311. $L_1(0,\infty)$ ノ函數ノ Fourier Transform II.

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$L_1(0,\infty)$ ノ函数ノ conjugate function ノ研究カラ始メル。ソノ方法ハ Titchmarsh (Reciprocal formulae involving series and integrals, Math. Zeits, Bd. 25) ニ倣フ、他ノ方法ヲ用ヘバ或ハモット簡單ニユクカモ知レナイ。

$$\phi(x) = \frac{x}{|\log x|^{1+\varepsilon}+1}$$ トオク。之ハ $0 < x < c_0$ デ monotone increasing 且ツ convex デアル、斯様ナ $c_0 (<1)$ ノ存在ハ 2nd deriative ヲトッテ容易ニ verify スル事ガ出

キル。$\sum_{m=1}^{\infty} a_m$ ヲ absolutely convergent series トスルト次ノ定理ガ得ラレル。

<u>Theorem 3.</u>

$$\sum_{n=1}^{\infty} \frac{\left|\sum_{m=1}^{\infty} \frac{a_m}{m-n+\frac{1}{2}}\right|}{\left|\log\left|\sum_{m=1}^{\infty} \frac{a_m}{m-n+\frac{1}{2}}\right|\right|^{1+\varepsilon}+1} \leqq A \sum_{n=1}^{\infty} |a_n|.$$

proof I ト同ジ notation ヲ用ヒテ

$$\sum_{n=1}^{\infty} \frac{\left|\sum_{m=1}^{\infty} \frac{a_m}{m-n+\frac{1}{2}}\right|}{\left|\log\left|\sum_{m=1}^{\infty} \frac{a_m}{m-n+\frac{1}{2}}\right|\right|^{1+\varepsilon}+1} = \sum_{n=1}^{\infty} \phi\left(\left|\sum_{m=1}^{\infty} \frac{a_m}{m-n+\frac{1}{2}}\right|\right)$$

$$\leqq A \sum_{n=1}^{\infty} \psi\left(\left|\sum_{m=1}^{\infty} \frac{a_m}{m-n+\frac{1}{2}}\right|\right) \leqq A \sum_{n=1}^{\infty} \psi\left(\left|\sum_{m=1}^{\left[\frac{n}{2}\right]} \frac{a_m}{m-n+\frac{1}{2}}\right|\right.$$

$$\left. + \left|\sum_{m=\left[\frac{n}{2}\right]+1}^{n+\left[\frac{n}{2}\right]} \frac{a_m}{m-n+\frac{1}{2}}\right| + \left|\sum_{m=n+\left[\frac{n}{2}\right]+1}^{\infty} \frac{a_m}{n-m+\frac{1}{2}}\right|\right)$$

$$\leqq A \sum_{n=1}^{\infty} \frac{\sum_{m=1}^{\left[\frac{n}{2}\right]} \frac{|a_m|}{\left|m-n+\frac{1}{2}\right|}}{\left|\log \sum_{m=1}^{\left[\frac{n}{2}\right]} \frac{|a_m|}{\left|m-n+\frac{1}{2}\right|}\right|^{1+\varepsilon}+1}$$

$$+ A \sum_{n=1}^{\infty} \frac{\sum_{m=\left[\frac{n}{2}\right]+1}^{n+\left[\frac{n}{2}\right]} \frac{|a_m|}{\left|m-n+\frac{1}{2}\right|}}{\left|\log \sum_{m=\left[\frac{n}{2}\right]+1}^{n+\left[\frac{n}{2}\right]} \frac{|a_m|}{\left|m-n+\frac{1}{2}\right|}\right|^{1+\varepsilon}+1}$$

$$+A\sum_{n=1}^{\infty}\frac{\displaystyle\sum_{m=n+\left[\frac{n}{2}\right]+1}^{\infty}\frac{|a_m|}{\left|n-m+\frac{1}{2}\right|}}{\left|\log\displaystyle\sum_{m=n+\left[\frac{n}{2}\right]+1}^{\infty}\frac{|a_m|}{\left|n-m+\frac{1}{2}\right|}\right|^{1+\varepsilon}+1}$$

$$=S_1+S_2+S_3$$

トオク。

$$S_1\leqq A\sum_{n=1}^{\infty}\frac{\displaystyle\sum_{m=1}^{\infty}|a_m|}{n\left(\left|\log\frac{A}{n}\right|^{1+\varepsilon}+1\right)}\leqq A\sum_{m=1}^{\infty}|a_m|\sum_{n=1}^{\infty}\frac{1}{n(\log n)^{1+\varepsilon}}$$

$$\leqq A\sum_{m=1}^{\infty}|a_m|.$$

$$S_3\leqq A\sum_{n=1}^{\infty}\frac{\displaystyle\sum_{m=n+\left[\frac{n}{2}\right]+1}^{\infty}|a_m|}{n(\log n)^{1+\varepsilon}}\leqq A\sum_{m=1}^{\infty}|a_m|\sum_{n=1}^{\infty}\frac{1}{n(\log n)^{1+\varepsilon}}$$

$$\leqq A\sum_{m=1}^{\infty}|a_m|.$$

次ニ S_2 ヲ計算スル、今 $n=1,2,3,\cdots\cdots$ ノ中 $|na_n|<\frac{1}{2}c_0$ ナル n ノ set ヲ E_1 トシ, $|na_n|\geqq\frac{c_0}{2}$ ナル n ノ set ヲ E_2 トスル。

$$S_2=A\sum_{n=1}^{\infty}\psi\left(\sum_{\substack{m\subset E_1\\ m\subset\left(\left[\frac{n}{2}\right]+1,\,n+\left[\frac{n}{2}\right]\right)}}\qquad+\sum_{\substack{m\subset E_2\\ m\subset\left(\left[\frac{n}{2}\right]+1,\,n+\left[\frac{n}{2}\right]\right)}}\qquad\right)$$

$$\leqq A\sum_{n=1}^{\infty}\phi\left(\sum_{\substack{m\subset E_1\\ m\subset\left(\left[\frac{n}{2}\right]+1,\,n+\left[\frac{n}{2}\right]\right)}}\qquad\right)+A\sum_{n=1}^{\infty}\phi\left(\sum_{\substack{m\subset E_2\\ m\subset\left(\left[\frac{n}{2}\right]+1,\,n+\left[\frac{n}{2}\right]\right)}}\qquad\right)$$

$$=\mathrm{K}_1+\mathrm{K}_2$$

トオク。

今 $m \subset E_1$ ナルトキ $a_m^* = a_m$, 他デハ $a_m^* = 0$ トスル。$\phi(x)$ ハ $0 < x < C_0$ デ convex. 且ツ $|m a_m^*| < \frac{C_0}{2}$ ナル故ニ $|n a_m^*| < C_0$, コヽニ $m \subset \left(\left[\frac{n}{2}\right]+1,\ n+\left[\frac{n}{2}\right]\right)$.

$$\mathrm{K}_1 = A\sum_{n=1}^{\infty} \frac{\displaystyle\sum_{m=\left[\frac{n}{2}\right]+1}^{n+\left[\frac{n}{2}\right]}{}' \frac{|a_m^*|}{\left|m-n+\frac{1}{2}\right|}}{\left|\log \sum \frac{|a_m^*|}{\left|m-n+\frac{1}{2}\right|}\right|^{1+\varepsilon}+1},$$

之ニ Jensen ノ不等式ヲ apply シテ

$$\mathrm{K}_1 \leqq A\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\left[\frac{n}{2}\right]+1}^{n+\left[\frac{n}{2}\right]} \frac{\frac{n|a_m^*|}{\left|m-n+\frac{1}{2}\right|}}{\left|\log n\frac{|a_m^*|}{\left|m-n+\frac{1}{2}\right|}\right|^{1+\varepsilon}+1}$$

$$\leqq A\sum_{n=1}^{\infty}\sum_{m=\left[\frac{n}{2}\right]}^{\infty}\frac{|a_m|}{\left|m-n+\frac{1}{2}\right|\log^{1+\varepsilon} n}$$

$$\leqq A\sum_{m=1}^{\infty}|a_m|\sum_{n=\left[\frac{m}{2}\right]}^{\infty}\frac{1}{\left|m-n+\frac{1}{2}\right|\log^{1+\varepsilon} n} \leqq A\sum_{n=1}^{\infty}|a_m|.$$

又 $m \subset E_2$ ナルトキ $a_m^{**} = a_m$, 他デハ $a_m^{**} = 0$ トスル。ソウスルト

$$\mathrm{K}_2 = A\sum_{n=1}^{\infty}\frac{\left|\displaystyle\sum_{m=\left[\frac{n}{2}\right]+1}^{n+\left[\frac{n}{2}\right]}\frac{a_m^{**}}{m-n+\frac{1}{2}}\right|}{\left|\log\left|\displaystyle\sum_{m=\left[\frac{n}{2}\right]+1}^{n+\left[\frac{n}{2}\right]}\frac{a_m^{**}}{m-n+\frac{1}{2}}\right|\right|^{1+\varepsilon}+1}$$

$$\leqq A\sum_{n=1}^{\infty}\sum_{m=\left[\frac{n}{2}\right]+1}^{n+\left[\frac{n}{2}\right]}\frac{\left|a_m^{**}\right|}{\left|m-n+\frac{1}{2}\right|}$$

$$\leqq A\sum_{m=1}^{\infty}\left|a_m^{**}\right|\sum_{\nu=1}^{m}\frac{1}{\nu}\leqq A\sum_{\substack{m=1\\ m\subset E_2}}^{\infty}\frac{A_m^{**}}{m}\leqq A\sum_{m=1}^{\infty}A_m^{**}\left|a_m\right|$$

$$\leqq A\sum_{m=1}^{\infty}\left|a_m\right|,$$

コヽニ $A_m^{**}=\sum_{n=1}^{m}\left|a_m^{**}\right|$. 之デ *theorem* ガ証明サレタ。

<u>*Lemma* 2.</u> $0\leqq X_1\leqq X_2<X_3\leqq X_4$ トシ

$$|f(t)|\leqq M \quad \text{for} \quad X_1\leqq t\leqq X_4,$$
$$f(t)=a \quad \text{for} \quad X_2\leqq t\leqq X_3,$$
$$f(t)=0 \quad \text{for} \quad t<X_1,\ t>X_4.$$

且ツ $X_2+\delta<x<X_3-\delta$, $\nu x<k\leqq \nu x+1$, $\nu>\frac{3}{\delta}$ トスル。ν, k ハ正ノ整数デアル。

今 $m_i=\nu\int_{\frac{i-1}{\nu}}^{\frac{i}{\nu}}f(t)\,dt$ トオケバ

$$\left|\int_0^{\infty}\frac{f(t)}{t-x}dt-\sum_{i=1}^{\infty}\frac{m_i}{i-k-\frac{1}{2}}\right|<\frac{6M}{\nu\delta-3}$$

proof. $\left|x-\frac{k\pm N}{\nu}\right|<\delta$ トスル。 $N+1\leqq\nu\delta$ ナラバ之ハ満足サレル。サテ $\frac{k-N}{\nu}<t<\frac{k+N}{\nu}$ デハ $f(t)=a$ デアルカラ

$$\left|\sum_{i=k-N}^{k+N}\frac{m_i}{i-k-\frac{1}{2}}\right|=a\left|\sum_{i=k-N}^{k+N}\frac{1}{i-k-\frac{1}{2}}\right|$$

$$= \frac{a}{N+\frac{1}{2}} < \frac{a}{\nu\delta-3} \leqq \frac{M}{\nu\delta-3},$$

又 $$\int_{\frac{k-N}{\nu}}^{\frac{k+N}{\nu}} \frac{f(t)}{t-x}dt = a\int_{x-\frac{k-N}{\nu}}^{\frac{k+N}{\nu}-x} \frac{dy}{y} \leqq \frac{2a\left(\frac{k}{\nu}-x\right)}{x-\frac{k-N}{\nu}}$$

$$\leqq \frac{2a}{N-1} \leqq \frac{2M}{\nu\delta-3}.$$

故＝ (1) $$\left|\int_{\frac{k-N}{\nu}}^{\frac{k+N}{\nu}} \frac{f(t)}{t-x}dt - \sum_{i=k-N}^{k+N} \frac{m_i}{i-k-\frac{1}{2}}\right| < \frac{3M}{\nu\delta-3}.$$

又 $i \geqq k+N+1$ トシテ

$$\left|\int_{\frac{i-1}{\nu}}^{\frac{i}{\nu}} \frac{f(t)}{t-x}dt - \frac{m_i}{i-\nu x-\frac{1}{2}}\right| = \left|\int_{\frac{i-1}{\nu}}^{\frac{i}{\nu}} \frac{f(t)}{t-x}dt - \frac{\nu}{i-\nu x-\frac{1}{2}}\int_{\frac{i-1}{\nu}}^{\frac{i}{\nu}} f(t)dt\right|$$

$$= \left|\int_{\frac{i-1}{\nu}}^{\frac{i}{\nu}} \left(\frac{1}{t-x} - \frac{1}{\frac{i-\frac{1}{2}}{\nu}-x}\right) f(t)dt\right| \leqq \left|\int_{\frac{i-1}{\nu}}^{\frac{i}{\nu}} \frac{\frac{i-\frac{1}{2}}{\nu}-t}{(t-x)\left(\frac{i-\frac{1}{2}}{\nu}-x\right)} f(t)dt\right|$$

$$\leqq \frac{M}{2\nu}\int_{\frac{i-1}{\nu}}^{\frac{i}{\nu}} \frac{dt}{\left(t-x-\frac{1}{2\nu}\right)^2}.$$

故＝ (2) $$\left|\int_{\frac{k+N}{\nu}}^{\infty} \frac{f(t)}{t-x}dt - \sum_{i=k+N+1}^{\infty} \frac{m_i}{i-\nu x-\frac{1}{2}}\right| \leqq \frac{M}{2\nu}\int_{\frac{k+N}{\nu}}^{\infty} \frac{dt}{\left(t-x-\frac{1}{2\nu}\right)^2}$$

$$= \frac{M}{2\left(k+N-\nu x-\frac{1}{2}\right)} < \frac{M}{2(N-1)} < \frac{M}{2(\nu\delta-3)}.$$

又 (3) $\left|\sum_{i=k+N+1}^{\infty}\left(\frac{m_i}{i-\nu x-\frac{1}{2}}-\frac{m_i}{i-k-\frac{1}{2}}\right)\right|<\frac{M}{\nu\delta-3}.$

故ニ (2)(3) カラ

$$\left|\int_{\frac{k+N}{\nu}}^{\infty}\frac{f(t)}{t-x}dt-\sum_{i=k+N+1}^{\infty}\frac{m_i}{i-k-\frac{1}{2}}\right|<\frac{3M}{2(\nu\delta-3)}.$$

又 $\left|\int_0^{\frac{k-N}{\nu}}\frac{f(t)}{t-x}dt-\sum_{i=1}^{k-N}\frac{m_i}{i-\nu x-\frac{1}{2}}\right|$

$$=\left|\int_0^{\frac{k-N}{\nu}}\frac{f(t)}{t-x}dt-\sum_{i=1}^{k-N}\frac{1}{(i-\frac{1}{2})\frac{1}{\nu}-x}\int_{\frac{i-1}{\nu}}^{\frac{i}{\nu}}f(t)\,dt\right|,$$

且ツ

$$\left|\int_{\frac{i-1}{\nu}}^{\frac{i}{\nu}}\frac{f(t)}{t-x}dt-\frac{m_i}{i-\nu x-\frac{1}{2}}\right|<\frac{M}{2\nu}\int_{\frac{i-1}{\nu}}^{\frac{i}{\nu}}\frac{dt}{(t-x-\frac{1}{2\nu})^2},$$

故ニ $\left|\int_0^{\frac{k-N}{\nu}}\frac{f(t)}{t-x}dt-\sum_{i=1}^{k-N}\frac{m_i}{i-\nu x-\frac{1}{2}}\right|<\frac{M}{2\nu}\int_0^{\frac{k-N}{\nu}}\frac{dt}{\left(t-x-\frac{1}{2\nu}\right)^2}<\frac{M}{\nu\delta}.$

故ニ以上ノ計算ヲ綜合シテ

$$\left|\int_0^{\infty}\frac{f(t)}{t-x}dt-\sum_{i=1}^{\infty}\frac{m_i}{i-k-\frac{1}{2}}\right|<\frac{6M}{\nu\delta-3}. \qquad q.e.d.$$

次ニ $f(t)=a_\mu$ for $0\leqq x_\mu\leqq t<x_{\mu+1}$ $(\mu=1,2,\cdots\cdots m-1)$.

$f(t)=0$ for $0\leqq t<x_1,\quad t>x_m.$

今 $0\leqq x_0<x_1,\ x_{m+1}>x_m$ トシ E ヲ $(x_\mu+\delta, x_{\mu+1}-\delta)$ ヨリ成ル set トシ E′ ヲ (x_0, x_{m+1}) ニ関スル E ノ comple-

mentary set トスル。

$$I=\int_0^\infty \frac{\left|\int_0^\infty \frac{f(t)}{t-x}dt-\sum_{i=1}^\infty \frac{m_i}{i-k-\frac{1}{2}}\right|}{\left|\log\left|\int_0^\infty \frac{f(t)}{t-x}dt-\sum_{i=1}^\infty \frac{m_i}{i-k-\frac{1}{2}}\right|\right|^{1+\varepsilon}+1}dx,$$

$$m_i=\nu\int_{\frac{i-1}{\nu}}^{\frac{i}{\nu}} f(t)\,dt$$

トスル。 $I=\int_E+\int_{E'}+\int_0^{x_0}+\int_{x_{m+1}}^\infty=I_1+I_2+I_3+I_4$

トオク。*Lemma* 2 デ $X_1=x_0$, $X_2=x_\mu$, $X_3=x_{\mu+1}$, $X_4=x_{m+1}$ トオクト

$$\int_{x_\mu+\delta}^{x_{\mu+1}-\delta} \frac{\left|\int_0^\infty \frac{f(t)}{t-x}dt-\sum_{i=1}^\infty \frac{m_i}{i-k-\frac{1}{2}}\right|}{\left|\log\left|\int_0^\infty \frac{f(t)}{t-x}dt-\sum_{i=1}^\infty \frac{m_i}{i-k-\frac{1}{2}}\right|\right|^{1+\varepsilon}+1}dx$$

$$\leqq A\int_{x_\mu+\delta}^{x_{\mu+1}-\delta}\varphi\left(\left|\int_0^\infty \frac{f(t)}{t-x}dt-\sum_{i=1}^\infty \frac{m_i}{i-k-\frac{1}{2}}\right|\right)dx$$

$$\leqq A\int_{x_\mu+\delta}^{x_{\mu+1}-\delta}\varphi\left(\frac{6M}{\nu\delta-3}\right)dx$$

$$\leqq A\phi\left(\frac{6M}{\nu\delta-3}\right)(x_{\mu+1}-x_\mu-2\delta)$$

$$=A\frac{\frac{M}{\nu\delta-3}}{\log^{1+\varepsilon}\frac{6M}{\nu\delta-3}+1}(x_{\mu+1}-x_\mu-2\delta).$$

故ニ

$$\left|I_1\right| \leqq \frac{\frac{AM}{\nu\delta-3}}{\left(\log\frac{6M}{\nu\delta-3}\right)^{1+\varepsilon}+1}(x_{m+1}-x_0) \leqq A(x_{m+1}-x_0)\frac{1}{\nu\delta(\log^{1+\varepsilon}\nu\delta+1)}$$

$$\left|I_2\right| \leqq A\int_{E'}\phi\left(\left|\int_0^\infty \frac{f(t)}{t-x}dt-\sum_{i=1}^\infty \frac{m_i}{i-k-\frac{1}{2}}\right|\right)dx$$

$$\leqq A\int_{E'}\phi\left(\left|\int_0^\infty \frac{f(t)}{t-x}dt\right|\right)dx+A\int_{E'}\phi\left(\left|\sum_{i=1}^\infty \frac{m_i}{i-k-\frac{1}{2}}\right|\right)dx$$

$\int_0^\infty \frac{f(t)}{t-x}dt$ ハ finite interval デ ϕ ニ関シテ integrable デアル（何者, logarithmic infinity ノ有限個ノミヲ持ツカラ）。故ニ $\delta \to 0$ ノトキ第一項ハ 0 ニ tend スル。又

$$\left|\sum_{i=1}^\infty \frac{m_i}{i-k-\frac{1}{2}}\right| < 4M\log\{2\nu(x_m-x_1)+4\}$$

（何者, $m_i=0$ for $i<\nu x_1$, $i>1+\nu x_m$）. 故ニ

$$\left|I_2\right| \leqq o(1)+\frac{AM\log\{2\nu(x_m-x_1)+4\}m\delta}{\left|\log|\log\{2\nu(x_m-x_1)+4\}|\right|^{1+\varepsilon}+1}$$

$$\leqq Am\delta\log 2\nu+o(1), \quad (\delta\to 0).$$

$$\left|I_3\right| < A\int_0^{x_0}\phi\left(\left|\int_0^\infty \frac{f(t)}{t-x}dt\right|\right)dx+A\int_0^{x_0}\phi\left(\left|\sum_{i=1}^\infty \frac{m_i}{i-k-\frac{1}{2}}\right|\right)dx.$$

然ルニ

$$\int_0^\infty \frac{f(t)}{t-x}dt \leqq M\int_{x_1}^{x_m}\frac{dt}{t-x} \leqq M\frac{x_m-x_1}{x_1-X}.$$

$$\left|\sum_{i=1}^{\infty}\frac{m_i}{i-k-\frac{1}{2}}\right| < M\,\frac{\nu x_m-\nu x_1+1}{\nu x_1-\nu x-2}\,.$$

故ニ

$$|I_3| \leqq \frac{A\left|\dfrac{x_m-x_1}{x_1-x_0}\right|}{\left|\log\dfrac{x_m-x_1}{x_1-x}\right|^{1+\varepsilon}+1}+\frac{A\,\dfrac{\nu x_m-\nu x_1+1}{\nu x_1-\nu x-2}}{\left|\log\dfrac{\nu x_m-\nu x_1+1}{\nu x_1-\nu x-2}\right|^{1+\varepsilon}+1}$$

$$\leqq A\,\frac{1}{x_1-x_0}\;\;\frac{1}{\left|\log\dfrac{x_m-x_1}{x_1-x}\right|^{1+\varepsilon}+1}\leqq\frac{A}{|\log x_0|^{1+\varepsilon}}$$

同様ニ $\quad |I_4| \leqq \dfrac{A}{|\log x_m|^{1+\varepsilon}}\,.$

上ノ計算ヲ綜合シテ

$$|I| \leqq A\,\frac{x_m-x_0}{\nu\delta(\log^{1+\varepsilon}\nu\delta+1)}+O(1)+A\delta\log\nu+\frac{A}{|\log x_0|^{1+\varepsilon}}$$

$$+\frac{A}{|\log x_m|^{1+\varepsilon}}\,.$$

今 $\quad \delta=\dfrac{1}{\sqrt{\nu}}$， $x_0=\dfrac{1}{\sqrt{\nu}}$， $x_{m+1}=\sqrt{\nu}$ 　トスルト $x_{m+1}-x_0=O(\sqrt{\nu})$，

故ニ

$$|I| \leqq \frac{A}{\log^{1+\varepsilon}\nu}+O(1)+\frac{A\log\nu}{\sqrt{\nu}}+\frac{A}{\log^{1+\varepsilon}\nu}+\frac{A}{\log^{1+\varepsilon}\nu}\longrightarrow 0.$$

故ニ

$$(4)\quad \lim_{\nu\to\infty}\int_0^{\infty}\phi\left(\left|\int_0^{\infty}\frac{f(t)}{t-x}dt-\sum_{i=1}^{\infty}\frac{m_i}{i-k-\frac{1}{2}}\right|\right)dx=0.$$

<u>*Lemma 3.*</u> $\quad \chi(x)$ ヲ *continuous monotone*

increasing function トシ $\chi(0)=0$, $\chi(2x) \leqq A\chi(x)$ ナラバ

$$\lim_{m\to\infty}\int_0^\infty \chi(f_m(x)-f(x))dx$$

ナラバ

$$\lim_{m\to\infty}\int_0^\infty \chi(f_m(x))dx=\int_0^\infty \chi(f(x))dx.$$

コレハ良ク知ラレテ居ル。(拙著, 数物會誌, 綜合報告).

コノ Lemma ト (4) カラ

$$\int_0^\infty \phi\left(\left|\int_0^\infty \frac{f(t)}{t-x}dt\right|\right)dx=\lim_{\nu\to\infty}\int_0^\infty \phi\left(\left|\sum_{i=1}^\infty \frac{m_i}{i-k-\frac{1}{2}}\right|\right)dx$$

Theorem 3 ニヨリ

$$\int_0^\infty \phi\left(\left|\sum_{i=1}^\infty \frac{m_i}{i-k-\frac{1}{2}}\right|\right)dx=\frac{1}{\nu}\sum_{k=1}^\infty \phi\left(\left|\sum_{i=1}^\infty \frac{m_i}{i-k-\frac{1}{2}}\right|\right)$$

$$\leqq \frac{1}{\nu}A\sum_{i=1}^\infty |m_i|$$

故ニ

$$\int_0^\infty \phi\left(\left|\int_0^\infty \frac{f(t)}{t-x}dt\right|\right)dx \leqq A\sum_{i=1}^\infty \int_{\frac{i-1}{\nu}}^{\frac{i}{\nu}} |f(t)|\,dt=A\int_0^\infty |f(t)|dt.$$

Theorem 4. $f(t) \subset L_1(0,\infty)$ トシソノ conjugate function ヲ $g(t)$ トスルト

$$\int_0^\infty \frac{|g(t)|}{\left|\log|g(t)|\right|^{1+\varepsilon}+1}dt \leqq A\int_0^\infty |f(t)|\,dt.$$

Proof. 上述ノコトカラ argument ガ充分大ナルトキ 0 デアルヤウナ step function ニ付テコノ Theorem ガ証明サレタ。次ニ一般ノ場合ヲ証明スル。$f(t) \subset L_1(0,\infty)$. ソウスルト step functions ノ sequence $f_n(x)$ ガアツテ

$$\lim_{n\to\infty}\int_0^\infty |f(x)-f_n(x)|\,dx = 0.$$

サテ I ノ Theorem 1 カラ，N ヲ fixed constant トスルト（Theorem ハ $(0, 2\pi)$ トシテヤツテキルガ finite interval デヨイ）

$$\int_0^N \phi\left(\left|\int_0^N \frac{f_n(t)}{t-x}\,dt - \int_0^N \frac{f_m(t)}{t-x}\,dt\right|\right)dx$$
$$\leqq A\int_0^N |f_n(t)-f(t)\,dt \longrightarrow 0 \quad (n\to\infty)$$

故ニヨク知ラレタ mean convergence ノ Theorem カラ (n_i) ナル (n) ノ subsequence ガ exist シテ $(0,N)$ ノ殆ンドスベテノ x ニ付テ

$$\lim_{i\to\infty}\int_0^N \frac{f_{n_i}(t)}{t-x}\,dt = \int_0^N \frac{f(t)}{t-x}\,dt$$

$x < N-\delta\ (\delta>0)$, $\displaystyle\int_N^\infty \frac{f_{n_i}(t)-f(t)}{t-x}\,dt \leqq \frac{1}{\delta}\int_N^\infty |f_{n_i}(t)-f(t)|\,dt \to 0.$

故ニ $(0,N)$ ノ殆ンドスベテノ x ニ付テ

$$(5)\quad \lim_{i\to\infty}\int_0^\infty \frac{f_{n_i}(t)}{t-x}\,dt = \int_0^\infty \frac{f(t)}{t-x}\,dt.$$

ソウスルト diagonal method ニヨリ一ツノ (n'_i) ナル seq ガアッテ，$(0,\infty)$ ノ殆ドスベテノ x ニ對シテ

$$(6)\quad \lim_{i\to\infty}\int_0^\infty \frac{f_{n'_i}(t)}{t-x}dt=\int_0^\infty \frac{f(t)}{t-x}dt$$

サテ $f_{n'_k}(t)-f_{n'_i}(t)$ ハ step function デアルカラ

$$\int_0^\infty \phi\left(\left|\int_0^\infty \frac{f_{n'_k}(t)}{t-x}dt-\int_0^\infty \frac{f_{n'_i}(t)}{t-x}dt\right|\right)dt$$

$$\leqq A\int_0^\infty \left|f_{n'_k}(t)-f_{n'_i}(t)\right|dt\to 0\quad (i,k\to\infty).$$

故ニ

$$\int_0^\infty \phi\left(\left|F(x)-\int_0^\infty \frac{f_{n'_i}(t)}{t-x}dt\right|\right)dt\to 0$$

ナル ϕ ニ關シテ integrable ナ $F(x)$ ガアル。(6) カラ

$$F(x)=\int_0^\infty \frac{f(t)}{t-x}dt \text{ (almost everywhere).}$$

故ニ

$$\lim_{i\to\infty}\int_0^\infty \phi\left(\left|\int_0^\infty \frac{f_{n'_i}(t)}{t-x}dt\right|\right)dx=\int_0^\infty \phi\left(\left|\int_0^\infty \frac{f(t)}{t-x}dt\right|\right)dx.$$

且ツ $$\int_0^\infty \phi\left(\left|\int_0^\infty \frac{f_{n'_i}(t)}{t-x}dt\right|\right)dx\leqq A\int_0^\infty |f(t)|dt$$

之デ Theorem ガ完全ニ證明サレタ。