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56.　素數ノ分布ニ就イテ

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$$\pi(x) > \alpha_1 \frac{x}{\log x}$$

ノα_1ノ値ニ就テ13号ニ市原哲治氏ガ述ベラレタ御質問ニ對シテ，ソノ後私ガ求メタ値ヲ御知ラセシタイ。

$$\pi(x) = \int_2^x \frac{du}{\log u} + O\left(xe^{-\alpha_2\sqrt{\log x \log\log x}}\right)$$

ナドノヨク知ラレタ定理ニ依ツテ充分大ナルxニ對シテ，$\alpha_1 = 1$ 即チ

$$\pi(x) > \frac{x}{\log x}$$

ガ成立スルコトハ明カデアル。併シ(2)ナドノ式ヲ不等式ニ書キ改メテ，充分大ナルト云フノデナク，アル定マツタx_0ヨリ大ナルxニ對シテ(3)ガ成立スル事ヲ證明シヤウト云フ時，x_0ノ値ハLehmerノ素數表位ノモノデ届ク程ニ小サクハナラナイ。x_0ガ10100000位ニナレバソレ以下ニツイテ(3)ガ成立スルコトガ證明出來ルカラ（後ニ證明ヲ述ベル）ウマイノデアルガソウ行カナイ。$x \geqq 5$ ニ對シテ $\alpha_1 = 0.918$ ガ成立スルコトハ

$$\vartheta(x) = \sum_{p \leqq x} \log p > ax - \frac{12}{5}a\sqrt{x} - \frac{3}{2}\log^2 x - 13\log x - 15$$

$$\vartheta(x) < \frac{6}{5}ax + 3\log^2 x + 8\log x + 5,$$

$$a = \log \frac{2^{\frac{1}{2}}\, 3^{\frac{1}{3}}\, 5^{\frac{1}{5}}}{30^{\frac{1}{30}}} = 0.92129$$

ヲ用ヒテ証明シタガ，ソノ方法ハ非常ニ初等的デアッテ，ソレデハ如何ニ精密ニ解イテモ ϑ_1 ガ a ヲコエナイ．ソレデ次ニ Math. Zeit. 34 ニ R. Breusch ガ $x \geqq 4$ ニ対シテ

$$\pi\left(\frac{9}{8}x\right) - \pi(x) > 0$$

ヲ証明シテヰルソノ方法ヲ用ヒテミタラ次ノ様ナ結果ガ得ラレタカラソレヲオ知ラセスル．

定理：　$x \geqq 11$ ニ対シテ

$$\pi(x) > 0.93219 \frac{x}{\log x} \tag{5}$$

下ニ之ヲ証明スル．尚 $\psi(x)$, $\Pi(x)$ ハ凡テ Landau ノ Primzahlen ニアルモノデアル．

補助定理 1：　$x \geqq 5$，n ハ $\left[\frac{x}{5}\right]$ ヨリ大ナラザル任意ノ整数ナリトス．シカシ

$$\pi(x)\log x > \psi(x) - 2\psi(\sqrt{x}) + 0.918\frac{n-\log n+2}{n}\frac{x}{\log x}$$

証明

$$\vartheta(x) = \sum_{p \leqq x} \log p = \sum_{i=1}^{n} \sum_{\frac{i-1}{n}x < p \leqq \frac{i}{n}x} \log p$$

$$\leqq \sum_{i=1}^{n}\left\{\pi\left(\frac{i}{n}x\right) - \pi\left(\frac{i-1}{n}x\right)\right\}\log\frac{i}{n}x$$

$$= \pi(x)\log x - \sum_{i=1}^{n-1}\left(\log\frac{i+1}{n}x - \log\frac{i}{n}x\right)\pi\left(\frac{i}{n}x\right)$$

$$= \pi(x)\log x - \sum_{i=1}^{n-1}\log\frac{i+1}{i}\,\pi\left(\frac{i}{n}x\right)$$

$$\pi(x)\log x \geqq \vartheta(x) + \sum_{i=1}^{n-1}\log\frac{i+1}{i}\,\pi\left(\frac{i}{n}x\right)$$

$$> \vartheta(x) + \sum_{i=1}^{n-1}\log\frac{i+1}{i}\cdot 0.918\frac{\frac{i}{n}x}{\log\frac{i}{n}x}$$

$$= \vartheta(x) + 0.918\frac{x}{n}\sum_{i=1}^{n-1}\frac{i\log\frac{i+1}{i}}{\log\frac{i}{n}x}$$

$$\geqq \vartheta(x) + 0.918 \frac{x}{n} \sum_{i=1}^{n-1} \frac{1-\frac{1}{i}}{\log x}$$

$$\geqq \vartheta(x) + 0.918 \frac{n-1-\log n-1}{n} \frac{x}{\log x}$$

シカルニ $\vartheta(x) \geqq \psi(x) - 2\psi(\sqrt{x})$

$$\therefore \quad \pi(x)\log x > \psi(x) - 2\psi(\sqrt{x}) + 0.918 \frac{n-\log n-2}{n} \frac{x}{\log x}$$

補助定理２． $\rho = \frac{1}{2} + i\rho_2$ ヲ $\zeta(s)$ ノ零点トスレバ

$$\frac{2}{3} y^3 \sum_{x<\nu\leqq(1+y)^4 x} a_\nu > x\left[y^4 - (2+y)^4 \cdot 3.32\cdot 10^{-8}\right]$$

$$-\sqrt{x}\left[(2+y)^4 \cdot 3.32\cdot 10^{-8} + 2\sum_{0<\rho_2<200} \frac{|(1+y)^\rho - 1|^4}{|\rho|^4}\right] - \frac{1}{12x^2}$$

但シ $a_\nu = \psi(\nu) - \psi(\nu-1)$

証明 Brunach ノ論文参照． Math. Zeit. 34. S. 506 − 513.

補助定理３、 $\psi((1+y)^4 x) - \psi(x) > x\cdot A(y) - \sqrt{x}\, B(y) - \frac{1}{8x^2 y^3}$ (7)

但シ $A(y) = \frac{3}{2y^3}\left[y^4 - (2+y)^4 \; 3.22\cdot 10^{-8}\right]$ (8)

$B(y) = \frac{3}{2y^3}\left[(2+y)^4 \cdot 3.22\cdot 10^{-8} + 2\sum_{0<\rho_2<200} \frac{|(1+y)^\rho - 1|^4}{|\rho|^4}\right]$ (9)

証明、補助定理２ヨリ明カデアル、

補助定理３ニ於テ $y = \frac{1}{26}, \frac{1}{32}, \frac{1}{28}, \frac{1}{21}$ トオケバ次ノ四式ヲ得ル、

$$\begin{cases} \psi\left(\frac{x}{6}\right) - \psi\left(\frac{x}{7}\right) \geqq \psi\left(\left(1+\frac{1}{26}\right)^4 \frac{x}{7}\right) - \psi\left(\frac{x}{7}\right) > \frac{x}{7} A\left(\frac{1}{26}\right) - \sqrt{\frac{x}{7}}\, B\left(\frac{1}{26}\right) - \frac{7^2\cdot 26}{8x^2} \\ \psi\left(\frac{x}{19}\right) - \psi\left(\frac{x}{17}\right) \geqq \psi\left(\left(1+\frac{1}{32}\right)^4 \frac{x}{17}\right) - \psi\left(\frac{x}{17}\right) > \frac{x}{17} A\left(\frac{1}{32}\right) - \sqrt{\frac{x}{17}}\, B\left(\frac{1}{32}\right) - \frac{17^2\cdot 32^3}{8x^2} \\ \psi\left(\frac{x}{20}\right) - \psi\left(\frac{x}{23}\right) \geqq \psi\left(\left(1+\frac{1}{28}\right)^4 \frac{x}{23}\right) - \psi\left(\frac{x}{23}\right) > \frac{x}{23} A\left(\frac{1}{28}\right) - \sqrt{\frac{x}{23}}\, B\left(\frac{1}{28}\right) - \frac{23^2\cdot 28^3}{8x^2} \\ \psi\left(\frac{x}{24}\right) - \psi\left(\frac{x}{29}\right) \geqq \psi\left(\left(1+\frac{1}{21}\right)^4 \frac{x}{27}\right) - \psi\left(\frac{x}{29}\right) > \frac{x}{29} A\left(\frac{1}{21}\right) - \sqrt{\frac{x}{29}}\, B\left(\frac{1}{21}\right) - \frac{29^2\cdot 21^3}{8x^2} \end{cases}$$ —(10)—

次ニヨク知ラレタ公式

$$\mho(x) = \psi(x) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{10}\right) + \psi\left(\frac{x}{11}\right) - \psi\left(\frac{x}{12}\right) + \psi\left(\frac{x}{13}\right) - \psi\left(\frac{x}{15}\right)$$
$$+ \psi\left(\frac{x}{17}\right) - \psi\left(\frac{x}{18}\right) + \psi\left(\frac{x}{19}\right) - \psi\left(\frac{x}{20}\right) + \psi\left(\frac{x}{23}\right) - \psi\left(\frac{x}{24}\right) + \psi\left(\frac{x}{29}\right) - \psi\left(\frac{x}{30}\right)$$
$$\leqq \psi(x) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{17}\right) - \psi\left(\frac{x}{20}\right) + \psi\left(\frac{x}{23}\right) - \psi\left(\frac{x}{24}\right) + \psi\left(\frac{x}{29}\right)$$

$\mho(x) \geqq ax - 5(\log x \cdots)$ ヨリ

$$\psi(x) \geqq ax - 5(\log x + 1) \quad \left(\frac{x}{6}\right) - \psi\left(\frac{x}{7}\right) + \psi\left(\frac{x}{15}\right) - \psi\left(\frac{x}{17}\right) + \psi\left(\frac{x}{20}\right) - \psi\left(\frac{x}{23}\right)$$
$$+ \psi\left(\frac{x}{24}\right) - \psi\left(\frac{x}{29}\right)$$

ヲ得ル．一方又

$$\psi(x) < \frac{6}{5}ax + (3\log x + 5)(\log x + 1)$$

ナルニヨリ $\quad \psi(\sqrt{x}) < \frac{6}{5}a x + \frac{3}{4}\log^2 x + 4\log x + 5$

依ッテ (6), (10), (13), (14) ヨリ

$$\pi(x)\log x > ax - 5(\log x + 1) + \psi\left(\frac{x}{6}\right) - \psi\left(\frac{x}{7}\right) + \psi\left(\frac{x}{15}\right) - \psi\left(\frac{x}{17}\right) + \psi\left(\frac{x}{20}\right) - \psi\left(\frac{x}{23}\right)$$
$$+ \psi\left(\frac{x}{24}\right) - \psi\left(\frac{x}{29}\right) - 2\left(\frac{6}{5}a\sqrt{x} + \frac{3}{4}\log^2 x + 4\log x + 5\right) + 0.918\,\frac{n - \log n - 2}{n}\,\frac{x}{\log x}$$
$$> ax - 5(\log x + 1) + \frac{x}{7}A\left(\frac{1}{26}\right) - \sqrt{\frac{x}{7}}\,B\left(\frac{1}{26}\right) - \frac{7^2 \cdot 26^3}{8x^2} + \frac{x}{17}A\left(\frac{1}{32}\right)$$
$$- \sqrt{\frac{x}{17}}\,B\left(\frac{1}{32}\right) - \frac{17^2 \cdot 32^3}{8x^2} + \frac{x}{23}A\left(\frac{1}{28}\right) - \sqrt{\frac{x}{23}}\,B\left(\frac{1}{28}\right) - \frac{23^2 \cdot 28^3}{8x^2} + \frac{x}{29}A\left(\frac{1}{21}\right) - \sqrt{\frac{x}{29}}\,B\left(\frac{1}{20}\right)$$
$$- \frac{29^2 \cdot 21^3}{8x^2} + 0.918\,\frac{n - \log n - 2}{n}\,\frac{x}{\log x} - 2\left(\frac{6}{5}a\sqrt{x} + \frac{3}{4}\log^2 x + 4\log x + 5\right)$$
$$= x\left\{a + \frac{1}{7}A\left(\frac{1}{26}\right) + \frac{1}{17}A\left(\frac{1}{32}\right) + \frac{1}{23}A\left(\frac{1}{28}\right) + \frac{1}{29}A\left(\frac{1}{21}\right) + 0.918\,\frac{n - \log n - 2}{n}\,\frac{x}{\log x}\right.$$
$$- \sqrt{x}\left\{\frac{1}{\sqrt{7}}B\left(\frac{1}{26}\right) + \frac{1}{\sqrt{17}}B\left(\frac{1}{32}\right) + \frac{1}{\sqrt{23}}B\left(\frac{1}{28}\right) + \frac{1}{\sqrt{29}}B\left(\frac{1}{21}\right) + \frac{12}{5}a\right\}$$
$$- \left\{\frac{3}{2}\log^2 x + 13\log x + \frac{2973\,2285}{8x^2} + 15\right\}$$

$$\therefore \frac{\pi(x)\log x}{x} > \left\{a + \frac{1}{7}A\left(\frac{1}{26}\right) + \frac{1}{17}A\left(\frac{1}{32}\right) + \frac{1}{23}A\left(\frac{1}{28}\right) + \frac{1}{29}A\left(\frac{1}{21}\right)\right\}$$
$$+ 0.918\,\frac{n - \log n - 2}{n}\,\frac{1}{\log x} - \frac{1}{\sqrt{x}}\left\{\frac{1}{\sqrt{7}}B\left(\frac{1}{26}\right) + \frac{1}{\sqrt{17}}B\left(\frac{1}{32}\right) + \frac{1}{\sqrt{23}}B\left(\frac{1}{28}\right)\right.$$

$+\frac{1}{\sqrt{29}}B\left(\frac{1}{21}\right)+\frac{12}{5}a\Big\}-\Big\{\frac{3}{2}\frac{\log^2 x}{x}+13\frac{\log x}{x}+\frac{29732285}{8x^3}+\frac{15}{x}\Big\}$

(8) ニ依ッテ $A\left(\frac{1}{26}\right), A\left(\frac{1}{32}\right), A\left(\frac{1}{28}\right)$,

$$\left.\begin{aligned} A\left(\tfrac{1}{26}\right) &= 0.042579\cdots\cdot \\ A\left(\tfrac{1}{32}\right) &= 0.019094\cdots\cdot \\ A\left(\tfrac{1}{28}\right) &= 0.034796\cdots\cdot \\ A\left(\tfrac{1}{21}\right) &= 0.063321\cdots\cdot \end{aligned}\right\}$$

終ニ又(9)ニ於テ $y \geqq 21$ ナラバ

$$\begin{aligned} |(1+y)^{\rho}-1|^4 &= \left|(1+y)^{\frac{1}{2}}\cos\overline{\rho_2\log(1+y)}-1+i(1+y)^{\frac{1}{2}}\sin\overline{\rho_2\log(1+y)}\right|^4 \\ &\leqq \left[\left\{\left|(1+y)^{\frac{1}{2}}\cos\overline{\rho_2\log(1+y)}\right|+1\right\}^2+\left\{(1+y)^{\frac{1}{2}}\sin\overline{\rho_2\log(1+y)}\right\}^2\right]^2 \\ &= \left[(1+y)+1+2\left|(1+y)^{\frac{1}{2}}\cos\overline{\rho_2\log(1+y)}\right|\right]^2 \\ &\leqq \left[2+y+2(1+y)^{\frac{1}{2}}\right] < 17 \end{aligned}$$

且ツ $0<\rho_2<200$ ρ_2 値ノ分布ハ

$14 \leqq \rho_2^{(1)} \leqq 20 \leqq \rho_2^{(2)} \leqq 24 \leqq \rho_2^{(3)} \leqq 30 \leqq \rho_2^{(4)},\ \rho_2^{(5)} \leqq 34 \leqq \rho_2^{(6)}$ ヨリ

$\rho_2^{(10)} \leqq 50 \leqq \rho_2^{(11)}$ ヨリ $\rho_2^{(29)} \leqq 100 \leqq \rho_2^{(30)}$ ヨリ $\rho_2^{(79)} < 200$

ナルニヨリ

$$\begin{aligned} \sum_{0<\rho_2<200}\frac{|(1+y)^{\rho}-1|}{|\rho|^4} &< 17\sum_{0<\rho_2<200}\frac{1}{|\rho|^4} \\ &< 17\left\{\frac{1}{14^4}+\frac{1}{20^4}+\frac{1}{24^4}+\frac{2}{30^4}+\frac{5}{34^4}+\frac{17}{50^4}+\frac{1}{16}\right. \\ &< 0.00076585 \end{aligned}$$

之ヨリ $B\left(\frac{1}{26}\right), B\left(\frac{1}{32}\right), B\left(\frac{1}{28}\right), B\left(\frac{1}{21}\right)$ ノ値ヲ計算スレバ

$$\left.\begin{aligned} &B\left(\tfrac{1}{26}\right) < 40.397 \qquad B\left(\tfrac{1}{32}\right) < 75.304 \\ &B\left(\tfrac{1}{28}\right) < 50.624 \qquad B\left(\tfrac{1}{21}\right) < 21.286 \end{aligned}\right\}$$

依ッテ $x > 10^7$ ニ対シテ

$$\frac{1}{\sqrt{x}}\left\{\frac{1}{\sqrt{7}}B\left(\frac{1}{26}\right)+\frac{1}{\sqrt{17}}B\left(\frac{1}{32}\right)+\frac{1}{\sqrt{23}}B\left(\frac{1}{28}\right)+\frac{1}{\sqrt{29}}B\left(\frac{1}{21}\right)+\frac{13}{5}a\right\}$$
$$+\left\{\frac{3}{2}\frac{\log^2 x}{x}+13\frac{\log x}{x}+\frac{29732285}{8x^3}+\frac{15}{x}\right\}<0.0175 \qquad (20)$$

他方 $n=10<\left[\frac{x}{5}\right]$ ヲトレバ

$$0.918\,\frac{n-\log n-2}{n}\,\frac{1}{\log x}>0.032 \qquad (21)$$

依ッテ (15), (16), (20), (21) ヨリ $x>10^7$ ニ対シテ

$$\frac{\pi(x)\log x}{x}>0.93219$$

ヲ得ル．之デ $x>10^7$ ニ対シテ定理ノ成立スルコトガ分ル．$17 \leqq x \leqq 10^7$ ニ対シテ成立スルコトハ次ノ事柄ニ含マレル．

$17 \leqq x < 10100000$ デ (3) ガ成立スル．ソレヲ云フニハ $17 \leqq x_1 \leqq x_2$ デ $\pi(x_1) > \frac{x_2}{\log x_2}$ ガ成立スルナラバ $x_1 \leqq x \leqq x_2$ ナル x ニツイテ (3) ガ成立スルト云フ論法ヲ用ヒレバヨイ．x_1, x_2 トシテハ次ノ数列ヲトレバ十分デアル．

$101\cdot10^5$, $94\cdot10^5$, $88\cdot10^5$, $82\cdot10^5$, $77\cdot10^5$, $72\cdot10^5$, $67\cdot10^5$, $63\cdot10^5$,
$585\cdot10^4$, $545\cdot10^4$, $505\cdot10^4$, $470\cdot10^4$, $435\cdot10^4$, $405\cdot10^4$, $375\cdot10^4$, $350\cdot10^4$,
$325\cdot10^4$, $300\cdot10^4$, $280\cdot10^4$, $260\cdot10^4$, $240\cdot10^4$, $225\cdot10^4$, $210\cdot10^4$, $195\cdot10^4$,
$180\cdot10^4$, $170\cdot10^4$, $160\cdot10^4$, $150\cdot10^4$, $140\cdot10^4$, $130\cdot10^4$, $120\cdot10^4$, $111\cdot10^4$
$102\cdot10^4$, $94\cdot10^4$, $861\cdot10^3$, $788\cdot10^3$, $721\cdot10^3$, $657\cdot10^3$, $600\cdot10^3$, $548\cdot10^3$,
$500\cdot10^3$, $456\cdot10^3$, $415\cdot10^3$, $378\cdot10^3$, $344\cdot10^3$, $313\cdot10^3$, $284\cdot10^3$, $258\cdot10^3$,
$234\cdot10^3$, $212\cdot10^3$, $191\cdot10^3$, $173\cdot10^3$, $156\cdot10^3$, $140\cdot10^3$, $126\cdot10^3$, $114\cdot10^3$,
$103\cdot10^3$, $94\cdot10^3$, 84300, 75500, 67500, 60200, 53800, 48000
42600, 37800, 33600, 29800, 26400, 23200, 21200, 18700,

16500, 14600, 12800, 11200, 9800, 8600, 7500, 6500,
5700, 5000, 3700, 3200, 2700, 2300, 2000, 1700,
1500, 1300, 1100, 930, 800, 660, 560, 470,
400, 340, 280, 230, 200, 170, 140, 110,
90, 73, 61, 47, 41, 37, 31, 29,
23, 19, 17

$11 \leqq x < 17$ デ (5) ガ成立スルコトモ容易ニ分ル. 之デ定理ハ完全ニ証明サレタ.
尚 $\psi\left(\frac{x}{10}\right)-\psi\left(\frac{x}{11}\right)$, $\psi\left(\frac{x}{12}\right)-\psi\left(\frac{x}{13}\right)$ 等ノ項ヲ加ヘレバモットヨイ結果ガ出ルベキデアルガ, $A(y)$ ガ負トナッテ Brensch ノ方法ガ用ヒラレナイ. $B(y)$ ノテ (17) (18) 等ノ計算ガ ρ ノ値ヲ精密ニ吟味スル事ナドニ依ッテ遥ニ精密ナ値ガ出ルカラ之等ノ項ヲ加ヘテモ何等差支ヘナイ. $A(y)$ ガ早ク負ニナッテ之等ノ項ヲヲ加ヘル事ガ意味ヲナサナクナルノハ $(2+y)^4$ ノ係数ガ大キ過ギルカラデアル. シカシ Brensch ノ論文ノ前半ノ大部分ガ之ヲ計算スルノニ用ヒラレテヰルノデスガ尚ヨクナルナラバ当前(4)モヨイ結果デオキカヘラレルワケデアル. 之ハナカナカ容易デハナイト思フ.

トマレモッシヨイ結果ガ出ソウナモノデアルガコンナ方法デハ α_1 ガ 1 ニ等シクナル事ハ絶対ニナイノダカラ止ムヲ得ナイカモシレナイ.

(11月10日 受取)