42 平均値定理ノ拡張ニ就テ

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私ガ東北数学雑誌（1934, p.200）ニテ

$$\int_{x_0}^{x}\int_{x_1}^{x}\cdots\int_{x_{n-1}}^{x}\varphi^{(n)}(x)\,dx^n=\varphi^{(n)}(\xi)\int_{x_0}^{x}\int_{x_1}^{x}\cdots\int_{x_{n-1}}^{x}dx^n$$

ナルガ如キξガ存在スルコトヲ証明ナシデ用ヒマシタ所、岡村博氏ガ必ズシモ成立セズ。例ヘバ $n=2$, $\varphi^{(n)}(x)=2x^3+2x+1$, $x_0=-1$, $x_1=0$, $x_2=1$ ノ如キモノアリト御注意下サイマシタ。此レハ全ク私ノ気付カナカッタ所デ真ニ赤面ノ至リデス。此レハ次ノ如ク訂正シタラヨイト思ヒマス。一般ノ御批評ヲ御願ヒ致シマス。

便宜上 $(x_1,x_2,\cdots,x_n)$ ヲ以テ n 個ノ点 $x_1,x_2,\cdots,x_n$ ヲ含ム総テノ区間ノ Durchschnitt ヲ表ハス。次ニ

$$\xi_1\in(x_0,x),\quad \xi_2\in(x_1,\xi_1),\quad \cdots,\quad \xi_n\in(x_{n-1},\xi_{n-1})$$

ナル条件ヲ充タス n 次元空間ノ点 $(\xi_1,\xi_2,\cdots,\xi_n)$ ノ Punktmenge ヲ B ヲ以テ表ハストスル。B ニテ

$$v(\xi_1,\xi_2,\cdots,\xi_n)=\mathrm{sign}(\xi_1-x_0)(\xi_2-x_0)\cdots(\xi_n-x_{n-1})$$

ナル函数ヲ定義スレバ

$$\int_{x_0}^{x}\int_{x_1}^{x}\cdots\int_{x_{n-1}}^{x}\varphi^{(n)}(x)\,dx^n=\int_B\varphi^{(n)}(\xi)\,v(\xi_1,\xi_2,\cdots,\xi_n)\,d\xi_1\cdots d\xi_n$$

ナルコトガ容易ニ知ラレル。故ニ $v(\xi_1,\cdots\xi_n)$ ガ B ニテ符号ヲ変ヘナケレバ $=\varphi^{(n)}(\xi)\int_B v(\xi_1,\cdots,\xi_n)\,d\xi_1\cdots d\xi_n=\varphi^{(n)}(\xi)\int_{x_0}^{x}\cdots\int_{x_{n-1}}^{x}dx^n$

ナルξガ存在スル。即チ平均値定理ガ存在スル。

故ニx_iガ$(x, x_0, \cdots, x_{i-1})$ $(i=1,2,\cdots,n-1)$ノ内部ニナケレバ平均値定理ガ成立ス。又x_iガ$(x, x_0, \cdots, x_{i-1})$ノ内部ニ来タトキハ $x_i = x_{i+1} = \cdots = x_{i+k} \neq x_{i+k+1}$ (kハ偶数) ナルモノトスレバ平均値定理ガ成立ス。即チ

$$\varphi(x) = \varphi(x_0) + \varphi'(x_1)\int_{x_0}^{x} dx + \cdots + \varphi^{(n-1)}(x_{n-1})\int_{x_0}^{x}\cdots\int_{x_{n-2}}^{x} dx^{n-1}$$

ナルξガ存在ス。故ニ $x = x_0 + h$, $x_0 = x_1 = \cdots = x_{n-3}$, $x_{n-2} = x_{n-1} = x_0 + \frac{h}{n-1}$ トスレバ Mazzoni ノ結果

$$\varphi(x_0+h) = \varphi(x_0) + \cdots + \varphi^{(n-3)}(x_0)\frac{h^{n-3}}{\underline{|n-3}}$$
$$+ \varphi^{(n-2)}\left(x_0 + \frac{h}{n-1}\right)\frac{h^{n-2}}{\underline{|n-2}} + \varphi^{(n)}(x_0+\theta h)\frac{n-2}{\underline{|n}\,(n-1)}h^n$$

ガ得ラレ又$k+1$番目ノ項ヲ消スニハ

$$x = x_0 + h,\ x_0 = x_1 = x_2 = \cdots = x_{k-2} = x_{k-1} = \cdots = x_{n-1}$$
$$x_{k-1} = x_k = x_0 + \frac{h}{k} \qquad (k \leqq n-1)$$

トセバ $\varphi(x_0+h) = \varphi(x_0) + \varphi'(x_0)h + \cdots + \varphi^{(k-2)}(x_0)\frac{h^{k-2}}{\underline{|k-2}}$

$$+ \varphi^{(k-1)}\left(x_0 + \frac{h}{k}\right)\frac{h^{k-1}}{\underline{|k-1}} + \varphi^{(k+1)}(x_0)\frac{k-1}{\underline{|k+1}\,k}h^{k+1} + \cdots\cdots$$

$$+ \varphi^{(n-1)}(x_0)\left(\frac{1}{\underline{|n-1}} - \frac{1}{k^{n-k}\underline{|(n-k)}\ \underline{|k-1}}\right)h^{n-1}$$

$$+ \varphi^{(n)}(x_0+\theta h)\left(\frac{1}{\underline{|n}} - \frac{1}{k^{n-k+1}\underline{|(n-k+1)}\ \underline{|k-1}}\right)$$

トナル。次ニ任意ノ $\varphi(x)$ ニ対シテ平均値定理

$$\int_{x_0}^{x}\int_{x_1}^{x}\cdots\int_{x_{n-1}}^{x}\varphi^{(n)}(x)\,dx^n=\varphi^{(n)}(\xi)\int_{x_0}^{x}\int_{x_1}^{x}\cdots\int_{x_{n-1}}^{x}dx^n$$

ガ成立スル為ニ $x, x_0, x_1, \cdots, x_{n-1}$ ガ充タス可キ必要且充分ナ條件ヲ求メテミヨウ。

$$\int_{x_{n-2}}^{\xi_{n-2}}\int_{x_{n-1}}^{\xi_{n-1}}\varphi^{(n)}(\xi_n)\,d\xi_n\,d\xi_{n-1}=\int_{x_{n-1}}^{\xi_{n-2}}\int_{\xi_n}^{\xi_{n-2}}\varphi^{(n)}(\xi_n)\,d\xi_{n-1}\,d\xi_n$$

$$-\int_{x_{n-1}}^{x_{n-2}}\int_{\xi_n}^{x_{n-2}}\varphi^{(n)}(\xi_n)\,d\xi_{n-1}\,d\xi_n$$

$$=\int_{x_{n-1}}^{\xi_{n-2}}\varphi^{(n)}(\xi_n)\,d\xi_n\int_{\xi_n}^{\xi_{n-2}}d\xi_{n-1}+\int_{x_{n-1}}^{x_{n-2}}\varphi^{(n)}(\xi_n)\,d\xi_n\int_{x_{n-2}}^{\xi_n}d\xi_{n-1}$$

$$\therefore\ \int_{x_0}^{x}\int_{x_1}^{x}\varphi^{(n)}(x)\,dx^n=\int_{x_{n-1}}^{x_{n-2}}\varphi^{(n)}(\xi_n)\,d\xi_n\int_{x_{n-2}}^{\xi_n}d\xi_{n-1}\int_{x_0}^{x}\cdots\int_{x_{n-3}}^{x}dx^{n-2}$$

$$+\int_{x_{n-1}}^{x_{n-3}}\varphi^{(n)}(\xi_n)\,d\xi_n\int_{x_{n-3}}^{\xi_n}\int_{\xi_n}^{\xi_{n-2}}d\xi_{n-1}\,d\xi_{n-2}\int_{x_0}^{x}\cdots\int_{x_{n-4}}^{x}dx^{n-3}$$

$$+\int_{x_{n-1}}^{x_{n-4}}\varphi^{(n)}(\xi_n)\,d\xi_n\int_{x_{n-4}}^{\xi_n}\int_{\xi_n}^{\xi_{n-3}}\int_{\xi_n}^{\xi_{n-2}}d\xi_{n-1}\,d\xi_{n-2}\,d\xi_{n-3}\int_{x_0}^{x}\cdots\int_{x_{n-5}}^{x}dx^{n-4}$$

$$+\cdots+\int_{x_{n-1}}^{x_0}\varphi^{(n)}(\xi_n)\,d\xi_n\int_{x_0}^{\xi_n}\int_{\xi_n}^{\xi_1}\cdots\int_{\xi_n}^{\xi_{n-2}}d\xi_{n-1}\,d\xi_{n-2}\cdots d\xi_1$$

$$+\int_{x_{n-1}}^{x}\varphi^{(n)}(\xi_n)\,d\xi_n\int_{x_0}^{\xi_n}\int_{\xi_n}^{\xi_1}\cdots\int_{\xi_n}^{\xi_{n-2}}d\xi_{n-1}\,d\xi_{n-2}\cdots d\xi_1$$

今 $v_i(x)=\begin{cases}\mathrm{sign}\,(x_i-x_{i+1}), & x\in(x_i,x_{n-1})\\ 0, & x\notin(x_i,x_{n-1})\end{cases}$

$(i=-1, 0, \cdots, n-1)$ 且 $x = x_{-1}$ トス

ナル函数ヲ $-\infty < x < \infty$ ニテ定義セバ

$$\int_{x_0}^{x}\int_{x_1}^{x}\int_{x_{n-1}}^{x}\varphi^{(n)}(x)\,dx^n$$

$$=\int_{-\infty}^{+\infty}\varphi^{(n)}(\xi)\left\{\sum_{i=1}^{n}v_i(\xi)\int_{x_i}^{\xi}\int_{\xi}^{\xi_{i+1}}\cdots\int_{\xi}^{\xi_{n-2}}d\xi_{n-1}\,d\xi_{n-2}\cdots d\xi_{i+1}\int_{x_0}^{x}\cdots\int_{x_{i-1}}^{x}dx^i\right\}d\xi$$

故ニ平均値定理ノ成立スル為ノ必充條件ハ

$$\sum_{i=0}^{n}v_i(\xi)\int_{x_i}^{\xi}\int_{\xi}^{\xi_{i+1}}\cdots\int_{\xi}^{\xi_{n-2}}d\xi_{n-1}\,d\xi_{n-2}\cdots d\xi_i\int_{x_0}^{x}\cdots\int_{x_{i-1}}^{x}dx^i+v_{-1}(\xi)\int_{\xi}^{x}\int_{\xi}^{\xi_1}\cdots\int_{\xi}^{\xi_{n-2}}d\xi_{n-1}\cdots d\xi_1$$

$$=\sum_{i=0}^{n}v_i(\xi)\int_{x_i}^{\xi}\frac{(\xi_i-\xi)^{n-i-2}}{\lfloor n-i-2}\,d\xi_i\int_{x_0}^{x}\cdots\int_{x_{i-1}}^{x}dx^i+v_{-1}(\xi)\frac{(x-\xi)^{n-1}}{\lfloor n-1}$$

$$=-\sum_{i=0}^{n}v_i(\xi)\frac{(x_i-\xi)^{n-i}}{\lfloor n-i}\int_{x_0}^{x}\cdots\int_{x_{i-1}}^{x}dx^i+v_{-1}(\xi)\frac{(x-\xi)^{n-1}}{\lfloor n-1}$$

$$=v_{-1}(\xi)\frac{(x-\xi)^{n-1}}{\lfloor n-1}-\sum_{i=0}^{n-2}v_i(\xi)\frac{(x_i-\xi)^{n-i-1}}{\lfloor n-i-1}\int_{x_0}^{x}\cdots\int_{x_{i-1}}^{x}dx^i$$

ガ $-\infty < \xi < \infty$ ニテ $x, x_0, x_1, \cdots, x_n$ ト If ケバ一定符号ナルコトナリ。

(9, 10, 12 參照)