu : \widehat{\mathbb{Q}\pi_1(S)} \rightarrow \widehat{\mathbb{Q}\pi_1(S)} \widehat{\otimes} \widehat{\mathbb{Q}\hat{\pi}(S)}$. Here $\widehat{\mathbb{Q}\pi_1(S)}$ and $\widehat{\mathbb{Q}\hat{\pi}(S)}$ are the completions of the group ring $\mathbb{Q}\pi_1(S)$ and the Goldman-Turaev Lie bialgebra $\mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\mathbf{1}$, respectively, with respect to the augmentation ideal of $\mathbb{Q}\pi_1(S)$. Then we can consider $\log \gamma = \sum_{i=1}^{\infty} ((-1)^{i+1}/i)(\gamma - 1)^i \in \widehat{\mathbb{Q}\pi_1(S)}$.

As the following result shows, if γ is simple then one can compute $\mu(\log \gamma)$ explicitly and the formula involves the Bernoulli numbers.

Theorem 2. *Let $\gamma \in \pi$ be represented by a simple loop, and assume that the pair $(\dot{\gamma}(0), \dot{\gamma}(1))$ is a positive basis of the tangent space T_*S . Then we have*

$$(4) \quad \mu(\log \gamma) = -\sum_{m=0}^{\infty} \frac{B_m}{m!} \sum_{p=0}^m (-1)^p \binom{m}{p} (\log \gamma)^p \widehat{\otimes} |(\log \gamma)^{m-p}|.$$

3. PROOF OF THEOREM 1 AND THEOREM 2

First of all, we describe a preliminary construction.

Let $\mathbb{Q}[[Z]]$ (resp. $\mathbb{Q}[[X, Y]]$) be the commutative ring of formal power series in an indeterminate Z (resp. in indeterminates X and Y). For a non-negative integer p , let F_p^Z (resp. $F_p^{X, Y}$) be the set of formal power series in $\mathbb{Q}[[Z]]$ (resp. $\mathbb{Q}[[X, Y]]$) which has only terms of (total) degree $\geq p$. We have natural isomorphisms $\mathbb{Q}[[Z]] \cong \varprojlim_p \mathbb{Q}[[Z]]/F_p^Z$ and $\mathbb{Q}[[X, Y]] \cong \varprojlim_p \mathbb{Q}[[X, Y]]/F_p^{X, Y}$.

Set $z := e^Z = \sum_{i=0}^{\infty} (1/i!)Z^i$. Then the Laurent polynomial ring $\mathbb{Q}[z, z^{-1}]$ is a subring of $\mathbb{Q}[[Z]]$. The augmentation ideal I is defined by

$$I = \text{Ker}(\mathbb{Q}[z, z^{-1}] \rightarrow \mathbb{Q}, \sum_j a_j z^j \mapsto \sum_j a_j).$$

Then I gives a filtration $\{I^p\}_p$ of $\mathbb{Q}[z, z^{-1}]$. By the inclusion map $\mathbb{Q}[z, z^{-1}] \hookrightarrow \mathbb{Q}[[Z]]$, the filtration $\{F_p^Z\}_p$ restricts to $\{I^p\}_p$. Moreover, we have a natural isomorphism $\mathbb{Q}[[Z]] \cong \varprojlim_p \mathbb{Q}[z, z^{-1}]/I^p$.

Motivated by the formula (3), we define a \mathbb{Q} -linear map $\hat{\mu}: \mathbb{Q}[z, z^{-1}] \rightarrow \mathbb{Q}[[X, Y]]$ by

$$(5) \quad \hat{\mu}(z^k) = \begin{cases} -\sum_{i=1}^k e^{iX} e^{(k-i)Y} & (k > 0) \\ 0 & (k = 0) \\ \sum_{i=0}^{|k|-1} e^{-iX} e^{(k+i)Y} & (k < 0). \end{cases}$$

From the definition of $\hat{\mu}$ it is easy to see that

$$(e^{-X} e^Y - 1)\hat{\mu}(z^k) = e^{kX} - e^{kY}, \quad k \in \mathbb{Z}.$$

Therefore, we have

$$(6) \quad (e^{-X} e^Y - 1)\hat{\mu}(f(z)) = f(e^X) - f(e^Y)$$

for any Laurent polynomial $f(z) \in \mathbb{Q}[z, z^{-1}]$. Consider

$$\Phi(X, Y) := \sum_{i=0}^{\infty} \frac{B_i}{i!} (-X + Y)^i.$$

Then we have $(e^{-X} e^Y - 1)\Phi(X, Y) = -X + Y$. Multiplying $\Phi(X, Y)$ to both sides of (6), we have

$$(7) \quad (-X + Y)\hat{\mu}(f(z)) = (f(e^X) - f(e^Y))\Phi(X, Y)$$

for any $f(z) \in \mathbb{Q}[z, z^{-1}]$.

Lemma 1. *There is a unique continuous extension $\hat{\mu}: \mathbb{Q}[[Z]] \rightarrow \mathbb{Q}[[X, Y]]$ of the map $\hat{\mu}$ in (5).*

Proof. It is sufficient to prove that $\hat{\mu}(I^p) \subset F_{p-1}^{X, Y}$ for any $p \geq 1$. Suppose $f(z) \in I^p$. Then $f(e^X)$ and $f(e^Y)$ lie in $F_p^{X, Y}$. This means that the right hand side of (7) is an element of $F_p^{X, Y}$. Therefore, $\hat{\mu}(f(z)) \in F_{p-1}^{X, Y}$. \square

Now for each $k \geq 1$ we can put $f(z) = (\log z)^k = Z^k$ in (7), and we obtain

$$(-X + Y)\hat{\mu}(Z^k) = (X^k - Y^k)\Phi(X, Y).$$

This shows that $\hat{\mu}(Z^k) \in F_{k-1}^{X,Y}$. Setting $k = 1$, we have

$$(8) \quad \hat{\mu}(Z) = -\Phi(X, Y) = -\sum_{i=0}^{\infty} \frac{B_i}{i!} \sum_{j=0}^i (-1)^j \binom{i}{j} X^j Y^{i-j}.$$

This formula is essentially the same as the assertion of Theorem 2:

Proof of Theorem 2. We identify the ring $\mathbb{Q}[[X, Y]]$ with the complete tensor product $\mathbb{Q}[[Z]] \hat{\otimes} \mathbb{Q}[[Z]]$ by the map $X \mapsto Z \hat{\otimes} 1$ and $Y \mapsto 1 \hat{\otimes} Z$. Then the computation (8) implies

$$(9) \quad \hat{\mu}(\log z) = -\sum_{m=0}^{\infty} \frac{B_m}{m!} \sum_{p=0}^m (-1)^p \binom{m}{p} (\log z)^p \hat{\otimes} (\log z)^{m-p}.$$

From (3) and (5) it follows that the substitution $z \mapsto \gamma$ commutes with μ and $\hat{\mu}$. Thus we obtain (4). \square

Further, by expanding the left hand side of (8) in terms of $\hat{\mu}(z^k)$'s modulo higher degree terms, we have the following:

Proposition 1. *Let m, n, a be integers satisfying $0 \leq a \leq m \leq n$. Then it holds that*

$$B_m = (-1)^a \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \left[\sum_{i=1}^{k-1} i^a (k-i)^{m-a} + \delta_{a,m} k^m \right].$$

Here $\delta_{a,m}$ is the Kronecker delta.

Proof. In what follows, \equiv means an equality in $\mathbb{Q}[[X, Y]]$ modulo $F_{n+1}^{X,Y}$. For $k = 1, \dots, n+1$, we have

$$(10) \quad \hat{\mu}(z^k) = \hat{\mu}(e^{kZ}) = \sum_{i=1}^{\infty} \frac{k^i}{i!} \hat{\mu}(Z^i) \equiv \sum_{i=1}^{n+1} \frac{k^i}{i!} \hat{\mu}(Z^i).$$

Consider the square matrix $D = (D_{ki})_{k,i}$ of order $n+1$, where $D_{ki} = k^i/i!$. Then D is invertible since $\det D$ is a non-zero multiple of Vandermonde's determinant $\det(k^{i-1})_{k,i}$. The inverse matrix of D has the first row (a_1, \dots, a_{n+1}) , where

$$a_k = \frac{(-1)^{k+1}}{k} \binom{n+1}{k}.$$

(To see this, for instance, one can use Lemma 2 below to get $(a_1, \dots, a_{n+1})D = (1, \dots, 0)$.) From (10) we have

$$(11) \quad \hat{\mu}(Z) \equiv \sum_{k=1}^{n+1} a_k \hat{\mu}(z^k) = \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \hat{\mu}(z^k).$$

Furthermore, for $k = 1, \dots, n+1$, from (5) we have

$$(12) \quad \hat{\mu}(z^k) = - \sum_{i=1}^{k-1} \sum_{a,b=0}^{\infty} \frac{i^a (k-i)^b}{a!b!} X^a Y^b - \sum_{a=0}^{\infty} \frac{k^a}{a!} X^a.$$

By (11) and (12), the coefficient of $X^a Y^{m-a}$ in $\hat{\mu}(Z)$ is

$$\sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left[\sum_{i=1}^{k-1} \frac{i^a (k-i)^{m-a}}{a!(m-a)!} + \delta_{m,a} \frac{k^m}{m!} \right].$$

On the other hand, by (8), this coincides with

$$(-1)^{a+1} \frac{B_m}{m!} \binom{m}{a} = \frac{(-1)^{a+1}}{a!(m-a)!} B_m.$$

This completes the proof. \square

Now, we can derive Theorem 1 from Proposition 1 by applying the following lemma. Although it might be well known, we give its proof for the sake of completeness.

Lemma 2. *Let m, n be integers satisfying $0 \leq m \leq n$. Then it holds that*

$$\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} k^m = \begin{cases} 0 & \text{if } m \geq 1, \\ -1 & \text{if } m = 0. \end{cases}$$

Proof. Set $f(x) := (e^x - 1)^{n+1}$. Since $m \leq n$, the coefficient of x^m in the series expansion of $f(x)$ is zero.

On the other hand, we compute

$$\begin{aligned} f(x) &= \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} e^{kx} \\ &= (-1)^{n+1} \left[\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} e^{kx} + 1 \right] \\ &= (-1)^{n+1} \left[\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \sum_{a=0}^{\infty} \frac{k^a}{a!} x^a + 1 \right]. \end{aligned}$$

Since the coefficient of x^m in the last expression is equal to

$$\begin{cases} \frac{(-1)^{n+1}}{m!} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} k^m & \text{if } m \geq 1, \\ (-1)^{n+1} \left[\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} + 1 \right] & \text{if } m = 0, \end{cases}$$

the assertion follows. \square

4. ANOTHER PROOF OF THEOREM 1

Introducing a generating function of two variables, we give another self-contained proof of Theorem 1. Since we have Lemma 2, it is sufficient to prove Proposition 1.

Let $f(x, y)$ and $g(x, y)$ be functions in variables x and y defined by

$$f(x, y) := \int_x^y (e^t - 1)^{n+1} dt, \quad \text{and} \quad g(x, y) := \frac{f(x, y)}{e^{y-x} - 1}.$$

We will examine the coefficient of $x^a y^{m-a}$ in the series expansion of $g(x, y)$.

First we compute $f(x, y)$ as follows:

$$\begin{aligned} f(x, y) &= \int_x^y (e^t - 1)^{n+1} dt \\ &= \int_x^y \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} e^{kt} dt \\ &= (-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} (e^{ky} - e^{kx}) + (-1)^{n+1} (y - x). \end{aligned}$$

Since

$$\frac{e^{ky} - e^{kx}}{e^{y-x} - 1} = \frac{e^{kx}(e^{k(y-x)} - 1)}{e^{y-x} - 1} = \sum_{i=1}^{k-1} e^{ix} e^{(k-i)y} + e^{kx},$$

we can compute $g(x, y)$ as follows:

$$\begin{aligned} g(x, y) &= \frac{f(x, y)}{e^{y-x} - 1} \\ &= (-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \frac{(e^{ky} - e^{kx})}{e^{y-x} - 1} + (-1)^{n+1} \frac{y - x}{e^{y-x} - 1} \\ &= (-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left[\sum_{i=1}^{k-1} e^{ix} e^{(k-i)y} + e^{kx} \right] \\ &\quad + (-1)^{n+1} \sum_{b=0}^{\infty} \frac{B_b}{b!} (y - x)^b. \end{aligned}$$

Then using the identities:

$$e^{ix} e^{(k-i)y} = \sum_{b,c=0}^{\infty} \frac{i^b (k-i)^c}{b!c!} x^b y^c \quad \text{and} \quad e^{kx} = \sum_{b=0}^{\infty} \frac{k^b}{b!} x^b,$$

we see that the coefficient of $x^a y^{m-a}$ in $g(x, y)$ is given by

$$\begin{aligned} & (-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left[\sum_{i=1}^{k-1} \frac{i^a}{a!} \frac{(k-i)^{m-a}}{(m-a)!} + \delta_{a,m} \frac{k^m}{m!} \right] \\ & + (-1)^{n+1+a} \frac{B_m}{m!} \binom{m}{a}. \end{aligned}$$

This is equal to $((-1)^{n+1+a}/m!) \binom{m}{a}$ times

$$(13) \quad (-1)^a \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left[\sum_{i=1}^{k-1} i^a (k-i)^{m-a} + \delta_{a,m} k^m \right] + B_m.$$

Secondly, we expand $g(x, y)$ in a different way. Put $g_1(x, y) = f(x, y)/(y-x)$. Then we have

$$g(x, y) = \frac{f(x, y)}{y-x} \frac{y-x}{e^{y-x} - 1} = g_1(x, y) \sum_{b=0}^{\infty} \frac{B_b}{b!} (y-x)^b.$$

Writing $(e^t - 1)^{n+1} = \sum_{i \geq n+1} a_i t^i$, we have

$$f(x, y) = \int_x^y (e^t - 1)^{n+1} dt = \sum_{i \geq n+1} \frac{a_i}{i+1} (y^{i+1} - x^{i+1}).$$

Thus the series expansion of $g_1(x, y)$ has all terms of degree $\geq n+1$, so does that of $g(x, y)$. In particular, the coefficient of $x^a y^{m-a}$ in this expansion is zero. Therefore, the expression (13) is zero, and we obtain Proposition 1. \square

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