Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N}=4$ gauge theories

based on joint work with A.Braverman, M.Finkelberg

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Shanghai Conference on Representation Theory

20151210

Fake Mathematical motivation

☆ Realize interesting (non)commutative algebras by convolution algebras

$$M \longrightarrow X \qquad\qquad Z = M \times_X M \qquad\qquad H_*^G(Z)$$

$$G$$

e.g.
- Steinberg variety $\longrightarrow$ affine Hecke algebra
- Lusztig's canonical base $\longrightarrow$ KLR algebra
- quiver variety $\longrightarrow$ quantum loop algebra

Other examples ? ◦ S. Kato exotic cone $\rightarrow$ affine Hecke alg with unequal param.

Unfortunately not much .....

Why ?, Lots of examples of M, X, but difficult to identify $H_*^G(Z)$ usually

Today A new class of examples, for which we have (expected) answers from physicists' research !

From now I ignore physics any more,

— explain the construction,

and — give (expected) answers without any explanation.

§1. Construction

G : complex reductive group

$\mathbf{N}$: representation

$\mathcal{O} = \mathbb{C}[[z]] \subset \mathcal{K} = \mathbb{C}((z))$

$\mathrm{Gr}_G := G_{\mathcal{K}} / G_{\mathcal{O}}$: affine Grassmannian

$\mathcal{T} := G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathbf{N}_{\mathcal{O}}$: ∞-rank vector bundle over Gr_G

$$\begin{array}{l} G_{\mathcal{K}} \circlearrowright \\ \Pi : \mathcal{T} \longrightarrow \mathbf{N}_{\mathcal{K}} \\ \quad\; \rotatebox{90}{$\in$} \qquad\quad \rotatebox{90}{$\in$} \\ \;[g, s] \longmapsto gs \end{array}$$

"analog of Springer resolution"

$T^*\mathrm{flag} \longrightarrow$ nilpotent cone

Fake Def. $St := \mathcal{T} \times_{N_K} \mathcal{T}$ and consider $H_*^{G_K}(St)$ + convolution product

loop rotation $\mathbb{C}^\times \curvearrowright \mathcal{O}, K, \mathrm{Gr}_G$ etc. $H_*^{G_K \rtimes \mathbb{C}^\times}(St)$ can be also considered.

<u>Remark</u> We cannot make sense of $H_*^{G_K}(St)$ + convolution product rigorously.

Therefore we need to replace $St \curvearrowleft G_K$ by

$\mathcal{R} = \{([g,s],[1,gs]) \in St\} \curvearrowleft G_\mathcal{O}$ instead. $\left(G_K \backslash St \cong G_\mathcal{O} \backslash \mathcal{R} \right)$

$$\boxed{\begin{aligned} \mathcal{A} &:= \left(H_*^{G_\mathcal{O}}(\mathcal{R}) \ \text{convolution product} \right) \\ \mathcal{A}_\hbar &:= \left(H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R}) + \ \text{''} \ \right) \end{aligned}}$$

<u>NB</u> $H^*_{\mathbb{C}^\times}(pt) = \mathbb{C}[\hbar]$

"Planck constant"

Properties 1) $\mathcal{A}$ is commutative & $\mathcal{A}_\hbar$ is its quantization
((flat) noncommutative deformation)

$\Longrightarrow$ $\mathcal{A} = \mathcal{A}_\hbar|_{\hbar=0}$ is a Poisson algebra $\{f, g\} = \lim_{\hbar \to 0} \dfrac{fg - gf}{\hbar}$

2) $\mathcal{A}$ is integral, finitely generated and normal.

$\therefore$ $\mathcal{M}_C := \operatorname{Spec} \mathcal{A}$: normal affine algebraic variety

3) $\mathcal{A}_\hbar \supset H^*_{G \times \mathbb{C}^\times}(pt) = \mathbb{C}[\mathfrak{g}]^G \otimes \mathbb{C}[\hbar]$ commutative subalg.

$\rightsquigarrow \varpi : \mathcal{M}_C \longrightarrow \mathfrak{g}/\!/G = \mathfrak{t}/W$ Poisson commuting

integrable system

generic fiber $= T^\vee$
dual of max. torus

In particular, $\dim \mathcal{M}_C = 2 \dim \mathfrak{t} = \operatorname{rank} \mathfrak{g}$

4) $\mathcal{M}_C \underset{\text{birational}}{\approx} (T^\vee \times \mathfrak{t})/W$

$\varpi \searrow$ $\mathfrak{t}/W$ $\swarrow$ 2nd projection

5) (flavor symmetry
in physics literature) Suppose $\mathbb{N}$ is a representation of $\tilde{G} \rhd G$

$\Rightarrow H^*_{\tilde{G}_\Theta}(\mathcal{R})$ is a deformation (parameter space $= H^*_{\tilde{G}/G}(pt)$)

Remark

$\mathbb{M} = \mathbb{N} \oplus \mathbb{N}^*$: symplectic representation of G

$\rightsquigarrow \mathcal{M}_H = \mathbb{M} /\!/\!/ G = \mu^{-1}(0) /\!/ G$ Hamiltonian (symplectic) reduction

quantization = quantum Hamiltonian reduction of $\mathrm{Diff}(\mathbb{N})$ by G

- In many situation, it is equipped with an integrable system.
- All known examples of "symplectic dual" are $\mathcal{M}_H$ vs $\mathcal{M}_C$.

[Braden-Licata-Proudfoot-Webster]

(This further helps to understand $\mathcal{M}_C$)

§2. (Expected) answers

☆ $\mathbf{N}=0$ [Bezrukavnikov-Finkelberg-Mirković '05]

$\mathcal{A}_\hbar =$ Toda lattice , $\mathcal{M}_C = T^*G^\vee /\!\!/\!\!/_{\chi\chi\chi} N^\vee \times N^\vee$

☆ $\mathbf{N}=\mathfrak{g}$ [BFM '05]

(cf. [Vasserot 05]
[Varagnolo-Vasserot '10]

$\mathcal{M}_C = \mathfrak{t} \times T^\vee / W$

$\mathcal{A}_\hbar \stackrel{?}{=}$ spherical part of the graded Cherednik DAHA

☆ $G=T$: torus $\Rightarrow$ $\mathcal{A}_\hbar$, $\mathcal{M}_C$ have linear basis + explicit structure constants

e.g. $\mathcal{M}_C$: toric hyperkähler manifold
(symplectic dual of $\mathcal{M}_H = \mathbf{N} \oplus \mathbf{N}^* /\!\!/\!\!/ G$)

☆ quiver gauge theory

$Q = (Q_0, Q_1)$: quiver

V, W : Q_0-graded vector spaces

$$(\mathbf{N}, G) = \left(\begin{Bmatrix} \text{framed quiver} & \text{linear} \\ \text{representations} & \text{+ bases} \end{Bmatrix}, \ \begin{matrix} \text{change} \\ \text{of basis} \end{matrix} \right)$$

$$= \left(\bigoplus_{h \in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}(W_i, V_i) \ , \ \prod_{i \in Q_0} GL(V_i) \right)$$

<u>Rem</u> $\mathcal{M}_H$ = quiver variety

☆ Suppose Q is of type ADE

$$W = 0 \ \Rightarrow \ \mathcal{M}_C = \{ \text{based maps } f : \underset{\infty}{\mathbb{P}^1} \longrightarrow \underset{1}{\text{flag variety}} \text{ of type ADE} \}$$

$$\infty \longmapsto 1$$

$$\deg f = \overrightarrow{\dim} V$$

$\mathcal{A}_h \overset{?}{=}$ quotient of $Y^{\geq 0}$ (Borel part of the Yangian)
of type ADE

$W \neq 0$ and assume $\mu := \dim W - C \cdot \dim V \in \mathbb{Z}_{\geq 0}^{I}$

$\mathcal{M}_C =$ slice of $\mathrm{Gr}^{\mu}_{G_{ADE}}$ in $\overline{\mathrm{Gr}^{\lambda}_{G_{ADE}}}$ $\qquad$ Gr^{λ} : $G_{ADE}[[z]]$- orbit in Gr corresponding to cochar λ

When μ: not dominant similar description

☆ Suppose Q: Jordan quiver
$$\begin{tikzcd} V \arrow[loop right] \\ W \arrow[u] \end{tikzcd}$$

Let $\dim V = k$, $\dim W = n$ $\Longrightarrow$ $\mathcal{M}_C \cong S^k(\mathbb{C}^2/\mathbb{Z}_n)$

$\mathcal{A}_\hbar \overset{?}{\cong}$ spherical part of the symplectic reflection algebra for the wreath product $\mathfrak{S}_k \ltimes (\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n)$

affine quiver $\rightsquigarrow$ $\mathcal{M}_C \overset{?}{\cong} G_{ADE}$ -instanton moduli space for multi-Taub NUT space

(Jordan quiver $\leftrightarrow$ $G_{ADE} = \mathbb{C}^*$)

$\mathcal{A}_\hbar \cong ???$

§3. even more general construction

Recall $\mathcal{A} = H^{G_\mathcal{O}}_*(\mathcal{R})$ and we have $\pi : \mathcal{R} \longrightarrow \mathrm{Gr}_G \xrightarrow{p} \mathrm{pt}$

$\therefore \quad \mathcal{A} = H^*_{G_\mathcal{O}}(p_* \pi_* \omega_\mathcal{R}) \qquad \omega_\mathcal{R}$ = dualizing cpx on $\mathcal{R}$

Consider $\pi_* \omega_\mathcal{R} \in D_{G_\mathcal{O}}(\mathrm{Gr}_G)$

tensor category under the convolution product $\star$

The convolution product on $\mathcal{A}$ is induced from

$m : \pi_* \omega_\mathcal{R} \star \pi_* \omega_\mathcal{R} \longrightarrow \pi_* \omega_\mathcal{R}$ satisfying

- commutativity
- associativity

i.e. comm. ring object in $D_{G_\mathcal{O}}(\mathrm{Gr}_G)$

Conversely if we have $\mathcal{C} \in D_{G_\mathcal{O}}(\mathrm{Gr}_G)$ and $m : \mathcal{C} \star \mathcal{C} \to \mathcal{C}$
commutative ring object

$\Longrightarrow \qquad H^*_{G_\mathcal{O}}(\mathrm{Gr}_G, \mathcal{C})$: commutative algebra

<u>Example</u>

$$(\mathrm{Perv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G), \star) \underset{\text{geometric Satake}}{\cong} (\mathrm{Rep}\, G^\vee, \otimes)$$

$$\begin{tikzcd}
\mathrm{Perv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G) \arrow[d, phantom, "\cap"] & \\
D_{G_{\mathcal{O}}}(\mathrm{Gr}_G) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \mathrm{Rep}\, G^\vee \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\mathcal{C}_{ABG} & \mathbb{C}[G^\vee] \arrow[l, mapsto]
\end{tikzcd}$$

$\mathbb{C}[G^\vee]$: regular rep.

[Arkhipov – Bezrukavnikov – Ginzburg '04]

$$\begin{array}{ccccc}
\mathrm{Ext}^*_{D_{G_{\mathcal{O}}}(\mathrm{Gr}_G)}(\overset{\substack{\text{skyscraper sheaf at the origin} \\ \in}}{1_{\mathrm{Gr}}}, & \mathcal{C}_{ABG}) & \cong & \mathbb{C}[\mathfrak{g}^{\vee *}] \\
\uparrow & \uparrow & & \uparrow \\
H^*_G(pt) & H^*_G(pt) & \cong \mathbb{C}[\mathfrak{t}/W] \cong & \mathbb{C}[\mathfrak{t}^{\vee *}/W]
\end{array}$$

<u>NB</u> $H^*(\mathrm{Gr}, \mathcal{R}) \cong \mathbb{C}[G^\vee]$ by geom. Satake

○ gluing $\mathcal{C}_1, \mathcal{C}_2, \cdots$: commutative ring objects

$$\Rightarrow \mathcal{C}_1 \overset{!}{\otimes} \mathcal{C}_2 \overset{!}{\otimes} \cdots = i_\Delta^!(\mathcal{C}_1 \boxtimes \mathcal{C}_2 \boxtimes \cdots) \qquad i_\Delta : \mathrm{Gr}_G \to \mathrm{Gr}_G \times \mathrm{Gr}_G \times \cdots$$

is also a comm. ring object (cf. Ginzburg – Kazhdan)