SUPPLEMENTARY MATERIAL TO "EFFICIENCY OF ESTIMATORS FOR LOCALLY ASYMPTOTICALLY NORMAL QUANTUM STATISTICAL MODELS"

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APPENDIX A: ASYMPTOTIC REPRESENTATION THEOREM FOR CLASSICAL LAN

This section gives a comprehensible proof of the asymptotic representation theorem for classical LAN models (Theorem 1.1). This also provides an alternative view for the 'randomized' statistics appeared in the theorem.

In constructing a statistic T that enjoys $T^{(n)} \overset{h}{\rightsquigarrow} T$ for all h, van der Vaart [1] emphasized that one must invoke external information. This prescription reminds us of a quantum POVM in which one makes use of an ancillary system in realizing it. In what follows, therefore, we identify the randomized statistic T with a σ-finite measure on $\mathbb{R}^s \times \mathbb{R}^d$ that gives the desired limit distribution $\mathcal{L}_h$ for every $h \in \mathbb{R}^d$.

PROOF. For each $t \in \mathbb{R}^s$, let

$$M^{(n)}(t,\omega) := \mathbb{1}_{T^{(n)-1}((-\infty,t])}(\omega), \qquad (n \in \mathbb{N},\ \omega \in \Omega^{(n)}). \tag{A.1}$$

Referring to the diagram

$$\begin{array}{ccccc} & & \Omega^{(n)} & \xrightarrow{\Delta^{(n)}} & \mathbb{R}^d \\ [0,1] & \xleftarrow{P^{(n)}_{\theta_0}} & \mathcal{F}^{(n)} & \xleftarrow{\Delta^{(n)-1}} & \mathcal{B}(\mathbb{R}^d) \end{array},$$

we define, for each $t \in \mathbb{R}^s$, a finite Borel measure $\mu_t^{(n)}$ on $\mathbb{R}^d$ as follows:

$$\mu_t^{(n)}(B) := \int_{\Delta^{(n)-1}(B)} M^{(n)}(t,\omega) dP^{(n)}_{\theta_0}(\omega), \qquad (B \in \mathcal{B}(\mathbb{R}^d)). \tag{A.2}$$

Note that the set $\{\mu_t^{(n)}\}_n$ is tight. In fact, since $\Delta^{(n)} \overset{0}{\rightsquigarrow} N(0,J)$, the sequence $\Delta^{(n)}$ is tight under $P^{(n)}_{\theta_0}$, that is, for any $\varepsilon > 0$, there exists a $K > 0$ such that for all n,

$$P^{(n)}_{\theta_0}(\Delta^{(n)} \notin [-K,K]^d) < \varepsilon.$$

Consequently,

$$\begin{aligned} \mu_t^{(n)}(\mathbb{R}^d \backslash [-K,K]^d) &= \int_{\Delta^{(n)} \notin [-K,K]^d} M^{(n)}(t,\omega) dP^{(n)}_{\theta_0}(\omega) \\ &\le \int_{\Delta^{(n)} \notin [-K,K]^d} dP^{(n)}_{\theta_0}(\omega) \\ &= P^{(n)}_{\theta_0}(\Delta^{(n)} \notin [-K,K]^d) < \varepsilon, \end{aligned}$$

proving the tightness of $\{\mu_t^{(n)}\}_n$. It then follows from the Prohorov theorem that there is a subsequence $\{\mu_t^{(n_k)}\}_k$ that is weakly convergent for all $t \in \mathbb{Q}^s$, i.e.,

$$\mu_t^{(n_k)} \overset{0}{\rightsquigarrow} {}^{\exists}\mu_t, \qquad (\forall t \in \mathbb{Q}^s). \tag{A.3}$$

Observe that for any continuity point $x \in \mathbb{R}^d$ of μ_t,

$$\begin{aligned}\mu_t((-\infty, x]) &= \lim_{k\to\infty} \int_{\Delta^{(n_k)} \le x} M^{(n_k)}(t,\omega) dP_{\theta_0}^{(n_k)}(\omega) \\ &= \lim_{k\to\infty} P_{\theta_0}^{(n_k)}(\{T^{(n_k)} \le t\} \cap \{\Delta^{(n_k)} \le x\}).\end{aligned}$$

Let us extend μ_t to all $t \in \mathbb{R}^s$ so that $\mu_t((-\infty, x])$ is right-continuous in t for each $x \in \mathbb{R}^d$, and denote the extension by $\overline{\mu}_t$, that is,

$$\overline{\mu}_t((-\infty, x]) := \inf\{\mu_\alpha((-\infty, x]) \mid \alpha \in \mathbb{Q}^s,\ \alpha > t\}. \tag{A.4}$$

Specifically, since $T^{(n)} \overset{0}{\rightsquigarrow} \mathcal{L}_0$, the total mass of $\overline{\mu}_t$ for a continuity point t of $\mathcal{L}_0$ is given by

$$\overline{\mu}_t(\mathbb{R}^d) = \mu_t(\mathbb{R}^d) = \lim_{k\to\infty} P_{\theta_0}^{(n_k)}(T^{(n_k)} \le t) = \mathcal{L}_0((-\infty, t]).$$

Further, since $\Delta^{(n)} \overset{0}{\rightsquigarrow} N(0, J)$, we have from the joint tightness of $(\Delta^{(n)}, T^{(n)})$ that

$$\overline{\mu}_\infty(B) := \lim_{t\to\infty} \mu_t(B) = \lim_{k\to\infty} P_{\theta_0}^{(n_k)}(\Delta^{(n_k)} \in B) = \int_B g_0(x) dx.$$

Here, $g_h(x)$ denotes the density of $N(Jh, J)$ with respect to the Lebesgue measure dx. Put differently, $\overline{\mu}_\infty \sim N(0, J)$.

Since $\overline{\mu}_t(B) \le \overline{\mu}_\infty(B)$ for all $t \in \mathbb{R}^s$, we find that $\overline{\mu}_t$ is absolutely continuous to $\overline{\mu}_\infty$, and hence to the Lebesgue measure. This guarantees the existence of the density

$$M_t(x) := \frac{d\overline{\mu}_t}{d\overline{\mu}_\infty}(x) = \frac{1}{g_0(x)} \frac{d\overline{\mu}_t}{dx}(x). \tag{A.5}$$

Note that $0 \le M_t(x) \le 1$ and $M_t(x) \uparrow M_\infty(x) = 1$ for each $x \in \mathbb{R}^d$.

We prove that this is the one that gives the desired limit distribution, in that

$$\mathcal{L}_h((-\infty, t]) = \int_{\mathbb{R}^d} g_h(x)\, M_t(x)\, dx \tag{A.6}$$

for any $h \in \mathbb{R}^d$ and any continuity point $t \in \mathbb{R}^s$ of $\mathcal{L}_h$.

Because of (1), we have

$$P_{\theta_0 + h/\sqrt{n}}^{(n)} \lhd\rhd P_{\theta_0}^{(n)}, \tag{A.7}$$

which, in particular, entails that $\dfrac{dP_{\theta_0+h/\sqrt{n}}^{(n)}}{dP_{\theta_0}^{(n)}}$ is uniformly integrable under $P_{\theta_0}^{(n)}$, and hence under $\mu_t^{(n)}$ for any $t \in \mathbb{R}^s$. Consequently, for any continuity point $t \in \mathbb{R}^s$ of $\mathcal{L}_h$, we have

$$\begin{aligned}\mathcal{L}_h((-\infty, t]) &= \lim_{k\to\infty} \int_{\Omega^{(n_k)}} M^{(n_k)}(t,\omega)\, dP_{\theta_0 + h/\sqrt{n_k}}^{(n_k)}(\omega) \\ &= \lim_{k\to\infty} \int_{\Omega^{(n_k)}} M^{(n_k)}(t,\omega)\, \frac{dP_{\theta_0+h/\sqrt{n_k}}^{(n_k)}}{dP_{\theta_0}^{(n_k)}}\, dP_{\theta_0}^{(n_k)}(\omega) \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^d} e^{h^i \Delta_i^{(n_k)} - \frac{1}{2} h^i h^j J_{ij}}\, d\mu_t^{(n_k)}(\Delta^{(n_k)}).\end{aligned} \tag{A.8}$$

Here, the first equality follows from the assumption that $T^{(n)} \overset{h}{\rightsquigarrow} \exists\mathcal{L}_h$, the second from (A.7) and Lemma A.1 below, and the third from (1) and (A.2). Now we prove that

$$\lim_{k\to\infty}\int_{\mathbb{R}^d} e^{h^i\Delta_i^{(n_k)}-\frac{1}{2}h^ih^jJ_{ij}}\,d\mu_t^{(n_k)}(\Delta^{(n_k)}) = \int_{\mathbb{R}^d}\frac{g_h(x)}{g_0(x)}\,d\overline{\mu}_t(x) \tag{A.9}$$

for any continuity point $t\in\mathbb{R}^s$ of $\mathcal{L}_h$. Given $\varepsilon>0$ arbitrarily, take another continuity point $t'\in\mathbb{R}^s$ of $\mathcal{L}_h$ and a rational point $\alpha\in\mathbb{Q}^s$ $(t<\alpha<t')$ such that

$$0\le \mathcal{L}_h((-\infty,t'])-\mathcal{L}_h((-\infty,t])<\varepsilon \tag{A.10}$$

and

$$0\le \int_{\mathbb{R}^d}\frac{g_h(x)}{g_0(x)}\,d\mu_\alpha(x)-\int_{\mathbb{R}^d}\frac{g_h(x)}{g_0(x)}\,d\overline{\mu}_t(x)<\varepsilon. \tag{A.11}$$

The existence of such t' is assured by the assumption that t is a continuity point of $\mathcal{L}_h$, and the existence of such $\alpha\in\mathbb{Q}^s$ by (A.4) and the monotone convergence theorem. Then

(A.12)
$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} e^{h^i\Delta_i^{(n_k)}-\frac{1}{2}h^ih^jJ_{ij}}\,d\mu_t^{(n_k)}(\Delta^{(n_k)})-\int_{\mathbb{R}^d}\frac{g_h(x)}{g_0(x)}\,d\overline{\mu}_t(x)\right|\\
&\quad\le\left|\int_{\mathbb{R}^d} e^{h^i\Delta_i^{(n_k)}-\frac{1}{2}h^ih^jJ_{ij}}\,d\mu_t^{(n_k)}(\Delta^{(n_k)})-\int_{\mathbb{R}^d} e^{h^i\Delta_i^{(n_k)}-\frac{1}{2}h^ih^jJ_{ij}}\,d\mu_\alpha^{(n_k)}(\Delta^{(n_k)})\right|\\
&\qquad+\left|\int_{\mathbb{R}^d} e^{h^i\Delta_i^{(n_k)}-\frac{1}{2}h^ih^jJ_{ij}}\,d\mu_\alpha^{(n_k)}(\Delta^{(n_k)})-\int_{\mathbb{R}^d}\frac{g_h(x)}{g_0(x)}\,d\mu_\alpha(x)\right|\\
&\qquad+\left|\int_{\mathbb{R}^d}\frac{g_h(x)}{g_0(x)}\,d\mu_\alpha(x)-\int_{\mathbb{R}^d}\frac{g_h(x)}{g_0(x)}\,d\overline{\mu}_t(x)\right|.
\end{aligned}$$

Firstly, due to (A.8) and (A.10), for sufficiently large k, the second line of (A.12) is evaluated from above by

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} e^{h^i\Delta_i^{(n_k)}-\frac{1}{2}h^ih^jJ_{ij}}\,d\mu_t^{(n_k)}(\Delta^{(n_k)})-\int_{\mathbb{R}^d} e^{h^i\Delta_i^{(n_k)}-\frac{1}{2}h^ih^jJ_{ij}}\,d\mu_{t'}^{(n_k)}(\Delta^{(n_k)})\right|\\
&\quad\le\left|\int_{\mathbb{R}^d} e^{h^i\Delta_i^{(n_k)}-\frac{1}{2}h^ih^jJ_{ij}}\,d\mu_t^{(n_k)}(\Delta^{(n_k)})-\mathcal{L}_h((-\infty,t])\right|\\
&\qquad+\left|\mathcal{L}_h((-\infty,t])-\mathcal{L}_h((-\infty,t'])\right|\\
&\qquad+\left|\mathcal{L}_h((-\infty,t'])-\int_{\mathbb{R}^d} e^{h^i\Delta_i^{(n_k)}-\frac{1}{2}h^ih^jJ_{ij}}\,d\mu_{t'}^{(n_k)}(\Delta^{(n_k)})\right|\\
&\quad<3\varepsilon.
\end{aligned}$$

Secondly, due to (A.3) and the Lemma A.2 below, for sufficiently large k, the third line of (A.12) gets smaller than ε. Finally, due to (A.11), the last line of (A.12) is bounded from above by ε. Putting these evaluations together, we find that

$$\left|\int_{\mathbb{R}^d} e^{h^i\Delta_i^{(n_k)}-\frac{1}{2}h^ih^jJ_{ij}}\,d\mu_t^{(n_k)}(\Delta^{(n_k)})-\int_{\mathbb{R}^d}\frac{g_h(x)}{g_0(x)}\,d\overline{\mu}_t(x)\right|<5\varepsilon,$$

proving (A.9).

Now that (A.8) and (A.9) are established, the desired identity (A.6) follows immediately from (A.5) and the assumption that $T^{(n)} \overset{h}{\rightsquigarrow} \exists\mathcal{L}_h$. $\square$

LEMMA A.1. *Let the probability measures P_n and Q_n on Ω_n satisfy $Q_n \triangleleft P_n$. Then, for any measurable subset F_n of Ω_n,*

$$\lim_{n\to\infty} E_{Q_n}[\mathbb{1}_{F_n}] = \lim_{n\to\infty} E_{P_n}\left[\mathbb{1}_{F_n}\frac{dQ_n}{dP_n}\right], \tag{A.13}$$

provided either of the limits exists.

PROOF. Let $Q_n = Q_n^{ac} + Q_n^{\perp}$ be the Lebesgue decomposition with respect to P_n, and let $A_n := \operatorname{supp} Q_n^{ac}$. Since $P_n(A_n^c) = 0$ for all n, we have from $Q_n \triangleleft P_n$ that $Q_n(A_n^c) \to 0$. Therefore,

$$\begin{aligned}
\int_{\Omega_n} \mathbb{1}_{F_n}(\omega)dQ_n &= \int_{A_n} \mathbb{1}_{F_n}(\omega)dQ_n + \int_{A_n^c} \mathbb{1}_{F_n}(\omega)dQ_n \\
&= \int_{\Omega_n} \mathbb{1}_{F_n}(\omega)\frac{dQ_n}{dP_n}\,dP_n + Q_n(A_n^c \cap F_n),
\end{aligned}$$

from which (A.13) immediately follows. □

LEMMA A.2. *Let $X_n \in L^1(P_n)$ for all n, and let $X \in L^1(P)$. Suppose that $\{X_n\}_n$ is uniformly integrable and $X_n \rightsquigarrow X$. Then*

$$\lim_{n\to\infty} E_{P_n}[X_n] = E_P[X].$$

PROOF. For $K \in [0,\infty)$, define a function $f_K : \mathbb{R} \to [-K, K]$ as follows:

$$f_K(x) := \begin{cases} K, & (x > K) \\ x, & (-K \le x \le K) \\ -K, & (x < -K) \end{cases}$$

Given $\varepsilon > 0$, we can choose K so that, by the uniform integrability,

$$E_{P_n}[|X_n - f_K(X_n)|] \le E_{P_n}[|X_n|\,;\,|X_n| > K] < \frac{\varepsilon}{3}, \quad (\forall n)$$

and

$$E_P[|X - f_K(X)|] \le E_P[|X|\,;\,|X| > K] < \frac{\varepsilon}{3}.$$

Further, since $X_n \rightsquigarrow X$, we can choose $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$,

$$|E_{P_n}[f_K(X_n)] - E_P[f_K(X)]| < \frac{\varepsilon}{3}.$$

The triangular inequality therefore implies that, for $n \ge n_0$,

$$|E_{P_n}[X_n] - E_P[X]| < \varepsilon,$$

and the proof is complete. □

APPENDIX B: GAUSSIAN STATES ON DEGENERATE CCR ALGEBRAS

This section gives a brief account of degenerate canonical commutation relations (CCR) and hybrid classical/quantum Gaussian states.

Let V be a real symplectic space with nonsingular symplectic form Δ. The unital $*$-algebra generated by elements of V satisfying

$$fg - gf = \sqrt{-1}\,\Delta(f,g), \quad f^* = f, \qquad (\forall f, g \in V)$$

is called the *canonical commutation relations (CCR) algebra*. There is a distinct, but closely related notion of the CCR. Let $\mathcal{H}$ be a separable Hilbert space and let $W : V \to B(\mathcal{H})$ satisfy the relations

$$W(f)W(g) = e^{-\frac{\sqrt{-1}}{2}\Delta(f,g)}W(f+g), \quad W(f)^* = W(-f), \qquad (f, g \in V).$$

These are called the *Weyl form* of the CCR. Specifically, the above relations imply that $W(f)$ is unitary and $W(0) = 1$.

One would like to represent the CCR by using selfadjoint operators on $\mathcal{H}$. We first treat the case when V is a two-dimensional symplectic space with symplectic basis $\{e_1, f_1\}$ satisfying $\Delta(e_1, f_1) = 1$. Then the above relation reduces to

$$W(te_1)W(sf_1) = e^{-\frac{\sqrt{-1}}{2}st}W(te_1 + sf_1) = e^{-\sqrt{-1}\,st}W(sf_1)W(te_1).$$

Let us regard $U(t) := W(te_1)$ and $V(s) := W(sf_1)$ as one-parameter unitary groups acting on $\mathcal{H}$. By Stone's theorem, there is a one-to-one correspondence between selfadjoint operators and (strongly continuous) one-parameter unitary groups. Thus one defines a pair of selfadjoint operators Q and P by

$$U(t) := e^{\sqrt{-1}tQ}, \quad V(s) = e^{\sqrt{-1}sP},$$

which fulfills the Weyl form of the CCR

$$e^{\sqrt{-1}tQ}e^{\sqrt{-1}sP} = e^{-\sqrt{-1}st}e^{\sqrt{-1}sP}e^{\sqrt{-1}tQ}.$$

Formally differentiating this identity with respect to t and s at $s = t = 0$, one has the Heisenberg form of the CCR

$$QP - PQ = \sqrt{-1}\,I.$$

The operators Q and P are called the *canonical observables* of the CCR.

There are variety of choices of Hilbert spaces $\mathcal{H}$ and irreducible representations of canonical observables on $\mathcal{H}$. However, according to the Stone-von Neumann theorem, they are unitarily equivalent [2]. Thus one may use any one of them. In this paper, we canonically use the Schrödinger representation on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Note that the von Neumann algebra generated by $\{e^{\sqrt{-1}tQ}\}$ is $L^\infty(\mathbb{R})$, and the von Neumann algebra generated by $\{e^{\sqrt{-1}(tQ+sP)}\}$ is $B(\mathcal{H})$.

Extending the above formulation to a generic even-dimensional symplectic space V is standard. This also allows us to use a more flexible formulation as follows. Given a regular $(2k) \times (2k)$ real skew-symmetric matrix $S = (S_{ij})$, let CCR(S) denote the von Neumann algebra generated by $\{e^{\sqrt{-1}\,t_1X_1}, \dots, e^{\sqrt{-1}\,t_{2k}X_{2k}}\}$ that satisfy the CCR

$$e^{\sqrt{-1}\,t_iX_i}e^{\sqrt{-1}\,t_jX_j} = e^{\sqrt{-1}\,t_it_jS_{ij}}e^{\sqrt{-1}(t_iX_i+t_jX_j)},$$

and call $X = (X_1, \dots, X_{2k})$ the canonical observables of the $\mathrm{CCR}(S)$. This is done by just finding a regular matrix T satisfying

$$T^\top ST = \frac{1}{2}\begin{bmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{bmatrix}$$

to obtain a suitable symplectic basis $\{e_i, f_i\}_{1\le i\le k}$ which generates $\{Q_i, P_i\}_{1\le i\le k}$ such that each X_i belongs to an $\mathbb{R}$-linear span of $\{Q_i, P_i\}_{1\le i\le k}$.

Now we formally extend this formulation to a generic $d \times d$ real skew-symmetric matrix $S = (S_{ij})$ as follows. We first find a regular matrix T satisfying

$$T^\top S T = \frac{1}{2}\left[\begin{array}{ccc|ccccccc} 0 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 0 & & & & & & & \\ \hline & & & 0 & -1 & & & & & \\ & & & 1 & 0 & & & & & \\ & & & & & 0 & -1 & & & \\ & & & & & 1 & 0 & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & 0 & -1 \\ & & & & & & & & 1 & 0 \end{array}\right],$$

to obtain a basis $\{\tilde{e}_1, \dots, \tilde{e}_r\} \sqcup \{e_i, f_i\}_{1\le i\le k}$, where $r + 2k = d$. We then extend $\{\tilde{e}_1, \dots, \tilde{e}_r\}$ to $\{\tilde{e}_i, \tilde{f}_i\}_{1\le i\le r}$ to form a symplectic basis $\{\tilde{e}_i, \tilde{f}_i\}_{1\le i\le r} \sqcup \{e_i, f_i\}_{1\le i\le k}$ of a $2(r + k)$-dimensional symplectic space V, which defines a von Neumann algebra $\mathcal{A}$, the canonical observables of which are denoted by $\{\tilde{Q}_i, \tilde{P}_i\}_{1\le i\le r} \sqcup \{Q_i, P_i\}_{1\le i\le k}$. Now we denote $\mathrm{CCR}(S)$ to be the von Neumann subalgebra of $\mathcal{A}$ generated by

$$\{e^{\sqrt{-1}\,\tilde{t}_i \tilde{Q}_i}\}_{1\le i\le r} \sqcup \{e^{\sqrt{-1}\,t_i Q_i}, e^{\sqrt{-1}\,s_i P_i}\}_{1\le i\le k}.$$

In summary, given a possibly degenerate $d \times d$ real skew-symmetric matrix $S = (S_{ij})$, let $\mathrm{CCR}(S)$ denote the algebra generated by the observables $X = (X_1, \dots, X_d)$ that satisfy the following Weyl form of the CCR

$$e^{\sqrt{-1}\xi^i X_i} e^{\sqrt{-1}\eta^j X_j} = e^{\sqrt{-1}\xi^\top S\eta} e^{\sqrt{-1}(\xi+\eta)^i X_i} \qquad (\xi, \eta \in \mathbb{R}^d)$$

which is formally rewritten in the Heisenberg form

$$\frac{\sqrt{-1}}{2}[X_i, X_j] = S_{ij} \qquad (1 \le i, j \le d).$$

This formulation is useful in handling hybrid classical/quantum Gaussian states. Given a possibly degenerate $d \times d$ real skew-symmetric matrix $S = (S_{ij})$, a state ϕ on $\mathrm{CCR}(S)$ with the canonical observables $X = (X_1, \dots, X_d)$ is called a *quantum Gaussian state*, denoted $\phi \sim N(\mu, \Sigma)$, if the characteristic function $\mathcal{F}_\xi\{\phi\} := \phi(e^{\sqrt{-1}\xi^i X_i})$ takes the form

$$\mathcal{F}_\xi\{\phi\} = e^{\sqrt{-1}\xi^i \mu_i - \frac{1}{2}\xi^i \xi^j V_{ij}}$$

where $\xi = (\xi^i)_{i=1}^d \in \mathbb{R}^d$, $\mu = (\mu_i)_{i=1}^d \in \mathbb{R}^d$, and $V = (V_{ij})$ is a $d \times d$ real symmetric matrix such that the Hermitian matrix $\Sigma := V + \sqrt{-1}S$ is positive semidefinite. When the canonical observables X need to be specified, we also use the notation $(X, \phi) \sim N(\mu, \Sigma)$.

When we discuss relationships between a quantum Gaussian state ϕ on a CCR and a state on another algebra, we need to use the *quasi-characteristic function* [4]

$$\text{(B.1)} \qquad \phi\left(\prod_{t=1}^r e^{\sqrt{-1}\xi_t^i X_i}\right) = \exp\left(\sum_{t=1}^r \left(\sqrt{-1}\xi_t^i \mu_i - \frac{1}{2}\xi_t^i \xi_t^j \Sigma_{ji}\right) - \sum_{t=1}^r \sum_{u=t+1}^r \xi_t^i \xi_u^j \Sigma_{ji}\right)$$

of a quantum Gaussian state, where $(X, \phi) \sim N(\mu, \Sigma)$ and $\{\xi_t\}_{t=1}^r \subset \mathbb{R}^d$. Note that (B.1) is analytically continued to $\{\xi_t\}_{t=1}^r \subset \mathbb{C}^d$.

The notion of quasi-characteristic function is exploited in discussing the quantum counterpart of the weak convergence [3, 4, 5]. For each $n \in \mathbb{N}$, let $\rho^{(n)}$ be a quantum state and $X^{(n)} = (X_1^{(n)}, \dots, X_d^{(n)})$ be a list of observables on a finite dimensional Hilbert space $\mathcal{H}^{(n)}$. We say the sequence $(X^{(n)}, \rho^{(n)})$ *converges in distribution* to $(X, \phi) \sim N(\mu, \Sigma)$, in symbols

$$(X^{(n)}, \rho^{(n)}) \rightsquigarrow (X, \phi) \quad \text{or} \quad X^{(n)} \overset{\rho^{(n)}}{\rightsquigarrow} N(\mu, \Sigma)$$

if

$$\lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} \left(\prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right) = \phi \left(\prod_{t=1}^{r} e^{\sqrt{-1}\xi_t^i X_i} \right)$$

holds for any $r \in \mathbb{N}$ and subset $\{\xi_t\}_{t=1}^r$ of $\mathbb{R}^d$.

APPENDIX C: D-EXTENDIBILITY FOR I.I.D. AND NON-I.I.D. MODELS

This section is a continuation of Remark 2.3, demonstrating the D-extendibility of i.i.d. models, the idea behind the terms 'asymptotic D-invariance' and 'D-extension', and a proper asymptotic treatment of the model presented in Example 2.1. We also give an example of a sequence of quantum statistical models that is non-i.i.d. but is, nevertheless, q-LAN and D-extendible.

Given a quantum state ρ on a finite dimensional Hilbert space $\mathcal{H}$, let $\mathcal{D}_\rho : B(\mathcal{H}) \to B(\mathcal{H})$ be Holevo's commutation operator [2] with respect to ρ defined by

$$\mathcal{D}_\rho = \sqrt{-1}\, \frac{\mathcal{L}_\rho - \mathcal{R}_\rho}{\mathcal{L}_\rho + \mathcal{R}_\rho},$$

where $\mathcal{L}_\rho$ and $\mathcal{R}_\rho$ are superoperators defined by

$$\mathcal{L}_\rho Z := \rho Z, \qquad \mathcal{R}_\rho Z := Z\rho, \qquad (Z \in B(\mathcal{H})).$$

They are positive (selfadjoint) operators with respect to the Hilbert-Schmidt inner product $\langle A, B\rangle_{\mathrm{HS}} := \operatorname{Tr} A^* B$ of $B(\mathcal{H})$ because

$$\langle Z, \mathcal{L}_\rho Z\rangle_{\mathrm{HS}} = \operatorname{Tr} Z^* \rho Z \geq 0$$

and

$$\langle Z, \mathcal{R}_\rho Z\rangle_{\mathrm{HS}} = \operatorname{Tr} Z^* Z \rho = \operatorname{Tr} Z \rho Z^* \geq 0$$

for all $Z \in B(\mathcal{H})$.

When ρ is not faithful, $\mathcal{D}_\rho$ is regarded as a superoperator acting on the quotient space $B(\mathcal{H})/K_\rho$, where[1]

$$K_\rho := \{K \in B(\mathcal{H}) : K\rho = \rho K = 0\}.$$

Since $\mathcal{D}_\rho$ sends selfadjoint operators to selfadjoint operators, it is also regarded as a superoperator on $B_{\mathrm{sa}}(\mathcal{H})/K_\rho$, where $B_{\mathrm{sa}}(\mathcal{H})$ is the set of selfadjoint operators.

A subspace V of $B_{\mathrm{sa}}(\mathcal{H})$ is called $\mathcal{D}_\rho$*-invariant* if V/K_ρ is $\mathcal{D}_\rho$-invariant. Given two lists of selfadjoint operators $(X_1, \dots, X_r)$ and $(L_1, \dots, L_d)$, the former is called a $\mathcal{D}_\rho$*-invariant extension* of the latter if $\operatorname{Span}_{\mathbb{R}} \{X_i\}_{i=1}^r \supset \operatorname{Span}_{\mathbb{R}} \{L_i\}_{i=1}^d$ and $\operatorname{Span}_{\mathbb{R}} \{X_i\}_{i=1}^r$ is $\mathcal{D}_\rho$-invariant.

The following theorem motivated us to adopt the term 'asymptotic D-invariance' in order to describe an asymptotic version of $\mathcal{D}_{\rho_{\theta_0}}$-invariance.

[1]In [2], commutation operator $\mathcal{D}_\rho$ was defined on the space $L^2(\rho)$ of square-summable operators, which is the completion of $B(\mathcal{H})$ with respect to the pre-inner product $\langle X, Y\rangle_\rho := \frac{1}{2} \operatorname{Tr} \rho(X^*Y + YX^*)$. Note that $\langle K, K\rangle_\rho = 0$ if and only if $K\rho = \rho K = 0$. The 'if' part is obvious, and the 'only if' part is proved by observing $2\langle K, K\rangle_\rho = \operatorname{Tr}(K\sqrt{\rho})^*(K\sqrt{\rho}) + \operatorname{Tr}(\sqrt{\rho}K)^*(\sqrt{\rho}K)$.

THEOREM C.1. *Given a quantum statistical model $\mathcal{S} := \{\rho_\theta : \theta \in \Theta \subset \mathbb{R}^d\}$ on a finite dimensional Hilbert space $\mathcal{H}$, let $(L_1, \ldots, L_d)$ be its SLDs at $\theta_0 \in \Theta$, and let $\mathcal{S}^{(n)} := \{\rho_\theta^{\otimes n} : \theta \in \Theta \subset \mathbb{R}^d\}$ be its i.i.d. extensions. Take a linearly independent $\mathcal{D}_{\rho_{\theta_0}}$-invariant extension $(X_1, \ldots, X_r)$ of $(L_1, \ldots, L_d)$ satisfying $\operatorname{Tr} \rho_{\theta_0} X_i = 0$ for all $i = 1, \ldots, r$, and let*

$$\Delta_i^{(n)} := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} I^{\otimes(k-1)} \otimes L_i \otimes I^{\otimes(n-k)}, \qquad (1 \le i \le d),$$

$$X_i^{(n)} := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} I^{\otimes(k-1)} \otimes X_i \otimes I^{\otimes(n-k)}, \qquad (1 \le i \le r).$$

Then $X^{(n)}$ satisfies conditions (4) – (6) *in Definition* 2.2.

We prove Theorem C.1 in a series of lemmas.

LEMMA C.2. *Given a quantum state ρ and a list of observables $(X_1, \ldots, X_d)$ on a finite dimensional Hilbert space $\mathcal{H}$, let A and J be $d \times d$ nonnegative matrices whose (i,j)th entries are $A_{ij} = \operatorname{Tr} \sqrt{\rho} X_j \sqrt{\rho} X_i$ and $J_{ij} = \operatorname{Tr} \rho X_j X_i$. Then, both A and $J \# J^\top$ are real matrices and satisfy*

$$A \le J \# J^\top,$$

where $\#$ denotes the operator geometric mean.

PROOF. $\overline{A} = A$ is obvious, and

$$\overline{J \# J^\top} = \overline{J \# \overline{J}} = \overline{J} \# J = J \# \overline{J} = J \# J^\top.$$

Now recall that the operator geometric mean $P \# Q$ for positive operators P and Q is characterized as [6]

$$P \# Q = \max\left\{ X \ge 0 : \begin{pmatrix} P & X \\ X & Q \end{pmatrix} \ge 0 \right\}.$$

Since the Gram matrix for $\{\sqrt{\rho} X_1, \ldots, \sqrt{\rho} X_d\} \cup \{X_1 \sqrt{\rho}, \ldots, X_d \sqrt{\rho}\}$ with respect to the Hilbert-Schmidt inner product is

$$\begin{pmatrix} J & A \\ A & J^\top \end{pmatrix},$$

the inequality $A \le J \# J^\top$ immediately follows. □

LEMMA C.3. *Let $J = V + \sqrt{-1} S$ be nonnegative matrix, and assume that $V = \operatorname{Re} J$ is strictly positive. Then*

$$J \# J^\top = V^{1/2} \left\{ I + \left(V^{-1/2} S V^{-1/2} \right)^2 \right\}^{1/2} V^{1/2}. \tag{C.1}$$

PROOF. By changing J into $J + \varepsilon I$ for $\varepsilon > 0$ and considering the limit $\varepsilon \downarrow 0$, it suffices to treat the case when $J > 0$. Set

$$S_V := V^{-1/2} S V^{-1/2} \quad \text{and} \quad X := V^{1/2} \left\{ I + S_V^2 \right\}^{1/2} V^{1/2}.$$

Then

$$\begin{aligned} XJ^{-1}X &= X\left\{V^{1/2}(I+\sqrt{-1}S_V)V^{1/2}\right\}^{-1}X \\ &= V^{1/2}\left\{I+S_V^2\right\}\left\{I+\sqrt{-1}S_V\right\}^{-1}V^{1/2} \\ &= V^{1/2}\left\{I-\sqrt{-1}S_V\right\}V^{1/2} \\ &= J^\top. \end{aligned}$$

This proves that $X = J\#J^\top$. □

LEMMA C.4. *Under the setting of Lemma C.2, assume further that $V = \operatorname{Re} J$ is strictly positive. Then the following conditions are equivalent.*

(i) $A = J\#J^\top$
(ii) $\mathrm{Span}_{\mathbb{C}}\left\{\sqrt{\rho}X_i + X_i\sqrt{\rho}\right\}_{i=1}^d \supset \mathrm{Span}_{\mathbb{C}}\left\{\sqrt{\rho}X_i - X_i\sqrt{\rho}\right\}_{i=1}^d$
(iii) $\mathrm{Span}_{\mathbb{C}}\left\{\rho X_i + X_i\rho\right\}_{i=1}^d \supset \mathrm{Span}_{\mathbb{C}}\left\{\rho X_i - X_i\rho\right\}_{i=1}^d$
(iv) $\mathrm{Span}_{\mathbb{C}}\left\{X_i\right\}_{i=1}^d$ *is $\mathcal{D}_\rho$-invariant.*
(v) $\mathrm{Span}_{\mathbb{R}}\left\{X_i\right\}_{i=1}^d$ *is $\mathcal{D}_\rho$-invariant.*

PROOF. We first prove that (i) $\Leftrightarrow$ (ii). Letting $J = V + \sqrt{-1}S$, the Gram matrix G for $\{\sqrt{\rho}X_i + X_i\sqrt{\rho}\}_{i=1}^d \cup \{\sqrt{\rho}X_i - X_i\sqrt{\rho}\}_{i=1}^d$ with respect to the Hilbert-Schmidt inner product is written as

$$G = 2\begin{pmatrix} V+A & \sqrt{-1}S \\ \sqrt{-1}S & V-A \end{pmatrix} = \begin{pmatrix} I & I \\ I & -I \end{pmatrix}\begin{pmatrix} J & A \\ A & J^\top \end{pmatrix}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}.$$

Condition (ii) is equivalent to saying that

$$\operatorname{rank} G = \operatorname{rank}(V+A) = d.$$

Since

$$\begin{pmatrix} J & A \\ A & J^\top \end{pmatrix} = \begin{pmatrix} V^{1/2} & 0 \\ 0 & V^{1/2} \end{pmatrix}\begin{pmatrix} I+\sqrt{-1}S_V & A_V \\ A_V & I-\sqrt{-1}S_V \end{pmatrix}\begin{pmatrix} V^{1/2} & 0 \\ 0 & V^{1/2} \end{pmatrix},$$

where $A_V := V^{-1/2}AV^{-1/2}$ and $S_V := V^{-1/2}SV^{-1/2}$, condition (ii) is further equivalent to saying that the nonnegative matrix

$$\begin{pmatrix} I+\sqrt{-1}S_V & A_V \\ A_V & I-\sqrt{-1}S_V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} \sqrt{-1}S_V & A_V \\ A_V & -\sqrt{-1}S_V \end{pmatrix}$$

is of rank d, that is, the matrix

$$\begin{pmatrix} \sqrt{-1}S_V & A_V \\ A_V & -\sqrt{-1}S_V \end{pmatrix}$$

has eigenvalues -1 and $+1$ each with multiplicity d. (Note that if $(x,y)^\top$ is an eigenvector corresponding to the eigenvalue -1, then $(y,-x)^\top$ is an eigenvector corresponding to the eigenvalue $+1$.) This is equivalent to

$$\begin{pmatrix} \sqrt{-1}S_V & A_V \\ A_V & -\sqrt{-1}S_V \end{pmatrix}^2 = \begin{pmatrix} A_V^2 - S_V^2 & \sqrt{-1}(S_VA_V - A_VS_V) \\ \sqrt{-1}(S_VA_V - A_VS_V) & A_V^2 - S_V^2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

or

$$A_V = \left\{I + S_V^2\right\}^{1/2}.$$

Due to Lemma C.3, this is further equivalent to

$$A = V^{1/2}\left\{I + S_V^2\right\}^{1/2} V^{1/2} = J\# J^\top.$$

We next prove that (ii) $\Leftrightarrow$ (iii). Condition (ii) says that $\mathrm{Span}_{\mathbb{C}}\{X_i\}_{i=1}^d$ is invariant under the action of

$$\mathcal{D}_1 = \frac{\sqrt{\mathcal{L}_\rho} - \sqrt{\mathcal{R}_\rho}}{\sqrt{\mathcal{L}_\rho} + \sqrt{\mathcal{R}_\rho}},$$

while condition (iii) says that $\mathrm{Span}_{\mathbb{C}}\{X_i\}_{i=1}^d$ is invariant under the action of

$$\mathcal{D}_2 = \frac{\mathcal{L}_\rho - \mathcal{R}_\rho}{\mathcal{L}_\rho + \mathcal{R}_\rho}.$$

Since both $\mathcal{D}_1$ and $\mathcal{D}_2$ are selfadjoint with respect to the Hilbert-Schmidt inner product, and $-I \le \mathcal{D}_1, \mathcal{D}_2 \le I$, continuous functional calculus shows that they are related as

$$\mathcal{D}_1 = \frac{\sqrt{1+\mathcal{D}_2} - \sqrt{1-\mathcal{D}_2}}{\sqrt{1+\mathcal{D}_2} + \sqrt{1-\mathcal{D}_2}} \quad \text{and} \quad \mathcal{D}_2 = \frac{2\mathcal{D}_1}{1+\mathcal{D}_1^2}.$$

Consequently, $\mathcal{D}_1$-invariance and $\mathcal{D}_2$-invariance are equivalent.

Further, since $\mathcal{D}_\rho = \sqrt{-1}\,\mathcal{D}_2$, we have (iii) $\Leftrightarrow$ (iv). Finally, (iv) $\Leftrightarrow$ (v) is obvious. □

PROOF OF THEOREM C.1. Firstly, condition (6) is obvious because $(X_1, \dots, X_r)$ is a $\mathcal{D}_{\rho_{\theta_0}}$-invariant extension of $(L_1, \dots, L_d)$. Secondly, condition (4) follows from the quantum central limit theorem for sums of i.i.d. observables [4] (cf., Lemma C.6 below), in that

$$X^{(n)} \overset{\rho_{\theta_0}^{\otimes n}}{\rightsquigarrow} N(0, \Sigma),$$

where $\Sigma_{ij} = \mathrm{Tr}\,\rho_{\theta_0} X_j X_i$. Now we prove the key condition (5).

Let us regard $\hat{\mathcal{H}} := B(\mathcal{H})$ as a Hilbert space endowed with the Hilbert-Schmidt inner product. We introduce selfadjoint operators $\mathcal{L}_{X_i}$ and $\mathcal{R}_{X_i}$ on $\hat{\mathcal{H}}$ for $i = 1, \dots, r$ by

$$\mathcal{L}_{X_i} Z := X_i Z, \qquad \mathcal{R}_{X_i} Z := Z X_i, \qquad (Z \in B(\mathcal{H})).$$

Further, let $\psi_0 := \sqrt{\rho_{\theta_0}}$ be a reference vector in $\hat{\mathcal{H}}$. Note that

$$\langle \psi_0, \mathcal{L}_{X_i} \psi_0 \rangle_{\mathrm{HS}} = \mathrm{Tr}\,\sqrt{\rho_{\theta_0}}(X_i \sqrt{\rho_{\theta_0}}) = \mathrm{Tr}\,\rho_{\theta_0} X_i = 0,$$

and $\langle \psi_0, \mathcal{R}_{X_i} \psi_0 \rangle_{\mathrm{HS}} = 0$ likewise. Now consider the operators on $\hat{\mathcal{H}}^{\otimes n}$ defined by

$$\mathcal{L}_{X_i}^{(n)} := \frac{1}{\sqrt{n}} \sum_{k=1}^n I^{\otimes(k-1)} \otimes \mathcal{L}_{X_i} \otimes I^{\otimes(n-k)},$$

$$\mathcal{R}_{X_i}^{(n)} := \frac{1}{\sqrt{n}} \sum_{k=1}^n I^{\otimes(k-1)} \otimes \mathcal{R}_{X_i} \otimes I^{\otimes(n-k)},$$

and apply the quantum central limit theorem to the i.i.d. extension states $(|\psi_0\rangle\langle\psi_0|)^{\otimes n}$, to obtain

$$\left(\mathcal{L}_{X_1}^{(n)}, \dots, \mathcal{L}_{X_r}^{(n)}, \mathcal{R}_{X_1}^{(n)}, \dots, \mathcal{R}_{X_r}^{(n)}\right) \overset{(|\psi_0\rangle\langle\psi_0|)^{\otimes n}}{\rightsquigarrow} N\left(0, \begin{pmatrix} \Sigma & A \\ A & \Sigma^\top \end{pmatrix}\right), \tag{C.2}$$

where

$$A_{ij} = \langle \psi_0, \mathcal{L}_{X_i} \mathcal{R}_{X_j} \psi_0 \rangle_{\mathrm{HS}} = \mathrm{Tr}\,\sqrt{\rho_{\theta_0}} X_i \sqrt{\rho_{\theta_0}} X_j.$$

Since $\mathrm{Span}_{\mathbb{R}}\{X_i\}_{i=1}^r$ is $\mathcal{D}_{\rho_0}$-invariant, we see from Lemma C.4 that

$$A = \Sigma \# \Sigma^\top.$$

Further, for all $\xi, \eta \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathrm{Tr}\sqrt{\rho_{\theta_0}^{\otimes n}} e^{\sqrt{-1}\xi^i X_i^{(n)}} \sqrt{\rho_{\theta_0}^{\otimes n}} e^{\sqrt{-1}\eta^i X_i^{(n)}} \\
&= \lim_{n\to\infty} \mathrm{Tr}\left(|\psi_0\rangle\langle\psi_0|\right)^{\otimes n} e^{\sqrt{-1}\left(\xi^i \mathcal{L}_{X_i}^{(n)} + \eta^i \mathcal{R}_{X_i}^{(n)}\right)} \\
&= \exp\left[-\frac{1}{2}\begin{pmatrix}\xi\\ \eta\end{pmatrix}^\top \begin{pmatrix}\Sigma & A\\ A & \Sigma^\top\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right],
\end{aligned}$$

where (C.2) is used in the second equality. This proves (5). □

REMARK C.5. Several remarks on the D-extendibility of the one-dimensional pure state model ρ_θ treated in Example 2.1 are now in order. Let us first show that

$$\Delta^{(n)} := \frac{1}{\sqrt{n}} \sum_{k=1}^n I^{\otimes(k-1)} \otimes \sigma_x \otimes I^{\otimes(n-k)}$$

is not asymptotically D-invariant at $\theta = 0$. To this end, it suffices to prove that $\Delta^{(n)}$ does *not* satisfy the identity

$$\lim_{n\to\infty} \mathrm{Tr}\sqrt{\rho_0^{\otimes n}} e^{\sqrt{-1}\xi\Delta^{(n)}} \sqrt{\rho_0^{\otimes n}} e^{\sqrt{-1}\eta\Delta^{(n)}} = e^{-\frac{1}{2}\begin{pmatrix}\xi\\ \eta\end{pmatrix}^\top \begin{pmatrix}J & J\\ J & J\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}}$$

for $\xi, \eta \in \mathbb{R}$, where $J := \mathrm{Tr}\,\rho_0\,\sigma_x^2 = 1$ is the SLD Fisher information of the model at $\theta = 0$. In fact, since $\mathrm{Tr}\sqrt{\rho_0}\,\sigma_x\sqrt{\rho_0}\,\sigma_x = 0$, we can compute in a quite similar way to the proof of Theorem C.1 that

$$\lim_{n\to\infty} \mathrm{Tr}\sqrt{\rho_0^{\otimes n}} e^{\sqrt{-1}\xi\Delta^{(n)}} \sqrt{\rho_0^{\otimes n}} e^{\sqrt{-1}\eta\Delta^{(n)}} = e^{-\frac{1}{2}\begin{pmatrix}\xi\\ \eta\end{pmatrix}^\top \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}}.$$

This proves the claim.

We next verify that $\Delta^{(n)}$ has a D-extension and therefore the model is D-extendible at $\theta = 0$. While this is a straightforward consequence of Theorem C.1, we demonstrate this by a direct computation. Let $(X_1, X_2) := (\sigma_x, \sigma_y)$, which is a $\mathcal{D}_{\rho_0}$-invariant extension of the SLD σ_x at ρ_0, and let

$$X_i^{(n)} := \frac{1}{\sqrt{n}} \sum_{k=1}^n I^{\otimes(k-1)} \otimes X_i \otimes I^{\otimes(n-k)}, \qquad (i = 1, 2).$$

Then by a direct computation similar to the above identity, we have

$$\lim_{n\to\infty} \mathrm{Tr}\sqrt{\rho_0^{\otimes n}} e^{\sqrt{-1}\xi^i X_i^{(n)}} \sqrt{\rho_0^{\otimes n}} e^{\sqrt{-1}\eta^j X_j^{(n)}} = e^{-\frac{1}{2}\begin{pmatrix}\xi\\ \eta\end{pmatrix}^\top \begin{pmatrix}\Sigma & 0\\ 0 & \Sigma^\top\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}}$$

for $\xi, \eta \in \mathbb{R}^2$, where

$$\Sigma = \left[\mathrm{Tr}\,\rho_0 X_j X_i\right]_{ij} = \begin{pmatrix}1 & -\sqrt{-1}\\ \sqrt{-1} & 1\end{pmatrix}.$$

Since $\Sigma\#\Sigma^\top = 0$, we see that $X^{(n)}$ is a D-extension of $\Delta^{(n)}$ $(= X_1^{(n)})$ with $F = (1, 0)^\top$.

Finally, we demonstrate a proper perspective on the local parameter i.i.d. model $\rho_{h/\sqrt{n}}^{\otimes n}$. As shown in Example 2.1, the sequence $M^{(n)} = \{\rho_0^{\otimes n},\, I^{(n)} - \rho_0^{\otimes n}\}$ of binary POVMs does not have a binary POVM on the 'classical' Gaussian shift model $N(h, 1)$ that gives the limiting distribution $\mathcal{L}_h = (e^{-\frac{1}{4}h^2},\, 1 - e^{-\frac{1}{4}h^2})$. This fact nullifies the naive conjecture presented just before Example 2.1, but it does not rule out the existence of a POVM on *another* CCR algebra that gives the above limiting distribution $\mathcal{L}_h$. In fact, Theorem 2.4 tells us that $M^{(n)}$ has a limiting binary POVM $M^{(\infty)} = \{M^{(\infty)}(0), M^{(\infty)}(1)\}$ on the 'quantum' Gaussian shift model

$$\phi_h \sim N((\operatorname{Re}\Sigma F)\, h, \Sigma) = N\left(\begin{pmatrix} h \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & -\sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix}\right)$$

that satisfies $\phi_h(M^{(\infty)}(0)) = \mathcal{L}_h(0)$ for every $h \in \mathbb{R}$. To be specific, let $\mathcal{H}^{(\infty)}$ be a separable Hilbert space that irreducibly represents the $\mathrm{CCR}(\operatorname{Im}\Sigma)$, and let $\rho_h^{(\infty)}$ be the density operator of the quantum Gaussian state ϕ_h on $\mathcal{H}^{(\infty)}$. Then, from the noncommutative Parseval identity [2], we see that the POVM $M^{(\infty)} := \{\rho_0^{(\infty)},\, I^{(\infty)} - \rho_0^{(\infty)}\}$ fulfills

$$\begin{aligned}\operatorname{Tr}\rho_h^{(\infty)} M^{(\infty)}(0) &= \sqrt{\frac{\det(\operatorname{Im}\Sigma)}{\pi^2}} \int_{\mathbb{R}^2} \overline{\mathcal{F}_\xi[\rho_h^{(\infty)}]}\, \mathcal{F}_\xi[M^{(\infty)}(0)]\, d\xi \\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} e^{-\sqrt{-1}\,\xi^1 h - \frac{1}{2}\|\xi\|^2} \cdot e^{-\frac{1}{2}\|\xi\|^2} d\xi = e^{-\frac{1}{4}h^2}\end{aligned}$$

for every $h \in \mathbb{R}$.

Let us proceed to the issue of handling non-i.i.d. quantum statistical models. We start with a slightly generalized version of the quantum central limit theorem.

LEMMA C.6 (Quantum central limit theorem for sums of non-i.i.d. observables). *For each $k \in \mathbb{N}$, let $\mathcal{H}^{(k)}$ be a finite dimensional Hilbert space, and let $\sigma^{(k)}$ and $A^{(k)} = (A_1^{(k)}, \dots, A_r^{(k)})$ be a quantum state and a list of observables on $\mathcal{H}^{(k)}$. Assume that $A^{(k)}$ are zero-mean:*

$$\operatorname{Tr}\sigma^{(k)} A_i^{(k)} = 0 \qquad (1 \le i \le r),$$

uniformly bounded:

$$\sup_{k\in\mathbb{N}, 1\le i\le r} \left\| A_i^{(k)} \right\| < \infty,$$

and there is an $r \times r$ nonnegative matrix Σ such that

$$\lim_{n\to\infty} \operatorname{Tr}\sigma^{(k)} A_j^{(k)} A_i^{(k)} = \Sigma_{ij} \qquad (1 \le i, j \le r).$$

Then under the tensor product states:

$$\rho^{(n)} := \bigotimes_{k=1}^{n} \sigma^{(k)},$$

the scaled sums of observables

$$X_i^{(n)} := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} I^{\otimes(k-1)} \otimes A_i^{(k)} \otimes I^{\otimes(n-k)}$$

exhibit

$$X^{(n)} \overset{\rho^{(n)}}{\rightsquigarrow} N(0, \Sigma).$$

PROOF. We need only check the convergence of quasi-characteristic functions

$$\lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} \prod_{t=1}^{T} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} = \exp\left\{-\frac{1}{2}\sum_{t=1}^{T} \xi_t^\top \Sigma \xi_t - \sum_{t=1}^{T}\sum_{s=t+1}^{T} \xi_s^\top \Sigma \xi_t\right\},$$

for all $T \in \mathbb{N}$ and $\{\xi_t\}_{t=1}^T \subset \mathbb{R}^r$. Observe

$$\begin{aligned}
&\operatorname{Tr} \rho^{(n)} \prod_{t=1}^{T} e^{\sqrt{-1}\xi_t^i X_i^{(n)}} \\
&= \prod_{k=1}^{n} \left\{\operatorname{Tr} \sigma^{(k)} \prod_{t=1}^{T} e^{\frac{\sqrt{-1}}{\sqrt{n}}\xi_t^i A_i^{(k)}}\right\} \\
&= \prod_{k=1}^{n} \left\{\operatorname{Tr} \sigma^{(k)} \sum_{m\in\mathbb{Z}_+^T} \prod_{t=1}^{T} \frac{1}{m_t!}\left(\frac{\sqrt{-1}}{\sqrt{n}}\xi_t^i A_i^{(k)}\right)^{m_t}\right\} \\
&= \prod_{k=1}^{n} \left\{1 - \frac{1}{n}\left(\frac{1}{2}\sum_{t=1}^{T} \xi_t^\top \Sigma^{(k)} \xi_t + \sum_{t=1}^{T}\sum_{s=t+1}^{T} \xi_s^\top \Sigma^{(k)} \xi_t\right) + c^{(k)}(n)\right\}
\end{aligned}$$

where $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$, $\Sigma_{ij}^{(k)} := \operatorname{Tr} \sigma^{(k)} A_j^{(k)} A_i^{(k)}$, and

$$c^{(k)}(n) := \sum_{m_1+\cdots+m_T\geq 3} \frac{(\sqrt{-1})^{m_1+\cdots+m_T}}{n^{(m_1+\cdots+m_T)/2}} \operatorname{Tr} \sigma^{(k)} \prod_{t=1}^{T} \frac{1}{m_t!}\left(\xi_t^i A_i^{(k)}\right)^{m_t}.$$

Note that, since $\{A_i^{(k)}\}_{i,k}$ are assumed to be uniformly bounded,

$$\max_{1\leq k\leq n} \left|c^{(k)}(n)\right| = O\left(\frac{1}{n\sqrt{n}}\right).$$

Consequently, we can further evaluate the quasi-characteristic function as

$$\begin{aligned}
&\log\left\{\operatorname{Tr} \rho^{(n)} \prod_{t=1}^{T} e^{\sqrt{-1}\xi_t^i X_i^{(n)}}\right\} \\
&= \sum_{k=1}^{n} \log\left\{1 - \frac{1}{n}\left(\frac{1}{2}\sum_{t=1}^{T} \xi_t^\top \Sigma^{(k)} \xi_t + \sum_{t=1}^{T}\sum_{s=t+1}^{T} \xi_s^\top \Sigma^{(k)} \xi_t\right) + O\left(\frac{1}{n\sqrt{n}}\right)\right\} \\
&= \sum_{k=1}^{n} \left\{-\frac{1}{n}\left(\frac{1}{2}\sum_{t=1}^{T} \xi_t^\top \Sigma^{(k)} \xi_t + \sum_{t=1}^{T}\sum_{s=t+1}^{T} \xi_s^\top \Sigma^{(k)} \xi_t\right) + O\left(\frac{1}{n\sqrt{n}}\right)\right\}.
\end{aligned}$$

Taking the limit $n \to \infty$, we have

$$\lim_{n\to\infty} \log\left\{\operatorname{Tr} \rho^{(n)} \prod_{t=1}^{T} e^{\sqrt{-1}\xi_t^i X_i^{(n)}}\right\} = -\frac{1}{2}\sum_{t=1}^{T} \xi_t^\top \Sigma \xi_t - \sum_{t=1}^{T}\sum_{s=t+1}^{T} \xi_s^\top \Sigma \xi_t.$$

This proves the claim. □

We are now ready to give an example of a sequence of quantum statistical models that is non-i.i.d. but is, nevertheless, q-LAN and D-extendible.

EXAMPLE C.7. Given a sequence $\{\sigma_\theta^{(k)} : \theta \in \Theta \subset \mathbb{R}^d\}_{k\in\mathbb{N}}$ of quantum statistical models on a fixed finite dimensional Hilbert space $\mathcal{H}$, let us consider their tensor products on $\mathcal{H}^{\otimes n}$ defined by

$$\rho_\theta^{(n)} := \bigotimes_{k=1}^{n} \sigma_\theta^{(k)}.$$

If $\sigma_\theta^{(k)}$ converges to a model $\sigma_\theta^{(\infty)}$ in a certain mode of convergence as $k \to \infty$, it is expected that the model $\rho_\theta^{(n)}$ will be q-LAN and D-extendible, because it is almost i.i.d. in the asymptotic limit. In what follows, we demonstrate a sufficient condition for realizing this scenario.

Assume that, for some $\theta_0 \in \Theta$,

$$\lim_{k\to\infty} \sigma_{\theta_0}^{(k)} = \sigma_{\theta_0}^{(\infty)} \tag{C.3}$$

and the SLDs $\{L_i^{(k)}\}_{i=1}^d$ of $\sigma_\theta^{(k)}$ at $\theta_0 \in \Theta$ is convergent:

$$\lim_{k\to\infty} L_i^{(k)} = L_i^{(\infty)} \qquad (i = 1,\dots,d). \tag{C.4}$$

Then $\rho_\theta^{(n)}$ with

$$\Delta_i^{(n)} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} I^{\otimes(k-1)} \otimes L_i^{(k)} \otimes I^{\otimes(n-k)}$$

is D-extendible at θ_0.

Assume further that the square-root likelihood ratios $R_h^{(k)} := \mathcal{R}(\sigma_{\theta_0+h}^{(k)} \mid \sigma_{\theta_0}^{(k)})$ around θ_0 satisfy

$$\sup_{k\in\mathbb{N}\cup\{\infty\}} \left\| R_h^{(k)} - I - \frac{1}{2} h^i L_i^{(k)} \right\| = o(\|h\|) \tag{C.5}$$

and

$$\sup_{k\in\mathbb{N}\cup\{\infty\}} \left(1 - \operatorname{Tr} \sigma_{\theta_0}^{(k)} R_h^{(k)^2}\right) = o(\|h\|^2). \tag{C.6}$$

In the left-hand side of (C.5), the norm $\|\cdot\|$ stands for the operator norm. Then $\rho_\theta^{(n)}$ is q-LAN at θ_0.

PROOF. Let $D^{(\infty)} = (D_1^{(\infty)}, \dots, D_r^{(\infty)})$ be a $\mathcal{D}_{\sigma^{(\infty)}}$-invariant extension of $L^{(\infty)} = (L_1^{(\infty)}, \dots, L_d^{(\infty)})$ such that $D_i^{(\infty)} = L_i^{(\infty)}$ for $i = 1,\dots,d$. Accordingly, we define, for each $k \in \mathbb{N}$, a set of observables $D^{(k)} = (D_1^{(k)}, \dots, D_r^{(k)})$ by

$$D_i^{(k)} = \begin{cases} L_i^{(k)} & (1 \le i \le d) \\ D_i^{(\infty)} - (\operatorname{Tr} \sigma_{\theta_0}^{(k)} D_i^{(\infty)}) I & (d+1 \le i \le r) \end{cases}.$$

It then follows from (C.3) and (C.4) that

$$\Sigma_{ij}^{(k)} := \operatorname{Tr} \sigma_{\theta_0}^{(k)} D_j^{(k)} D_i^{(k)}, \qquad A_{ij}^{(k)} =: \operatorname{Tr} \sqrt{\sigma_{\theta_0}^{(k)}} D_j^{(k)} \sqrt{\sigma_{\theta_0}^{(k)}} D_i^{(k)}$$

for $k \in \mathbb{N} \cup \{\infty\}$ satisfy

$$\lim_{k\to\infty} \Sigma^{(k)} = \Sigma^{(\infty)}, \qquad \lim_{k\to\infty} A^{(k)} = A^{(\infty)}.$$

Moreover, since $D^{(\infty)}$ is $\mathcal{D}_{\sigma^{(\infty)}}$-invariant, we see from Lemma C.4 that $A^{(\infty)} = \Sigma^{(\infty)} \# \Sigma^{(\infty)\top}$.

Let $X^{(n)} = (X_1^{(n)}, \dots, X_r^{(n)})$, where

$$X_i^{(n)} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} I^{\otimes(k-1)} \otimes D_i^{(k)} \otimes I^{\otimes(n-k)}.$$

In order to prove the D-extendibility, it suffices to verify the following:

(i) There is an $r \times d$ matrix F satisfying $\Delta_k^{(n)} = F_k^i X_i^{(n)}$ for all n.

(ii) $X^{(n)} \overset{\rho_{\theta_0}^{(n)}}{\rightsquigarrow} N(0, \Sigma^{(\infty)})$.

(iii) For all $\xi, \eta \in \mathbb{R}^r$,

$$\lim_{n\to\infty} \mathrm{Tr} \sqrt{\rho_{\theta_0}^{(n)}} e^{\sqrt{-1}\xi^i X_i^{(n)}} \sqrt{\rho_{\theta_0}^{(n)}} e^{\sqrt{-1}\eta^i X_i^{(n)}} = e^{-\frac{1}{2}\begin{pmatrix}\xi\\ \eta\end{pmatrix}^\top \begin{pmatrix}\Sigma^{(\infty)} & \Sigma^{(\infty)}\#\Sigma^{(\infty)\top} \\ \Sigma^{(\infty)}\#\Sigma^{(\infty)\top} & \Sigma^{(\infty)\top}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}}.$$

Firstly, by definition of $D^{(k)}$, (i) is satisfied by the following matrix

$$F = \begin{pmatrix} I \\ O \end{pmatrix},$$

where I is the $d \times d$ identity matrix and O is the $(r-d) \times d$ zero matrix. We next show (ii) and (iii) simultaneously by modifying the proof of Theorem C.1. Let us regard $\hat{\mathcal{H}} := \mathcal{B}(\mathcal{H})$ as a Hilbert space endowed with the Hilbert-Schmidt inner product $\langle A, B\rangle_{\mathrm{HS}} := \mathrm{Tr}\, A^* B$, and let us introduce, for each $X \in B(\mathcal{H})$, linear operators $\mathcal{L}_X$ and $\mathcal{R}_X$ $(1 \le i \le r)$ on $\hat{\mathcal{H}}$ by

$$\mathcal{L}_X Z = XZ, \qquad \mathcal{R}_X Z = ZX, \qquad (Z \in B(\hat{\mathcal{H}})).$$

Further, let $\psi_0^{(k)} := \sqrt{\sigma_{\theta_0}^{(k)}} \in \hat{\mathcal{H}}$ and introduce operators on $\hat{\mathcal{H}}^{\otimes n}$ by

$$\mathcal{L}_{X_i}^{(n)} := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} I^{\otimes(k-1)} \otimes \mathcal{L}_{X_i^{(k)}} \otimes I^{\otimes(n-k)},$$

$$\mathcal{R}_{X_i}^{(n)} := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} I^{\otimes(k-1)} \otimes \mathcal{R}_{X_i^{(k)}} \otimes I^{\otimes(n-k)}.$$

Then, applying the quantum central limit theorem (Lemma C.6) to the product states

$$\hat{\rho}_0^{(n)} := \bigotimes_{k=1}^{n} \left|\psi_0^{(k)}\right\rangle \left\langle \psi_0^{(k)}\right|,$$

we obtain

$$\left(\mathcal{L}_{X^{(n)}}^{(n)}, \mathcal{R}_{X^{(n)}}^{(n)}\right) \overset{\hat{\rho}_0^{(n)}}{\rightsquigarrow} N\left(0, \begin{pmatrix}\Sigma^{(\infty)} & \Sigma^{(\infty)}\#\Sigma^{(\infty)\top} \\ \Sigma^{(\infty)}\#\Sigma^{(\infty)\top} & \Sigma^{(\infty)\top}\end{pmatrix}\right).$$

This proves (ii) and (iii).

We next prove that, with additional assumptions (C.5) and (C.6), the model $\rho_\theta^{(n)}$ is q-LAN at θ_0. Let $J_{ij}^{(k)} := \mathrm{Tr}\, \sigma_{\theta_0}^{(k)} L_j^{(k)} L_i^{(k)}$ for $k \in \mathbb{N} \cup \{\infty\}$. Since

$$\Delta^{(n)} \overset{\rho_{\theta_0}^{(n)}}{\rightsquigarrow} N(0, J^{(\infty)})$$

has been shown in (ii) of the proof of D-extendibility, it suffices to prove that

$$\lim_{n\to\infty} \mathrm{Tr}\,\rho_{\theta_0}^{(n)} \left\{ e^{\frac{1}{2}\left(h^i\Delta_i^{(n)} - \frac{1}{2}h^\top J^{(\infty)}h\right)} - \bar{R}_h^{(n)} \right\}^2 = 0 \tag{C.7}$$

for $\bar{R}_h^{(n)} := \mathcal{R}(\rho_{\theta_0+h/\sqrt{n}}^{(n)} \mid \rho_{\theta_0}^{(n)})$. The sequence appeared in the left-hand side of (C.7) is rewritten as

$$\mathrm{Tr}\,\rho_{\theta_0}^{(n)} e^{h^i\Delta_i^{(n)} - \frac{1}{2}h^\top J^{(\infty)}h} + \mathrm{Tr}\,\rho_{\theta_0}^{(n)} \bar{R}_h^{(n)^2} - 2\,\mathrm{Re}\,\mathrm{Tr}\,\rho_{\theta_0}^{(n)} \bar{R}_h^{(n)} e^{\frac{1}{2}\left(h^i\Delta_i^{(n)} - \frac{1}{2}h^\top J^{(\infty)}h\right)}. \tag{C.8}$$

In order to prove (C.7), therefore, it suffices to verify the following:

(iv) $\lim_{n\to\infty} \mathrm{Tr}\,\rho_{\theta_0}^{(n)} e^{h^i\Delta_i^{(n)} - \frac{1}{2}h^\top J^{(\infty)}h} = 1.$

(v) $\lim_{n\to\infty} \mathrm{Tr}\,\rho_{\theta_0}^{(n)} \bar{R}_h^{(n)^2} = 1.$

(vi) $\lim_{n\to\infty} \mathrm{Tr}\,\rho_{\theta_0}^{(n)} \bar{R}_h^{(n)} e^{\frac{1}{2}\left(h^i\Delta_i^{(n)} - \frac{1}{2}h^\top J^{(\infty)}h\right)} = 1.$

Firstly, because of (C.4), the SLDs $\{L_i^{(k)}\}_{k\in\mathbb{N},1\le i\le d}$ are uniformly bounded, and thus

$$\begin{aligned}
\mathrm{Tr}\,\rho_{\theta_0}^{(n)} e^{h^i\Delta_i^{(n)}} &= \prod_{k=1}^n \mathrm{Tr}\,\sigma_{\theta_0}^{(k)} e^{\frac{1}{\sqrt{n}}h^i L_i^{(k)}} \\
&= \prod_{k=1}^n \left(1 + \frac{1}{2n}h^\top J^{(k)}h + O\left(\frac{1}{n\sqrt{n}}\right)\right).
\end{aligned}$$

In the second line, $\mathrm{Tr}\,\sigma_{\theta_0}^{(k)} L_i^{(k)} = 0$ was used, and the remainder term $O(1/n\sqrt{n})$ is uniform in k. Consequently,

$$\begin{aligned}
\lim_{n\to\infty} \log \mathrm{Tr}\,\rho_{\theta_0}^{(n)} e^{h^i\Delta_i^{(n)}} &= \lim_{n\to\infty} \sum_{k=1}^n \log\left(1 + \frac{1}{2n}h^\top J^{(k)}h + O\left(\frac{1}{n\sqrt{n}}\right)\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^n \left(\frac{1}{2n}h^\top J^{(k)}h + O\left(\frac{1}{n\sqrt{n}}\right)\right) \\
&= \frac{1}{2}h^\top J^{(\infty)}h,
\end{aligned}$$

proving (iv).

Secondly, taking the logarithm of

$$\mathrm{Tr}\,\rho_{\theta_0}^{(n)} \bar{R}_h^{(n)^2} = \prod_{k=1}^n \mathrm{Tr}\,\sigma_{\theta_0}^{(k)} R_{h/\sqrt{n}}^{(k)^2},$$

we have

$$\begin{aligned}
\log \mathrm{Tr}\,\rho_{\theta_0}^{(n)} \bar{R}_h^{(n)^2} &= \sum_{k=1}^n \log \mathrm{Tr}\,\sigma_{\theta_0}^{(k)} R_{h/\sqrt{n}}^{(k)^2} \\
&= \sum_{k=1}^n \log\left\{1 - \left(1 - \mathrm{Tr}\,\sigma_{\theta_0}^{(k)} R_{h/\sqrt{n}}^{(k)^2}\right)\right\} \\
&= \sum_{k=1}^n \log\left\{1 - o\left(\frac{1}{n}\right)\right\}.
\end{aligned}$$

In the last equality, we used (C.6). Since the last line converges to 0 as $n \to \infty$, we have (v).

In order to prove (vi), we need to show that

$$B^{(k)}(h) := I + \frac{1}{2} h^i L_i^{(k)} - R_h^{(k)}$$

satisfies

$$\operatorname{Tr} \sigma_{\theta_0}^{(k)} B^{(k)}(h/\sqrt{n}) = \frac{1}{8n} h^\top J^{(k)} h + o\left(\frac{1}{n}\right), \tag{C.9}$$

where the remainder term $o(1/n)$ is uniform in k. This is shown by observing the identity

$$\begin{aligned}
1 - \operatorname{Tr} \sigma_{\theta_0}^{(k)} R_{h/\sqrt{n}}^{(k)^2} &= 1 - \operatorname{Tr} \sigma_{\theta_0}^{(k)} \left(I + \frac{1}{2\sqrt{n}} h^i L_i^{(k)} - B^{(k)}(h/\sqrt{n}) \right)^2 \\
&= -\frac{1}{4n} h^\top J^{(k)} h - \operatorname{Tr} \sigma_{\theta_0}^{(k)} B^{(k)}(h/\sqrt{n})^2 \\
&\quad + 2 \operatorname{Tr} \sigma_{\theta_0}^{(k)} B^{(k)}(h/\sqrt{n}) + \frac{h^i}{\sqrt{n}} \operatorname{Re} \operatorname{Tr} \sigma_{\theta_0}^{(k)} L_i^{(k)} B^{(k)}(h/\sqrt{n}) \\
&= -\frac{1}{4n} h^\top J^{(k)} h + 2 \operatorname{Tr} \sigma_{\theta_0}^{(k)} B^{(k)}(h/\sqrt{n}) + o\left(\frac{1}{n}\right).
\end{aligned}$$

Here, (C.5) was used in the last equality. Since the above quantity is of order $o(1/n)$ due to (C.6), the equality (C.9) is proved.

Now we are ready to prove (vi), i.e.,

$$\operatorname{Tr} \rho_{\theta_0}^{(n)} \bar{R}_h^{(n)} e^{\frac{1}{2}\left(h^i \Delta_i^{(n)} - \frac{1}{2} h^\top J^{(\infty)} h\right)} = e^{-\frac{1}{4} h^\top J^{(\infty)} h} \prod_{k=1}^{n} \operatorname{Tr} \sigma_{\theta_0}^{(k)} R_{h/\sqrt{n}}^{(k)} e^{\frac{1}{2\sqrt{n}} h^i L_i^{(k)}} \to 1 \quad (n \to \infty).$$

We have from (C.5) and (C.9) that

$$\begin{aligned}
&\operatorname{Tr} \sigma_{\theta_0}^{(k)} R_{h/\sqrt{n}}^{(k)} e^{\frac{1}{2\sqrt{n}} h^i L_i^{(k)}} \\
&= \operatorname{Tr} \sigma_{\theta_0}^{(k)} \left(I + \frac{1}{2\sqrt{n}} h^i L_i^{(k)} - B^{(k)}(h/\sqrt{n}) \right) \\
&\qquad \times \left(I + \frac{1}{2\sqrt{n}} h^i L_i^{(k)} + \frac{1}{8n} \left(h^i L_i^{(k)} \right)^2 + o\left(\frac{1}{n}\right) \right) \\
&= 1 + \frac{1}{8n} h^\top J^{(k)} h + \frac{1}{4n} h^\top J^{(k)} h - \operatorname{Tr} \sigma_{\theta_0}^{(k)} B^{(k)}(h/\sqrt{n}) + o\left(\frac{1}{n}\right) \\
&= 1 + \frac{1}{4n} h^\top J^{(k)} h + o\left(\frac{1}{n}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \log \prod_{k=1}^{n} \operatorname{Tr} \sigma_{\theta_0}^{(k)} R_{h/\sqrt{n}}^{(k)} e^{\frac{1}{2\sqrt{n}} h^i L_i^{(k)}} &= \lim_{n\to\infty} \sum_{k=1}^{n} \log \left(1 + \frac{1}{4n} h^\top J^{(k)} h + o\left(\frac{1}{n}\right) \right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{4n} h^\top J^{(k)} h + o\left(\frac{1}{n}\right) \right) \\
&= \frac{1}{4} h^\top J^{(\infty)} h
\end{aligned}$$

or equivalently,

$$\lim_{n\to\infty} \operatorname{Tr} \rho_{\theta_0}^{(n)} \bar{R}_h^{(n)} e^{\frac{1}{2}h^i \Delta_i^{(n)}} = e^{\frac{1}{4}h^\top J^{(\infty)} h}.$$

This proves (vi), and the proof of (C.7) is complete. □

APPENDIX D: PROOFS OF LEMMAS IN SECTION 3

In this section, we give detailed proofs of lemmas presented in Section 3.

D.1. Proof of Lemma 3.1.

PROOF. Let ρ be the density operator of $N(0,J)$ on the irreducible representation Hilbert space $\mathcal{H}$. Then ρ is pure if and only if $\operatorname{Tr}\rho^2 = 1$. On the other hand, due to the noncommutative Parseval identity [2],

$$\operatorname{Tr}\rho^2 = \sqrt{\frac{\det S}{\pi^d}} \int_{\mathbb{R}^d} d\xi \, |\mathcal{F}_\xi[\rho]|^2 = \sqrt{\frac{\det S}{\pi^d}} \int_{\mathbb{R}^d} d\xi e^{-\xi^\top V \xi} = \sqrt{\frac{\det S}{\pi^d}} \sqrt{\frac{\pi^d}{\det V}} = \sqrt{\frac{\det S}{\det V}}.$$

As a consequence, ρ is pure if and only if $\det V = \det S$. □

D.2. Proof of Corollary 3.2.

PROOF. We first remark that the dimension d of the matrix J is even because the skew-symmetric matrix $S = \operatorname{Im} J$ is invertible. Now we set

$$\hat{J} := \begin{pmatrix} J & J\#J^\top \\ J\#J^\top & J^\top \end{pmatrix}.$$

Then

$$\det(\operatorname{Im}\hat{J}) = \det\begin{pmatrix} S & 0 \\ 0 & S^\top \end{pmatrix} = (\det S)^2.$$

On the other hand, $\hat{J}$ is rewritten as

$$\hat{J} = \frac{1}{2}\begin{pmatrix} I & I \\ I & -I \end{pmatrix} \begin{pmatrix} V + J\#J^\top & \sqrt{-1}S \\ \sqrt{-1}S & V - J\#J^\top \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}.$$

Therefore, setting $S_V := V^{-1/2} S V^{-1/2}$, we have

$$\begin{aligned}
\det(\operatorname{Re}\hat{J}) &= \det\begin{pmatrix} V + J\#J^\top & 0 \\ 0 & V - J\#J^\top \end{pmatrix} \\
&= \det(V + J\#J^\top)\det(V - J\#J^\top) \\
&= (\det V)^2 \det\left(I + \sqrt{I + S_V^2}\right) \det\left(I - \sqrt{I + S_V^2}\right) \\
&= (\det V)^2 \det(-S_V^2) \\
&= (\det S)^2.
\end{aligned}$$

Here we used Lemma C.3 in the third equality. It then follows from Lemma 3.1 that $N(0,\hat{J})$ is a pure state. □

D.3. Proof of Lemma 3.3.

PROOF. We first verify that the subspace $\mathring{\mathcal{H}} := \mathrm{Span}_{\mathbb{C}}\{\psi(\xi)\}_{\xi\in D}$ is dense in $\mathcal{H}$. Since ψ is a cyclic vector, the subspace $\mathrm{Span}_{\mathbb{C}}\{\psi(\xi)\}_{\xi\in\mathbb{R}^d}$ is dense in $\mathcal{H}$. Further, given $\xi\in\mathbb{R}^d$, take an arbitrary sequence $\xi^{(n)}\in D$ that is convergent to ξ. Then

$$\begin{aligned}\lim_{n\to\infty}\left\|\psi(\xi)-\psi(\xi^{(n)})\right\|^2 &= 2-2\lim_{n\to\infty}\mathrm{Re}\,\langle\psi(\xi),\psi(\xi^{(n)})\rangle\\ &= 2-2\lim_{n\to\infty}\mathrm{Re}\,\left\{e^{-\sqrt{-1}\xi^\top S\xi^{(n)}}\left\langle\psi, e^{-\sqrt{-1}\left(\xi-\xi^{(n)}\right)^i X_i}\psi\right\rangle\right\}\\ &= 0.\end{aligned}$$

This proves that $\mathring{\mathcal{H}}$ is dense in $\mathcal{H}$.

We next introduce a sesquilinear functional $F:\mathring{\mathcal{H}}\times\mathring{\mathcal{H}}\to\mathbb{C}$ by

$$F\left(\sum_{i=1}^n a_i\,\psi(\xi^{(i)}),\sum_{j=1}^m b_j\,\psi(\eta^{(j)})\right) := \sum_{i=1}^n\sum_{j=1}^m \bar{a}_i b_j\,\varphi(\xi^{(i)};\eta^{(j)}).$$

We need to verify that F is well-defined. Let

$$\sum_{i=1}^n a_i\,\psi(\xi^{(i)}) = \sum_{i=1}^{n'} a'_i\,\psi(\xi'^{(i)}) \quad\text{and}\quad \sum_{j=1}^m b_j\,\psi(\eta^{(j)}) = \sum_{j=1}^{m'} b'_j\,\psi(\eta'^{(j)}).$$

be different representations of the same vectors in $\mathring{\mathcal{H}}$. The well-definedness of F is proved by showing the following series of equalities:

$$\sum_{i=1}^n\sum_{j=1}^m \bar{a}_i b_j\,\varphi(\xi^{(i)};\eta^{(j)}) = \sum_{i=1}^n\sum_{j=1}^{m'} \bar{a}_i b'_j\,\varphi(\xi^{(i)};\eta'^{(j)}) = \sum_{i=1}^{n'}\sum_{j=1}^{m'} \bar{a}'_i b'_j\,\varphi(\xi'^{(i)};\eta'^{(j)}). \tag{D.1}$$

The first equality in (D.1) is equivalent to

$$\sum_{j=1}^m b_j\,\varphi(\xi^{(i)};\eta^{(j)}) - \sum_{j=1}^{m'} b'_j\,\varphi(\xi^{(i)};\eta'^{(j)}) = 0, \qquad (\forall\xi^{(i)}\in D),$$

which is further equivalent to the following proposition:

$$\sum_{k=1}^r c_k\,\psi(\xi^{(k)}) = 0 \quad\Longrightarrow\quad \sum_{k=1}^r c_k\,\varphi(\xi^{(0)};\xi^{(k)}) = 0, \qquad (\forall\xi^{(0)}\in D). \tag{D.2}$$

Since $0\prec\varphi\prec\varphi_I$, the antecedent of the above proposition (D.2) implies that for any $r\in\mathbb{N}$, $\{\xi^{(i)}\}_{0\le i\le r}\subset D$, and $\{c_i\}_{0\le i\le r}\subset\mathbb{C}$ with $c_0=0$,

$$0\le\sum_{i=0}^r\sum_{j=0}^r \bar{c}_i c_j\,\varphi(\xi^{(i)};\xi^{(j)}) \le \sum_{i=0}^r\sum_{j=0}^r \bar{c}_i c_j\,\varphi_I(\xi^{(i)};\xi^{(j)}) = \left\|\sum_{k=0}^r c_k\,\psi(\xi^{(k)})\right\|^2 = 0.$$

This shows that the vector $(c_0,c_1,\dots,c_r)^\top$ belongs to the kernel of the positive-semidefinite matrix

$$\left[\varphi(\xi^{(i)};\xi^{(j)})\right]_{0\le i,j\le r}.$$

As a consequence,

$$\sum_{k=1}^r c_k\,\varphi(\xi^{(0)};\xi^{(k)}) = \sum_{k=0}^r c_k\,\varphi(\xi^{(0)};\xi^{(k)}) = 0,$$

proving the proposition (D.2). The second equality in (D.1) is proved in the same way.

Now, fix an element $\phi_0 \in \mathring{\mathcal{H}}$ arbitrarily. Then the map $\phi_1 \mapsto F(\phi_1, \phi_0)$ is a bounded conjugate-linear functional on a dense subset $\mathring{\mathcal{H}}$ of $\mathcal{H}$, so that it is continuously extended to the totality of $\mathcal{H}$, and there is a vector $\phi_0^F \in \mathcal{H}$ such that

$$\langle \phi_1, \phi_0^F \rangle = F(\phi_1, \phi_0).$$

Since the map $\phi_0 \mapsto \phi_0^F$ is a bounded linear transformation on $\mathring{\mathcal{H}}$, it is continuously extended to $\mathcal{H}$, and there is a bounded operator A satisfying $\phi_0^F = A\phi_0$ for all $\phi_0 \in \mathcal{H}$. In summary,

$$F(\phi_1, \phi_0) = \langle \phi_1, A\phi_0 \rangle.$$

Since $F(\phi_1, \phi_0) = \overline{F(\phi_0, \phi_1)}$, the operator A is selfadjoint. Further, since $0 \prec \varphi \prec \varphi_I$, we see that $0 \leq A \leq I$. Finally, since

$$\varphi(\xi; \eta) = F(\psi(\xi), \psi(\eta)) = \langle \psi(\xi), A\psi(\eta) \rangle$$

on $D \times D$, it is continuously extended to $\mathbb{R}^d \times \mathbb{R}^d$. $\square$

D.4. Proof of Lemma 3.4.

PROOF. In view of (8), it suffices to show that $V(\varphi)$ and $e^{\sqrt{-1}(\zeta_c^i \hat{X}_{c,i} + \zeta_a^j \hat{X}_{a,j})}$ commute for any $\zeta_c \in \mathbb{R}^{d_c}$ and $\zeta_a \in \mathbb{R}^{d_a}$. For any $\xi_c, \eta_c \in \mathbb{R}^{d_c}$, $\xi_q, \eta_q \in \mathbb{R}^{d_q}$, and $\xi_a, \eta_a \in \mathbb{R}^{d_a}$,

$$\begin{aligned}
&\left\langle \psi(\xi_c, \xi_q, \xi_a), e^{-\sqrt{-1}(\zeta_c^i \hat{X}_{c,i} + \zeta_a^j \hat{X}_{a,j})} V(\varphi) e^{\sqrt{-1}(\zeta_c^i \hat{X}_{c,i} + \zeta_a^j \hat{X}_{a,j})} \psi(\eta_c, \eta_q, \eta_a) \right\rangle \\
&= e^{\sqrt{-1}(-\zeta_a^\top S_a \xi_a + \zeta_a^\top S_a \eta_a)} \langle \psi(\xi_c + \zeta_c, \xi_q, \xi_a + \zeta_a), V(\varphi) \psi(\eta_c + \zeta_c, \eta_q, \eta_a + \zeta_a) \rangle \\
&= e^{\sqrt{-1}(-\zeta_a^\top S_a \xi_a + \zeta_a^\top S_a \eta_a)} \varphi(\xi_c + \zeta_c, \xi_q, \xi_a + \zeta_a; \eta_c + \zeta_c, \eta_q, \eta_a + \zeta_a) \\
&= e^{\sqrt{-1}(-\zeta_a^\top S_a \xi_a + \zeta_a^\top S_a \eta_a)} e^{-\sqrt{-1}(\xi_a + \zeta_a)^\top S_a (\eta_a + \zeta_a)} \varphi(\xi_c - \eta_c, \xi_q, \xi_a - \eta_a; 0, \eta_q, 0) \\
&= e^{-\sqrt{-1}\xi_a^\top S_a \eta_a} \varphi(\xi_c - \eta_c, \xi_q, \xi_a - \eta_a; 0, \eta_q, 0)
\end{aligned}$$

Since the last line is independent of ζ_c and ζ_a, the proof is complete. $\square$

D.5. Proof of Lemma 3.5.

PROOF. Let $L^\infty_{c.a.e.}(\mathbb{R})$ denote the set of real-valued bounded Borel functions on $\mathbb{R}$ that are continuous almost everywhere. To prove Lemma 3.5, it suffices to verify that

$$\begin{aligned}
&\lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} e^{\sqrt{-1}\xi^i X_i^{(n)}} f(\zeta^i X_i^{(n)}) A^{(n)} e^{\sqrt{-1}\eta^i X_i^{(n)}} \\
&\qquad = \operatorname{Tr} \rho^{(\infty)} e^{\sqrt{-1}\xi^i X_i^{(\infty)}} f(\zeta^i X_i^{(\infty)}) A^{(\infty)} e^{\sqrt{-1}\eta^i X_i^{(\infty)}}
\end{aligned} \tag{D.3}$$

for all $\xi, \eta, \zeta \in \mathbb{R}^d$ and $f \in L^\infty_{c.a.e.}(\mathbb{R})$. In fact, since $f(\zeta^i X_i^{(n)}) A^{(n)}$ are also uniformly bounded, (11) can be derived by applying (D.3) and its complex conjugate recursively.

We first show that (10) can be extended to all $\xi \in \mathbb{R}^d$. For any $\varepsilon > 0$ and $\xi \in \mathbb{R}^d$, there exists a $\tilde{\xi} \in \mathbb{Q}^d$ such that

$$\operatorname{Tr} \rho^{(\infty)} \left| \left(e^{\sqrt{-1}\xi^i X_i^{(\infty)}} - e^{\sqrt{-1}\tilde{\xi}^i X_i^{(\infty)}} \right)^* \right|^2 < \varepsilon$$

Then, by using the Schwarz inequality,

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\operatorname{Tr}\rho^{(n)}\left(e^{\sqrt{-1}\xi^i X_i^{(n)}}-e^{\sqrt{-1}\tilde{\xi}^i X_i^{(n)}}\right)A^{(n)}e^{\sqrt{-1}\eta^i X_i^{(n)}}\right|^2\\
&\quad\le\limsup_{n\to\infty}\operatorname{Tr}\rho^{(n)}\left|\left(e^{\sqrt{-1}\xi^i X_i^{(n)}}-e^{\sqrt{-1}\tilde{\xi}^i X_i^{(n)}}\right)^*\right|^2\times\operatorname{Tr}\rho^{(n)}\left|A^{(n)}e^{\sqrt{-1}\eta^i X_i^{(n)}}\right|^2\\
&\quad\le M^2\operatorname{Tr}\rho^{(\infty)}\left|\left(e^{\sqrt{-1}\xi^i X_i^{(\infty)}}-e^{\sqrt{-1}\tilde{\xi}^i X_i^{(\infty)}}\right)^*\right|^2\\
&\quad< M^2\varepsilon,
\end{aligned}$$

where $M := \sup_n \|A^{(n)}\|$. Similarly, we have

$$\left|\operatorname{Tr}\rho^{(\infty)}\left(e^{\sqrt{-1}\xi^i X_i^{(\infty)}}-e^{\sqrt{-1}\tilde{\xi}^i X_i^{(\infty)}}\right)A^{(\infty)}e^{\sqrt{-1}\eta^i X_i^{(\infty)}}\right|^2 < M^2\varepsilon.$$

It then follows from Lemma E.1 that (10) holds for all $\xi\in\mathbb{R}^d$ and $\eta\in\mathbb{Q}^d$. By applying a similar argument to η, we see that (10) holds for all $\xi,\eta\in\mathbb{R}^d$.

We next show that

$$\begin{aligned}
\text{(D.4)}\qquad &\lim_{n\to\infty}\operatorname{Tr}\rho^{(n)}e^{\sqrt{-1}\xi^i X_i^{(n)}}e^{\sqrt{-1}\zeta^i X_i^{(n)}}A^{(n)}e^{\sqrt{-1}\eta^i X_i^{(n)}}\\
&\qquad=\operatorname{Tr}\rho^{(\infty)}e^{\sqrt{-1}\xi^i X_i^{(\infty)}}e^{\sqrt{-1}\zeta^i X_i^{(\infty)}}A^{(\infty)}e^{\sqrt{-1}\eta^i X_i^{(\infty)}}
\end{aligned}$$

for all $\xi,\eta,\zeta\in\mathbb{R}^d$. In fact, letting $S=\operatorname{Im} J$,

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\operatorname{Tr}\rho^{(n)}\left(e^{\sqrt{-1}\xi^i X_i^{(n)}}e^{\sqrt{-1}\zeta^i X_i^{(n)}}-e^{\sqrt{-1}\xi^\top S\zeta}e^{\sqrt{-1}(\xi+\zeta)^i X_i^{(n)}}\right)A^{(n)}e^{\sqrt{-1}\eta^i X_i^{(n)}}\right|^2\\
&\quad\le\limsup_{n\to\infty}\operatorname{Tr}\rho^{(n)}\left|\left(e^{\sqrt{-1}\zeta^i X_i^{(n)}}e^{\sqrt{-1}\xi^i X_i^{(n)}}-e^{\sqrt{-1}\xi^\top S\zeta}e^{\sqrt{-1}(\xi+\zeta)^i X_i^{(n)}}\right)^*\right|^2\\
&\qquad\qquad\times\operatorname{Tr}\rho^{(n)}\left|A^{(n)}e^{\sqrt{-1}\eta^i X_i^{(n)}}\right|^2\\
&\quad\le M^2\operatorname{Tr}\rho^{(\infty)}\left|\left(e^{\sqrt{-1}\zeta^i X_i^{(\infty)}}e^{\sqrt{-1}\xi^i X_i^{(\infty)}}-e^{\sqrt{-1}\xi^\top S\zeta}e^{\sqrt{-1}(\xi+\zeta)^i X_i^{(\infty)}}\right)^*\right|^2\\
&\quad=0.
\end{aligned}$$

By using this, (D.4) is proved as follows.

$$\begin{aligned}
&\lim_{n\to\infty}\operatorname{Tr}\rho^{(n)}e^{\sqrt{-1}\xi^i X_i^{(n)}}e^{\sqrt{-1}\zeta^i X_i^{(n)}}A^{(n)}e^{\sqrt{-1}\eta^i X_i^{(n)}}\\
&\qquad=e^{\sqrt{-1}\xi^\top S\zeta}\lim_{n\to\infty}\operatorname{Tr}\rho^{(n)}e^{\sqrt{-1}(\xi+\zeta)^i X_i^{(n)}}A^{(n)}e^{\sqrt{-1}\eta^i X_i^{(n)}}\\
&\qquad=e^{\sqrt{-1}\xi^\top S\zeta}\operatorname{Tr}\rho^{(\infty)}e^{\sqrt{-1}(\xi+\zeta)^i X_i^{(\infty)}}A^{(\infty)}e^{\sqrt{-1}\eta^i X_i^{(\infty)}}\\
&\qquad=\operatorname{Tr}\rho^{(\infty)}e^{\sqrt{-1}\xi^i X_i^{(\infty)}}e^{\sqrt{-1}\zeta^i X_i^{(\infty)}}A^{(\infty)}e^{\sqrt{-1}\eta^i X_i^{(\infty)}}.
\end{aligned}$$

In the second equality, (10) is used.

Now we are ready to prove (D.3). Let $Z^{(n)} := \zeta^i X_i^{(n)}$ for each $n\in\mathbb{N}\cup\{\infty\}$ and $\zeta\in\mathbb{R}^d$. According to (D.4), for any $f\in\operatorname{Span}_{\mathbb{C}}\{e^{\sqrt{-1}tx}\}_{t\in\mathbb{R}}$, we have

$$\begin{aligned}
\text{(D.5)}\qquad &\lim_{n\to\infty}\operatorname{Tr}\rho^{(n)}e^{\sqrt{-1}\xi^i X_i^{(n)}}f(Z^{(n)})A^{(n)}e^{\sqrt{-1}\eta^i X_i^{(n)}}\\
&\qquad=\operatorname{Tr}\rho^{(\infty)}e^{\sqrt{-1}\xi^i X_i^{(\infty)}}f(Z^{(\infty)})A^{(\infty)}e^{\sqrt{-1}\eta^i X_i^{(\infty)}}.
\end{aligned}$$

Our goal is to prove this identity for all $f \in L^{\infty}_{c.a.e.}(\mathbb{R})$.

Let

$$\rho_\xi^{(n)} := e^{-\sqrt{-1}\xi^i X_i^{(n)}} \rho^{(n)} e^{\sqrt{-1}\xi^i X_i^{(n)}},$$

and let $\mu_\xi^{(n)}$ be the classical probability measure on $\mathbb{R}$ that has the characteristic function

$$\varphi_\xi^{(n)}(t) := \operatorname{Tr} \rho_\xi^{(n)} e^{\sqrt{-1}tZ^{(n)}}.$$

It then follows from (9) that, for all $t \in \mathbb{R}$,

$$\begin{aligned}
\lim_{n\to\infty} \varphi_\xi^{(n)}(t) &= \lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} e^{\sqrt{-1}\xi^i X_i^{(n)}} e^{\sqrt{-1}t\,\zeta^i X_i^{(n)}} e^{-\sqrt{-1}\xi^i X_i^{(n)}} \\
&= \operatorname{Tr} \rho^{(\infty)} e^{\sqrt{-1}\xi^i X_i^{(\infty)}} e^{\sqrt{-1}t\,\zeta^i X_i^{(\infty)}} e^{-\sqrt{-1}\xi^i X_i^{(\infty)}} \\
&= e^{-2\sqrt{-1}t\,\zeta^\top S\xi} \operatorname{Tr} \rho^{(\infty)} e^{\sqrt{-1}t\,\zeta^i X_i^{(\infty)}} \\
&= \exp\left[\sqrt{-1}t\left(\zeta^\top h - 2\zeta^\top S\xi\right) - \frac{t^2}{2}\zeta^\top J\zeta\right].
\end{aligned}$$

This shows that that

$$\mu_\xi^{(n)} \rightsquigarrow N\left(\zeta^\top h - 2\zeta^\top S\xi,\, \zeta^\top J\zeta\right).$$

Let p_ξ be the density function of the classical Gaussian distribution $N(\zeta^\top h - 2\zeta^\top S\xi,\, \zeta^\top J\zeta)$, and let $\psi_\xi := \sqrt{p_\xi} \in L^2(\mathbb{R})$. Then, the portmanteau lemma shows that, for all $f \in L^{(\infty)}_{c.a.e.}(\mathbb{R})$,

$$\lim_{n\to\infty} \operatorname{Tr} \rho_\xi^{(n)} f(Z^{(n)}) = \lim_{n\to\infty} \int_{\mathbb{R}} f(z)\mu_\xi^{(n)}(dz) = \int_{\mathbb{R}} f(z)p_\xi(z)dz, = \langle \psi_\xi, f\psi_\xi\rangle$$

where

$$\langle \psi_1, \psi_2\rangle := \int_{\mathbb{R}} \overline{\psi_1(z)}\psi_2(z)dz \qquad (\psi_1, \psi_2 \in L^2(\mathbb{R})).$$

Now recall that

$$\overline{\operatorname{Span}_{\mathbb{C}}}^{\mathrm{SOT}} \left\{e^{\sqrt{-1}tx}\right\}_{t\in\mathbb{R}} = L^\infty(\mathbb{R}).$$

Thus, for all $\varepsilon > 0$ and $f \in L^{(\infty)}_{c.a.e.}(\mathbb{R})$, there exists a real-valued function $\tilde{f} \in \operatorname{Span}_{\mathbb{C}}\{e^{\sqrt{-1}tx}\}_{t\in\mathbb{R}}$ such that

$$\|(f - \tilde{f})\psi_\xi\|^2 = \langle \psi_\xi, (f - \tilde{f})^2\psi_\xi\rangle < \varepsilon.$$

Then by using the Schwarz inequality,

$$\begin{aligned}
\text{(D.6)} \qquad & \limsup_{n\to\infty} \left|\operatorname{Tr} \rho^{(n)} e^{\sqrt{-1}\xi^i X_i^{(n)}} \left\{f(Z^{(n)}) - \tilde{f}(Z^{(n)})\right\} A^{(n)} e^{\sqrt{-1}\eta^i X_i^{(n)}}\right|^2 \\
& \le \limsup_{n\to\infty} \operatorname{Tr} \rho_\xi^{(n)} \left\{f(Z^{(n)}) - \tilde{f}(Z^{(n)})\right\}^2 \times \operatorname{Tr} \rho^{(n)} \left|A^{(n)} e^{\sqrt{-1}\eta^i X_i^{(n)}}\right|^2 \\
& \le M^2 \langle \psi_\xi, (f - \tilde{f})^2\psi_\xi\rangle \\
& < M^2\varepsilon.
\end{aligned}$$

Likewise,

$$\text{(D.7)} \qquad \left|\operatorname{Tr} \rho^{(\infty)} e^{\sqrt{-1}\xi^i X_i^{(\infty)}} \left\{f(Z^{(\infty)}) - \tilde{f}(Z^{(\infty)})\right\} A^{(\infty)} e^{\sqrt{-1}\eta^i X_i^{(\infty)}}\right|^2 < M^2\varepsilon.$$

Now that (D.5) (D.6) (D.7) have been verified, the identity (D.3) is an immediate consequence of Lemma E.1. □

D.6. Proof of Lemma 3.6. We begin with a brief review of the uniform integrability. Given sequences of quantum states $\{\rho^{(n)}\}_{n\in\mathbb{N}}$ and observables $\{B^{(n)}\}_{n\in\mathbb{N}}$ on Hilbert spaces $\{\mathcal{H}^{(n)}\}_{n\in\mathbb{N}}$, we say that $B^{(n)}$ is *uniformly integrable* with respect to $\rho^{(n)}$ if for all $\varepsilon>0$, there exists $L>0$ that satisfies

$$\mathrm{Tr}\,\rho^{(n)}\left|B^{(n)}-h_L(B^{(n)})\right|<\varepsilon \tag{D.8}$$

for all n. Here, the function h_L is defined by

$$h_L(x)=\begin{cases}x & (|x|\le L)\\ 0 & (|x|>L)\end{cases}. \tag{D.9}$$

When $\mathrm{Tr}\,\rho^{(n)}|B^{(n)}|<\infty$ for all $n\in\mathbb{N}$, the uniform integrability is equivalent to saying that

$$\limsup_{n\to\infty}\mathrm{Tr}\,\rho^{(n)}\left|B^{(n)}-h_L(B^{(n)})\right|<\varepsilon.$$

Note that (D.8) implies $\mathrm{Tr}\,\rho^{(n)}|B^{(n)}|<L+\varepsilon$ for all n. In other words, uniform integrability entails uniform boundedness of $\mathrm{Tr}\,\rho^{(n)}|B^{(n)}|$.

PROOF OF LEMMA 3.6. Set

$$\tilde{A}^{(n)}:=\left\{\prod_{s=2}^{r_1}f_s(\xi_s^iX_i^{(n)})\right\}A^{(n)}\left\{\prod_{t=2}^{r_2}g_t(\eta_t^iX_i^{(n)})\right\}^*.$$

Then $\tilde{A}^{(n)}$ is uniformly bounded, i.e., there is an $\tilde{M}>0$ such that $\|\tilde{A}^{(n)}\|<\tilde{M}$ for all $n\in\mathbb{N}\cup\{\infty\}$. Further, set

$$Y^{(n)}:=\xi_1^iX_i^{(n)},\quad Z^{(n)}:=\eta_1^iX_i^{(n)},\quad \tilde{Y}^{(n)}:=Y^{(n)}+o_1^{(n)},\quad \tilde{Z}^{(n)}:=Z^{(n)}+o_2^{(n)}.$$

It then follows from the proof of Lemma 3.5 that, for any $s,t\in\mathbb{R}$,

$$\lim_{n\to\infty}\mathrm{Tr}\,\rho^{(n)}e^{\sqrt{-1}sY^{(n)}}\tilde{A}^{(n)}e^{\sqrt{-1}tZ^{(n)}}=\mathrm{Tr}\,\rho^{(\infty)}e^{\sqrt{-1}sY^{(\infty)}}\tilde{A}^{(\infty)}e^{\sqrt{-1}tZ^{(\infty)}}.$$

and therefore

$$\lim_{n\to\infty}\mathrm{Tr}\,\rho^{(n)}e^{\sqrt{-1}s\tilde{Y}^{(n)}}\tilde{A}^{(n)}e^{\sqrt{-1}t\tilde{Z}^{(n)}}=\mathrm{Tr}\,\rho^{(\infty)}e^{\sqrt{-1}sY^{(\infty)}}\tilde{A}^{(\infty)}e^{\sqrt{-1}tZ^{(\infty)}}.$$

We can further deduce from Lemma 3.5 that, for any $L>0$,

$$\lim_{n\to\infty}\mathrm{Tr}\,\rho^{(n)}f_L(\tilde{Y}^{(n)})\tilde{A}^{(n)}g_L(\tilde{Z}^{(n)})=\mathrm{Tr}\,\rho^{(\infty)}f_L(Y^{(\infty)})\tilde{A}^{(\infty)}g_L(Z^{(\infty)}), \tag{D.10}$$

where $f_L:=h_L\circ f_1$ and $g_L:=h_L\circ g_1$ are bounded functions. Our goal is to prove that f_L and g_L in (D.10) can be replaced with f_1 and g_1 if both $f_1(\tilde{Y}^{(n)})^2$ and $g_1(\tilde{Z}^{(n)})^2$ are uniformly integrable under $\rho^{(n)}$.

As stated in the preliminary remark of this subsection, there exists a $K>0$ that fulfills

$$\max\left\{\mathrm{Tr}\,\rho^{(n)}f_1(\tilde{Y}^{(n)})^2,\ \mathrm{Tr}\,\rho^{(n)}g_1(\tilde{Z}^{(n)})^2\right\}\le K \tag{D.11}$$

for all $n\in\mathbb{N}\cup\{\infty\}$. In addition, for any $\varepsilon>0$, there exists an $L>0$ such that

$$\max\left\{\mathrm{Tr}\,\rho^{(n)}\left(f_1(\tilde{Y}^{(n)})-f_L(\tilde{Y}^{(n)})\right)^2,\ \mathrm{Tr}\,\rho^{(n)}\left(g_1(\tilde{Z}^{(n)})-g_L(\tilde{Z}^{(n)})\right)^2\right\}<\varepsilon \tag{D.12}$$

for all $n \in \mathbb{N} \cup \{\infty\}$. Observe that, for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\text{(D.13)}\quad & \left| \operatorname{Tr} \rho^{(n)} f_1(\tilde{Y}^{(n)}) \tilde{A}^{(n)} g_1(\tilde{Z}^{(n)}) - \operatorname{Tr} \rho^{(\infty)} f_1(Y^{(\infty)}) \tilde{A}^{(\infty)} g_1(Z^{(\infty)}) \right| \\
& \le \left| \operatorname{Tr} \rho^{(n)} f_1(\tilde{Y}^{(n)}) \tilde{A}^{(n)} g_1(\tilde{Z}^{(n)}) - \operatorname{Tr} \rho^{(n)} f_L(\tilde{Y}^{(n)}) \tilde{A}^{(n)} g_L(\tilde{Z}^{(n)}) \right| \\
& \quad + \left| \operatorname{Tr} \rho^{(n)} f_L(\tilde{Y}^{(n)}) \tilde{A}^{(n)} g_L(\tilde{Z}^{(n)}) - \operatorname{Tr} \rho^{(\infty)} f_L(Y^{(\infty)}) \tilde{A}^{(\infty)} g_L(Z^{(\infty)}) \right| \\
& \quad + \left| \operatorname{Tr} \rho^{(\infty)} f_L(Y^{(\infty)}) \tilde{A}^{(\infty)} g_L(Z^{(\infty)}) - \operatorname{Tr} \rho^{(\infty)} f_1(Y^{(\infty)}) \tilde{A}^{(\infty)} g_1(Z^{(\infty)}) \right|.
\end{aligned}
$$

The second line in (D.13) is evaluated as follows. For any $\varepsilon > 0$, take $L > 0$ satisfying (D.12). Then by using (D.11),

$$
\begin{aligned}
\text{(D.14)}\quad & \left| \operatorname{Tr} \rho^{(n)} f_1(\tilde{Y}^{(n)}) \tilde{A}^{(n)} g_1(\tilde{Z}^{(n)}) - \operatorname{Tr} \rho^{(n)} f_L(\tilde{Y}^{(n)}) \tilde{A}^{(n)} g_L(\tilde{Z}^{(n)}) \right| \\
& \le \left| \operatorname{Tr} \rho^{(n)} \left\{ f_1(\tilde{Y}^{(n)}) - f_L(\tilde{Y}^{(n)}) \right\} \tilde{A}^{(n)} g_1(\tilde{Z}^{(n)}) \right| \\
& \quad + \left| \operatorname{Tr} \rho^{(n)} f_L(\tilde{Y}^{(n)}) \tilde{A}^{(n)} \left\{ g_1(\tilde{Z}^{(n)}) - g_L(\tilde{Z}^{(n)}) \right\} \right| \\
& \le \sqrt{ \operatorname{Tr} \rho^{(n)} \left\{ f_1(\tilde{Y}^{(n)}) - f_L(\tilde{Y}^{(n)}) \right\}^2 \times \operatorname{Tr} \rho^{(n)} \left| \tilde{A}^{(n)} g_1(\tilde{Z}^{(n)}) \right|^2 } \\
& \quad + \sqrt{ \operatorname{Tr} \rho^{(n)} \left| \left(f_L(\tilde{Y}^{(n)}) \tilde{A}^{(n)} \right)^* \right|^2 \times \operatorname{Tr} \rho^{(n)} \left\{ g_1(\tilde{Z}^{(n)}) - g_L(\tilde{Z}^{(n)}) \right\}^2 } \\
& < 2\tilde{M}\sqrt{\varepsilon K}.
\end{aligned}
$$

The last line in (D.13) is evaluated just by setting $n = \infty$ in (D.14). Finally, the third line in (D.13) is evaluated as follows: because of (D.10), for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $n \ge N$ implies

$$
\left| \operatorname{Tr} \rho^{(n)} f_L(\tilde{Y}^{(n)}) \tilde{A}^{(n)} g_L(\tilde{Z}^{(n)}) - \operatorname{Tr} \rho^{(\infty)} f_L(Y^{(\infty)}) \tilde{A}^{(\infty)} g_L(Z^{(\infty)}) \right| < \varepsilon.
$$

Putting these evaluations together, we have

$$
\left| \operatorname{Tr} \rho^{(n)} f_1(\tilde{Y}^{(n)}) \tilde{A}^{(n)} g_1(\tilde{Z}^{(n)}) - \operatorname{Tr} \rho^{(\infty)} f_1(Y^{(\infty)}) \tilde{A}^{(\infty)} g_1(Z^{(\infty)}) \right| < 4\tilde{M}\sqrt{\varepsilon K} + \varepsilon.
$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

D.7. Proof of Corollary 3.7.

PROOF. The first assertion (12) immediately follows from the conventional quantum Le Cam third lemma [5, Corollary 7.5]. We focus our attention on the proof of (13) and (14).

The basic observation for the proof of (13) is that the square-root likelihood ratio $R_h^{(n)}$ is an (unbounded) function of $X^{(n)}$. Therefore, in order to invoke the extended version of the sandwiched Lévy-Cramér continuity theorem (Lemma 3.6), we need to show the uniform integrability. As a matter of fact, uniform integrability of $(R_h^{(n)} + o_{L^2}(\rho_{\theta_0}^{(n)}))^2$ under $\rho_{\theta_0}^{(n)}$ has been shown in [5, Theorem 6.2]. For the sake of the reader's convenience, however, we give a simplified proof.

Let $\sigma^{(\infty)}$ be the density operator of $N(0, J)$ and let $\{\Delta_i^{(\infty)}\}_{i=1}^d$ be the corresponding canonical observables. Since the model $\rho_\theta^{(n)}$ is q-LAN at θ_0, for any $\varepsilon > 0$, there exists an

$L > 0$ such that

$$\begin{aligned}
&\limsup_{n\to\infty} \operatorname{Tr} \rho_{\theta_0}^{(n)} \left\{ \left(R_h^{(n)} + o_{L^2}(\rho_{\theta_0}^{(n)}) \right)^2 - h_L \left(\left\{ R_h^{(n)} + o_{L^2}(\rho_{\theta_0}^{(n)}) \right\}^2 \right) \right\} \\
&\quad \le \limsup_{n\to\infty} \left\{ 1 - \operatorname{Tr} \rho_{\theta_0}^{(n)} h_L \left(\left\{ R_h^{(n)} + o_{L^2}(\rho_{\theta_0}^{(n)}) \right\}^2 \right) \right\} \\
&\quad = \limsup_{n\to\infty} \left\{ 1 - \operatorname{Tr} \rho_{\theta_0}^{(n)} h_L \left(e^{h^i \Delta_i^{(n)} - \frac{1}{2} h^\top J h + o_D(h^i \Delta_i^{(n)}, \rho_{\theta_0}^{(n)})} \right) \right\} \\
&\quad = 1 - \operatorname{Tr} \sigma^{(\infty)} h_L \left(e^{h^i \Delta_i^{(\infty)} - \frac{1}{2} h^\top J h} \right) \\
&\quad < \varepsilon.
\end{aligned}$$

Here, the function h_L is defined by (D.9), and the last equality is guaranteed by the quantum Lévy-Cramér continuity theorem (cf., Lemma 3.5 with $A^{(n)} = I^{(n)}$). This proves that $(R_h^{(n)} + o_{L^2}(\rho_{\theta_0}^{(n)}))^2$ is uniformly integrable.

Now we prove (13). Since $\{X_k^{(n)}\}_{1\le k\le r}$ is a D-extension of $\{\Delta_i^{(n)}\}_{1\le i\le d}$,

$$\Delta_i^{(n)} = F_i^k X_k^{(n)}.$$

It then follows from the definition of q-LAN that, for all $h \in \mathbb{R}^d$,

$$R_h^{(n)} = \exp\left\{ \frac{1}{2} \left((Fh)^i X_i^{(n)} - \frac{1}{2} h^\top J h + o_h^{(n)} \right) \right\} - o_{L^2}(\rho_{\theta_0}^{(n)}),$$

where $o_h^{(n)} = o_D((Fh)^i X_i^{(n)}, \rho_{\theta_0}^{(n)})$, and $J = F^\top \Sigma F$. Since $(R_h^{(n)} + o_{L^2}(\rho_{\theta_0}^{(n)}))^2$ is uniformly integrable for all h under $\rho_{\theta_0}^{(n)}$, we can conclude from the extended version of the sandwiched Lévy-Cramér continuity theorem (Lemma 3.6) that

$$\begin{aligned}
&\lim_{n\to\infty} \operatorname{Tr} \rho_{\theta_0}^{(n)} R_{h_1}^{(n)} A^{(n)} R_{h_2}^{(n)} \\
&\quad = \lim_{n\to\infty} \operatorname{Tr} \rho_{\theta_0}^{(n)} e^{\frac{1}{2}\left((Fh_1)^i X_i^{(n)} - \frac{1}{2} h_1^\top J h_1 + o_{h_1}^{(n)} \right)} A^{(n)} e^{\frac{1}{2}\left((Fh_2)^i X_i^{(n)} - \frac{1}{2} h_2^\top J h_2 + o_{h_2}^{(n)} \right)} \\
&\quad = \operatorname{Tr} \rho_0^{(\infty)} e^{\frac{1}{2}\left((Fh_1)^i X_i^{(\infty)} - \frac{1}{2} h_1^\top J h_1 \right)} A^{(\infty)} e^{\frac{1}{2}\left((Fh_2)^i X_i^{(\infty)} - \frac{1}{2} h_2^\top J h_2 \right)} \\
&\quad = \operatorname{Tr} \rho_0^{(\infty)} R_{h_1}^{(\infty)} A^{(\infty)} R_{h_2}^{(\infty)}.
\end{aligned}$$

Finally, (14) immediately follows from (13) and the fact that the singular parts asymptotically vanish [5, Corollary 7.5], in that,

$$\lim_{n\to\infty} \operatorname{Tr} \left| \rho_{\theta_0 + h/\sqrt{n}}^{(n)} - R_h^{(n)} \rho_{\theta_0}^{(n)} R_h^{(n)} \right| = \lim_{n\to\infty} \left\{ 1 - \operatorname{Tr} \rho_{\theta_0}^{(n)} R_h^{(n)^2} \right\} = 0.$$

The proof is complete. □

D.8. Proof of Lemma 3.8.

PROOF. The Hilbert-Schmidt norm under consideration is calculated as

$$\begin{aligned}
&\left\| W^{(n)}(\xi) W^{(n)}(\eta) \sqrt{\rho^{(n)}} - e^{\sqrt{-1}\xi^\top S \eta}\, W^{(n)}(\xi + \eta) \sqrt{\rho^{(n)}} \right\|_{\mathrm{HS}}^2 \\
&\quad = \operatorname{Tr} \sqrt{\rho^{(n)}} W^{(n)}(\eta)^* W^{(n)}(\xi)^* W^{(n)}(\xi) W^{(n)}(\eta) \sqrt{\rho^{(n)}}
\end{aligned}$$

$$
\begin{aligned}
&+ \operatorname{Tr} \sqrt{\rho^{(n)}} W^{(n)}(\xi+\eta)^* W^{(n)}(\xi+\eta) \sqrt{\rho^{(n)}} \\
&- 2 \operatorname{Re} \left\{ e^{\sqrt{-1}\xi^\top S \eta} \cdot \operatorname{Tr} \sqrt{\rho^{(n)}} W^{(n)}(\eta)^* W^{(n)}(\xi)^* W^{(n)}(\xi+\eta) \sqrt{\rho^{(n)}} \right\} \\
= {} & 2 - 2 \operatorname{Re} \left\{ e^{\sqrt{-1}\xi^\top S \eta} \cdot \operatorname{Tr} \rho^{(n)} W^{(n)}(-\eta) W^{(n)}(-\xi) W^{(n)}(\xi+\eta) \right\}.
\end{aligned}
$$

Letting $W(\xi) := e^{\sqrt{-1}\xi^i X_i}$, we see from the assumption $(X^{(n)}, \rho^{(n)}) \rightsquigarrow N(0, J)$ that

$$
\begin{aligned}
&\lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} W^{(n)}(-\eta) W^{(n)}(-\xi) W^{(n)}(\xi+\eta) \\
&\quad = \operatorname{Tr} \rho^{(\infty)} W(-\eta) W(-\xi) W(\xi+\eta) \\
&\quad = e^{\sqrt{-1}(-\eta)^i(-\xi)^j S_{ij}} \operatorname{Tr} \rho^{(\infty)} W(-\eta-\xi) W(\xi+\eta) \\
&\quad = e^{\sqrt{-1}\eta^\top S \xi},
\end{aligned}
$$

where $\rho^{(\infty)}$ is the density operator of $N(0, J)$. Since S is real skew-symmetric,

$$
\begin{aligned}
&\lim_{n\to\infty} \left\| W^{(n)}(\xi) W^{(n)}(\eta) \sqrt{\rho^{(n)}} - e^{\sqrt{-1}\xi^\top S \eta}\, W^{(n)}(\xi+\eta) \sqrt{\rho^{(n)}} \right\|_{\mathrm{HS}}^2 \\
&\quad = 2 - 2 \operatorname{Re} \left\{ e^{\sqrt{-1}\xi^\top S \eta} \cdot e^{\sqrt{-1}\eta^\top S \xi} \right\} = 0.
\end{aligned}
$$

This proves the claim. □

APPENDIX E: PROOFS OF THEOREMS IN SECTION 5

In this section, we give detailed proofs of theorems presented in Section 5.

E.1. Chain of convergence. We begin with the following Lemma, which is elementary but is useful in later applications.

LEMMA E.1 (Chain of convergence). *Let X and Y be sets and $\langle Z, d\rangle$ be a metric space. Suppose that sequences of functions $F_n : X \to Z$ and $G_n : Y \to Z$ for $n \in \mathbb{N} \cup \{\infty\}$ satisfy the following conditions:*

$$
\lim_{n\to\infty} F_n(x) = F_\infty(x), \qquad (\forall x \in X),
$$

and for all $\varepsilon > 0$ and $y \in Y$, there exists $x \in X$ satisfying

$$
\limsup_{n\to\infty} d\left(G_n(y), F_n(x)\right) < \varepsilon \quad \text{and} \quad d\left(G_\infty(y), F_\infty(x)\right) < \varepsilon.
$$

Then

$$
\lim_{n\to\infty} G_n(y) = G_\infty(y), \qquad (\forall y \in Y).
$$

PROOF. Take $\varepsilon > 0$ and $y \in Y$ arbitrarily. Then, there exist $x \in X$ and $N \in \mathbb{N}$ such that

$$
d\left(F_n(x), F_\infty(x)\right) < \varepsilon \quad \text{and} \quad d\left(G_n(y), F_n(x)\right) < 2\varepsilon
$$

for all $n \ge N$, and

$$
d\left(G_\infty(y), F_\infty(x)\right) < \varepsilon.
$$

Thus

$$
d\left(G_n(y), G_\infty(y)\right) \le d\left(G_n(y), F_n(x)\right) + d\left(F_n(x), F_\infty(x)\right) + d\left(F_\infty(x), G_\infty(y)\right) < 4\varepsilon,
$$

proving the claim. □

E.2. Proof of Lemma 5.1.

PROOF. Let $\mathcal{L}$ be the classical distribution obtained by applying the shifted POVM $M-h$ to ϕ_h, that is, $\mathcal{L}(B) := \phi_h((M-h)(B))$, which is independent of h by assumption. Let m be the first moment of $\mathcal{L}$. Then, for each $i = 1, \dots, d$,

$$\begin{aligned}\int_{\mathbb{R}^d} (z-h)^i \phi_h((M-m)(dz)) &= \int_{\mathbb{R}^d+m} (x-m-h)^i \, \phi_h(M(dx)) \\ &= \int_{\mathbb{R}^d+m-h} (y-m)^i \phi_h((M-h)(dy)) \\ &= \int_{\mathbb{R}^d} (y-m)^i \mathcal{L}(dy) = 0.\end{aligned}$$

This implies that $M-m$ is an unbiased estimator for the parameter h of ϕ_h. It then follows from the quantum Cramér-Rao type inequality [2] that

$$\begin{aligned}c_G^{(H)} &\leq G_{ij} \int_{\mathbb{R}^d} (z-h)^i (z-h)^j \phi_h((M-m)(dz)) \\ &= G_{ij} \int_{\mathbb{R}^d+m-h} (y-m)^i (y-m)^j \phi_h((M-h)(dy)) \\ &= G_{ij} \left\{ \int_{\mathbb{R}^d} y^i y^j \mathcal{L}(dy) - m^i m^j \right\}.\end{aligned}$$

As a consequence,

$$G_{ij} \int_{\mathbb{R}^d} (x-h)^i \, (x-h)^j \, \phi_h(M(dx)) = G_{ij} \int_{\mathbb{R}^d-h} y^i y^j \phi_h((M-h)(dy)) \geq c_G^{(H)},$$

proving the claim. □

Note that this result is closely related to what Holevo established in [2] within the framework of group covariant measurement, where the achievability of the lower bound was also discussed.

E.3. Proof of Theorem 5.2.

PROOF. By applying the representation Theorem 2.4 to the sequence

$$N^{(n)} := M^{(n)h} + h = \sqrt{n}(M^{(n)} - \theta_0)$$

of POVMs that is independent of $h \in \mathbb{R}^d$, we see that there exists a POVM N on $\phi_h \sim N((\mathrm{Re}\,\tau)h, \Sigma)$ such that

$$\left(N^{(n)}, \rho^{(n)}_{\theta_0+h/\sqrt{n}}\right) \overset{h}{\rightsquigarrow} (N, \phi_h) \qquad (\forall h \in \mathbb{R}^d).$$

Let $\mathcal{L}_h$ denote the classical probability distribution of outcomes of N applied to ϕ_h. Then, by construction, for any $B \in \mathcal{B}(\mathbb{R}^d)$ that satisfies $\mathcal{L}_h(\partial B) = 0$,

$$\begin{aligned}\mathcal{L}_h(B) &= \phi_h(N(B)) \\ &= \lim_{n\to\infty} \mathrm{Tr}\, \rho^{(n)}_{\theta_0+h/\sqrt{n}} N^{(n)}(B) \\ &= \lim_{n\to\infty} \mathrm{Tr}\, \rho^{(n)}_{\theta_0+h/\sqrt{n}} (M^{(n)h} + h)(B) \\ &= \lim_{n\to\infty} \mathrm{Tr}\, \rho^{(n)}_{\theta_0+h/\sqrt{n}} M^{(n)h}(B-h) \\ &= \mathcal{L}(B-h).\end{aligned}$$

Here, $\mathcal{L}$ is the limit distribution of $M^{(n)h}$ under $\rho^{(n)}_{\theta_0+h/\sqrt{n}}$, which is independent of h by regularity. As a consequence,

$$\phi_h((N-h)(B)) = \phi_h(N(B+h)) = \mathcal{L}_h(B+h) = \mathcal{L}(B).$$

Since the last side is independent of h, N is equivalent in law. Thus, Lemma 5.1 yields

$$\int_{\mathbb{R}^d} G_{ij}(x-h)^i(x-h)^j \phi_h(N(dx)) \geq c_G^{(rep)},$$

which implies (25), and the portmanteau lemma proves (26). □

E.4. Proof of Theorem 5.3.

PROOF. Let $\{X_i\}_{i=1}^r$ be the canonical observables of $\phi_h \sim N((\operatorname{Re}\tau)h, \Sigma)$, and let

$$Y_{\star i} := (K_\star)_i^j X_j, \tag{E.1}$$

where $K_\star$ is the $r \times d$ matrix K that achieves the minimum in the definition (7) of $c_G^{(rep)}$. Thus,

$$c_G^{(rep)} = \operatorname{Tr} G \operatorname{Re} Z_\star + \operatorname{Tr} \left| \sqrt{G} \operatorname{Im} Z_\star \sqrt{G} \right| = \operatorname{Tr} G V_\star,$$

where

$$Z_\star = K_\star^\top \Sigma K_\star$$

is the complex $d \times d$ matrix whose (i,j)th entry is $\phi_0(Y_{\star j} Y_{\star i})$, and

$$V_\star := \operatorname{Re} Z_\star + \sqrt{G}^{-1} \left| \sqrt{G} \operatorname{Im} Z_\star \sqrt{G} \right| \sqrt{G}^{-1},$$

is a real $d \times d$ matrix. Note that $V_\star \geq Z_\star$, since

$$V_\star \geq \operatorname{Re} Z_\star + \sqrt{-1}\sqrt{G}^{-1}\sqrt{G} \operatorname{Im} Z_\star \sqrt{G}\sqrt{G}^{-1} = Z_\star.$$

By analogy to (E.1), we introduce a sequence of transformed observables

$$Y_{\star i}^{(n)} := (K_\star)_i^j X_j^{(n)}$$

on $\mathcal{H}^{(n)}$.

Let us consider another quantum Gaussian state $\tilde{\phi} \sim N(0, \tilde{Z}_\star)$ with $\tilde{Z}_\star := V_\star - Z_\star$ and canonical observables $\tilde{Y} = (\tilde{Y}_1, \dots, \tilde{Y}_d)$ on an ancillary Hilbert space $\tilde{\mathcal{H}}$. Accordingly, for each $n \in \mathbb{N}$, we introduce a quantum state $\sigma_\star^{(n)}$ and observables $\tilde{Y}_\star^{(n)} = (\tilde{Y}_{\star 1}^{(n)}, \dots, \tilde{Y}_{\star d}^{(n)})$ on an ancillary Hilbert space $\tilde{\mathcal{H}}^{(n)}$ satisfying

$$(\tilde{Y}_\star^{(n)}, \sigma_\star^{(n)}) \rightsquigarrow N(0, \tilde{Z}_\star). \tag{E.2}$$

A key observation is that the series of observables[2]

$$\bar{Y}_{\star i}^{(n)} := Y_{\star i}^{(n)} \otimes I + I \otimes \tilde{Y}_{\star i}^{(n)} \qquad (1 \leq i \leq d)$$

on the enlarged Hilbert spaces $\mathcal{H}^{(n)} \otimes \tilde{\mathcal{H}}^{(n)}$ exhibits

$$(\bar{Y}_\star^{(n)}, \rho^{(n)}_{\theta_0+h/\sqrt{n}} \otimes \sigma_\star^{(n)}) \overset{h}{\rightsquigarrow} N(h, V_\star). \tag{E.3}$$

[2]This construction was inspired by the optical heterodyne measurement [2].

This can be verified by calculating the limit of the quasi-characteristic function

$$\text{(E.4)}\qquad \mathrm{Tr}\left(\rho^{(n)}_{\theta_0+h/\sqrt{n}}\otimes\sigma^{(n)}_\star\right)\prod_{t=1}^{T}e^{\sqrt{-1}\xi_t^i\bar{Y}^{(n)}_{\star i}}$$
$$=\left\{\mathrm{Tr}\,\rho^{(n)}_{\theta_0+h/\sqrt{n}}\prod_{t=1}^{T}e^{\sqrt{-1}\xi_t^i(K_\star)_i^jX_j^{(n)}}\right\}\times\left\{\mathrm{Tr}\,\sigma^{(n)}_\star\prod_{t=1}^{T}e^{\sqrt{-1}\xi_t^i\tilde{Y}^{(n)}_{\star i}}\right\},$$

where $\{\xi_t\}_{t=1}^T\subset\mathbb{R}^d$. In fact, since

$$(X^{(n)},\rho^{(n)}_{\theta_0+h/\sqrt{n}})\overset{h}{\rightsquigarrow}N((\mathrm{Re}\,\tau)h,\Sigma),$$

which follows from the quantum Le Cam third lemma (Corollary 3.7), the first factor in the second line of (E.4) has the limit

$$\text{(E.5)}\qquad \exp\left[\sum_{t=1}^{T}\left\{\sqrt{-1}\,\xi_t^i(K_\star)_i^j(\mathrm{Re}\,\tau)_{jk}h^k-\frac{1}{2}\xi_t^i(K_\star)_i^k\cdot\xi_t^j(K_\star)_j^\ell\,\Sigma_{\ell k}\right\}\right.$$
$$\left.-\sum_{t=1}^{T}\sum_{u=r+1}^{T}\xi_t^i(K_\star)_i^k\cdot\xi_u^j(K_\star)_j^\ell\,\Sigma_{\ell k}\right]$$
$$=\exp\left[\sum_{t=1}^{T}\left\{\sqrt{-1}\,\xi_t^i\delta_{ij}h^j-\frac{1}{2}\xi_t^i\xi_t^j(Z_\star)_{ji}\right\}-\sum_{t=1}^{T}\sum_{u=r+1}^{T}\xi_t^i\xi_u^j(Z_\star)_{ji}\right].$$

Here, the equalities $K_\star^\top(\mathrm{Re}\,\tau)=I$ and $K_\star^\top\Sigma K_\star=Z_\star$ have been used. On the other hand, due to the assumption (E.2), the second factor in the second line of (E.4) has the limit

$$\text{(E.6)}\qquad \exp\left[\sum_{t=1}^{T}\left\{-\frac{1}{2}\xi_t^i\xi_t^j(\tilde{Z}_\star)_{ji}\right\}-\sum_{t=1}^{T}\sum_{u=r+1}^{T}\xi_t^i\xi_u^j(\tilde{Z}_\star)_{ji}\right].$$

Since $Z_\star+\tilde{Z}_\star=V_\star$, (E.4), (E.5), and (E.6) yield

$$\lim_{n\to\infty}\mathrm{Tr}\left(\rho^{(n)}_{\theta_0+h/\sqrt{n}}\otimes\sigma^{(n)}_\star\right)\prod_{t=1}^{T}e^{\sqrt{-1}\xi_t^i\bar{Y}^{(n)}_{\star i}}$$
$$=\exp\left[\sum_{t=1}^{T}\left\{\sqrt{-1}\,\xi_t^i\delta_{ij}h^j-\frac{1}{2}\xi_t^i\xi_t^j(V_\star)_{ji}\right\}-\sum_{t=1}^{T}\sum_{u=r+1}^{T}\xi_t^i\xi_u^j(V_\star)_{ji}\right].$$

This is nothing but the quasi-characteristic function of the classical Gaussian shift model $N(h,V_\star)$, proving (E.3).

We next construct a sequence $M_\star^{(n)}$ of POVMs by means of functional calculus for $\bar{Y}_\star^{(n)}$. Since $\{\bar{Y}^{(n)}_{\star i}\}_{i=1}^d$ do not in general commute, we need some elaboration. For each positive integer $m\in\mathbb{N}$, define an indicator function $S^{(m)}:\mathbb{R}\to\{0,1\}$ by

$$S^{(m)}(x):=\mathbb{1}_{(-\frac{1}{2m},\frac{1}{2m}]}(x)=\begin{cases}1, & \text{if } x\in(-\frac{1}{2m},\frac{1}{2m}]\\ 0, & \text{if } x\notin(-\frac{1}{2m},\frac{1}{2m}]\end{cases}.$$

Then, for each $x\in\mathbb{R}$ and $n\in\mathbb{N}$, the map enjoys the identity

$$\text{(E.7)}\qquad \sum_{k\in\mathbb{Z}}S^{(m)}\left(x-\frac{k}{m}\right)=1.$$

For each pair (n, m) of positive integers, define

$$M_\star^{(n,m)}(\omega) := \left(\prod_{i=1}^d S^{(m)}\left(\bar{Y}_{\star i}^{(n)} - \omega_i\right)\right)\left(\prod_{i=1}^d S^{(m)}\left(\bar{Y}_{\star i}^{(n)} - \omega_i\right)\right)^*,$$

where

$$\omega = (\omega_1, \dots, \omega_d) \in \Omega^{(m)} := \left\{\left(\frac{z_1}{m}, \dots, \frac{z_d}{m}\right) \mid z_1, \dots, z_d \in \mathbb{Z}\right\}.$$

It then follows from (E.7) that $M_\star^{(n,m)}$ is a POVM on $\mathcal{H}^{(n)} \otimes \tilde{\mathcal{H}}$ whose outcomes take values on $\Omega^{(m)}$.

Note that, due to (E.3) and the quantum Lévy-Cramér continuity theorem (cf., Lemma 3.5 with $A^{(n)} = I^{(n)}$), as well as the fact that the set of discontinuity points of $S^{(m)}$ has Lebesgue measure zero, the following equality holds:

$$\lim_{n\to\infty} \mathrm{Tr}\left(\rho_{\theta_0+h/\sqrt{n}}^{(n)} \otimes \sigma_\star^{(n)}\right) M_\star^{(n,m)}(\omega) = \int_{\mathbb{R}^d} p_h(x) \prod_{i=1}^d S^{(m)}(x_i - \omega_i) dx,$$

where $p_h(x)$ denotes the probability density function of $N(h, V_\star)$. Note also that for each $t = (t_i) \in \mathbb{R}^d$, the indicator function $\chi_t(x) := \mathbb{1}_{(-\infty,t]}(x)$ fulfills the following equality

$$\lim_{m\to\infty} \sum_{\omega\in\Omega^{(m)}} \chi_t(\omega) \prod_{i=1}^d S^{(m)}(x_i - \omega_i) = \chi_t(x)$$

for all $x \in \mathbb{R}^d$ but $x = t$. Combining these equalities, we have

$$\lim_{m\to\infty} \lim_{n\to\infty} \sum_{\omega\in\Omega^{(m)}} \chi_t(\omega) \mathrm{Tr}\left(\rho_{\theta_0+h/\sqrt{n}}^{(n)} \otimes \sigma_\star^{(n)}\right) M_\star^{(n,m)}(\omega) = \int_{\mathbb{R}^d} \chi_t(x) p_h(x) dx$$

for all $h \in \mathbb{R}^d$ and $t \in \mathbb{R}^d$.

As a consequence, the diagonal sequence trick shows that there exists a subsequence $\{m(n)\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \sum_{\omega\in\Omega^{(m(n))}} \chi_t(\omega) \mathrm{Tr}\left(\rho_{\theta_0+h/\sqrt{n}}^{(n)} \otimes \sigma_\star^{(n)}\right) M_\star^{(n,m(n))}(\omega) = \int_{\mathbb{R}^d} \chi_t(x) p_h(x) dx$$

for all $h \in \mathbb{Q}^d$ and $t \in \mathbb{Q}^d$. Setting $\bar{M}_\star^{(n)} := M_\star^{(n,m(n))}$ and $\bar{\Omega}^{(n)} := \Omega^{(m(n))}$, we get

$$\lim_{n\to\infty} \sum_{\omega\in\bar{\Omega}^{(n)}} \chi_t(\omega) \mathrm{Tr}\left(\rho_{\theta_0+h/\sqrt{n}}^{(n)} \otimes \sigma_\star^{(n)}\right) \bar{M}_\star^{(n)}(\omega) = \int_{\mathbb{R}^d} \chi_t(x) p_h(x) dx \tag{E.8}$$

for all $h \in \mathbb{Q}^d$ and $t \in \mathbb{Q}^d$. Moreover, since both sides of (E.8) are monotone increasing in t, and the right-hand side is continuous in t, (E.8) holds for all $t \in \mathbb{R}^d$ and all $h \in \mathbb{Q}^d$.

Now let, for each $t \in \mathbb{R}^d$,

$$M_{\star t}^{(n)} := \mathrm{Tr}_{\tilde{\mathcal{H}}^{(n)}}\left\{\sum_{\omega\in\bar{\Omega}^{(n)}} \chi_t(\omega) \bar{M}_\star^{(n)}(\omega)\right\}\left(I^{(n)} \otimes \sigma_\star^{(n)}\right),$$

where $\mathrm{Tr}_{\tilde{\mathcal{H}}^{(n)}}$ denotes the partial trace on $\tilde{\mathcal{H}}^{(n)}$. Then, $M_{\star t}^{(n)}$ is a resolution of identity on $\mathcal{H}^{(n)}$ satisfying

$$\begin{aligned}\lim_{n\to\infty} \mathrm{Tr}\, \rho_{\theta_0+h/\sqrt{n}}^{(n)} M_{\star t}^{(n)} &= \lim_{n\to\infty} \sum_{\omega\in\bar{\Omega}^{(n)}} \chi_t(\omega) \mathrm{Tr}\left(\rho_{\theta_0+h/\sqrt{n}}^{(n)} \otimes \sigma_\star^{(n)}\right) \bar{M}_\star^{(n)}(\omega) \\ &= \int_{\mathbb{R}^d} \chi_t(x) p_h(x) dx\end{aligned} \tag{E.9}$$

for all $t \in \mathbb{R}^d$ and all $h \in \mathbb{Q}^d$.

Finally, we extend the identity (E.9) to all $h \in \mathbb{R}^d$. Let

$$R_h^{(n)} := \mathcal{R}\left(\rho^{(n)}_{\theta_0+h/\sqrt{n}} \,\middle|\, \rho^{(n)}_{\theta_0}\right) \quad \text{and} \quad R_h^{(\infty)} := e^{\frac{1}{2}\left((Fh)^i X_i^{(\infty)} - \frac{1}{2} h^\top F^\top \Sigma F h\right)}$$

and fix $h \in \mathbb{R}^d$ and $t \in \mathbb{R}^d$ arbitrarily. Then for all $\varepsilon > 0$, there exists an $\tilde{h} \in \mathbb{Q}^d$ that satisfies

$$\phi_0\left(\left(R_h^{(\infty)} - R_{\tilde{h}}^{(\infty)}\right)^2\right) < \varepsilon.$$

On the other hand,

$$\begin{aligned}
\text{(E.10)} \qquad & \limsup_{n\to\infty} \left| \mathrm{Tr}\left(\rho^{(n)}_{\theta_0+h/\sqrt{n}} - \rho^{(n)}_{\theta_0+\tilde{h}/\sqrt{n}}\right) M^{(n)}_{\star t}\right| \\
&= \limsup_{n\to\infty} \left| \mathrm{Tr}\left(R_h^{(n)} \rho^{(n)}_{\theta_0} R_h^{(n)} - R_{\tilde{h}}^{(n)} \rho^{(n)}_{\theta_0} R_{\tilde{h}}^{(n)}\right) M^{(n)}_{\star t}\right| \\
&\le \limsup_{n\to\infty} \left\{ \left| \mathrm{Tr}\left(R_h^{(n)} - R_{\tilde{h}}^{(n)}\right) \rho^{(n)}_{\theta_0} R_h^{(n)} M^{(n)}_{\star t}\right| \right. \\
&\qquad\qquad \left. + \left| \mathrm{Tr}\, R_{\tilde{h}}^{(n)} \rho^{(n)}_{\theta_0} \left(R_h^{(n)} - R_{\tilde{h}}^{(n)}\right) M^{(n)}_{\star t}\right| \right\}.
\end{aligned}$$

Here, the second line follows from (14) in Corollary 3.7, which tells us that the contribution of the singular parts of $\rho^{(n)}_{\theta_0+h/\sqrt{n}}$ are asymptotically negligible. By using Corollary 3.7, the third line of (E.10) is evaluated as follows:

$$\begin{aligned}
& \limsup_{n\to\infty} \left| \mathrm{Tr}\left(R_h^{(n)} - R_{\tilde{h}}^{(n)}\right) \rho^{(n)}_{\theta_0} R_h^{(n)} M^{(n)}_{\star t}\right|^2 \\
&\quad \le \limsup_{n\to\infty} \mathrm{Tr}\, \rho^{(n)}_{\theta_0} \left(R_h^{(n)} - R_{\tilde{h}}^{(n)}\right)^2 \times \mathrm{Tr}\, \rho^{(n)}_{\theta_0} \left| \left(R_h^{(n)} M^{(n)}_{\star t}\right)^* \right|^2 \\
&\quad \le \limsup_{n\to\infty} \mathrm{Tr}\, \rho^{(n)}_{\theta_0} \left(R_h^{(n)} - R_{\tilde{h}}^{(n)}\right)^2 \\
&\quad = \phi_0\left(\left(R_h^{(\infty)} - R_{\tilde{h}}^{(\infty)}\right)^2\right) < \varepsilon.
\end{aligned}$$

Since the fourth line of (E.10) is evaluated similarly, we can conclude that

$$\text{(E.11)} \qquad \limsup_{n\to\infty} \left| \mathrm{Tr}\left(\rho^{(n)}_{\theta_0+h/\sqrt{n}} - \rho^{(n)}_{\theta_0+\tilde{h}/\sqrt{n}}\right) M^{(n)}_{\star t}\right| < 2\sqrt{\varepsilon}$$

In a quite similar way, we can prove that

$$\text{(E.12)} \qquad \left| \phi_h\left(M_t^{(\infty)}\right) - \phi_{\tilde{h}}\left(M_{\star t}^{(\infty)}\right) \right| < 2\sqrt{\varepsilon}$$

Now that we have established (E.9) (E.11) (E.12), Lemma E.1 leads us to

$$\lim_{n\to\infty} \mathrm{Tr}\, \rho^{(n)}_{\theta_0+h/\sqrt{n}} M^{(n)}_{\star t} = \phi_h\left(M^{(\infty)}_{\star t}\right) = \int_{\mathbb{R}^d} \chi_t(x) p_h(x) dx$$

for all $t \in \mathbb{R}^d$ and $h \in \mathbb{R}^d$. To put it differently, letting $M_\star^{(n)}$ be the POVM that corresponds to the resolution of identity $M^{(n)}_{\star t}$, we have

$$\left(M_\star^{(n)}, \rho^{(n)}_{\theta_0+h/\sqrt{n}}\right) \overset{h}{\rightsquigarrow} N(h, V_\star) \qquad (\forall h \in \mathbb{R}^d).$$

This completes the proof. $\square$

E.5. Proof of Theorem 5.5.

PROOF. For the quantum Gaussian shift model $\phi_h \sim N((\mathrm{Re}\,\tau)h, \Sigma)$, take a family of unitary operators $\{U(k)\}_{k\in\mathbb{R}^d}$ on $\mathcal{H}$ that satisfy

$$\phi_h(U(k)^* A U(k)) = \phi_{h+k}(A) \qquad (\forall A \in \mathrm{CCR}(\mathrm{Im}\Sigma)).$$

Given a POVM M, let $M_t := M(-\infty, t]$ be the corresponding resolution of identity, and let us define, for each $t \in \mathbb{R}^d$ and $L \in \mathbb{N}$, a bounded operator

$$\hat{N}_t^{(L)} := \frac{1}{(2L)^d} \int_{[-L,L]^d} U(k)^* M_{t+k} U(k) dk,$$

where the integration is taken in the weak operator topology (WOT). It is not difficult to show that $\{\hat{N}_t^{(L)}\}_{t\in\mathbb{R}^d}$ is a resolution of identity for all $L \in \mathbb{N}$. From this resolution of identity, we shall construct a POVM N that is equivalent in law and surpasses the original M. Here we follow the method used in Step 2 of the proof of Theorem 2.4.

Take a cyclic vector ψ on the Hilbert space $\mathcal{H}$, and consider the sandwiched coherent state representation

$$\varphi_t^{(L)}(\xi;\eta) := \left\langle e^{\sqrt{-1}\xi^i X_i}\psi, \hat{N}_t^{(L)} e^{\sqrt{-1}\eta^i X_i}\psi \right\rangle.$$

Since $\left|\varphi_t^{(L)}(\xi;\eta)\right| \le 1$ for all $L \in \mathbb{N}$, $t \in \mathbb{R}^d$, and $\xi, \eta \in \mathbb{R}^r$, the diagonal sequence trick shows that there is a subsequence $\{L_m\} \subset \{L\}$ through which $\varphi_\alpha^{(L_m)}(\xi;\eta)$ is convergent for all $\alpha \in \mathbb{Q}^d$ and $\xi, \eta \in \mathbb{Q}^r$, yielding a limiting function $\varphi_\alpha(\xi;\eta)$. Due to Lemma 3.3, this limiting function uniquely determines an operator $\hat{N}_\alpha$ that satisfies

$$\varphi_\alpha(\xi;\eta) = \left\langle e^{\sqrt{-1}\xi^i X_i}\psi, \hat{N}_\alpha e^{\sqrt{-1}\eta^i X_i}\psi \right\rangle.$$

In this way, we obtain the WOT-limit

$$\hat{N}_\alpha := \lim_{m\to\infty} \hat{N}_\alpha^{(L_m)} \tag{E.13}$$

for all $\alpha \in \mathbb{Q}^d$. Further, for each $t \in \mathbb{R}^d$, let

$$\bar{N}_t := \inf_{\alpha > t, \alpha\in\mathbb{Q}^d} \hat{N}_\alpha. \tag{E.14}$$

Then $\{\bar{N}_t\}_{t\in\mathbb{R}^d}$ determines a POVM $\bar{N}$ over $\bar{\mathbb{R}}^d$, and by transferring the measure at infinity $\bar{N}(\bar{\mathbb{R}}^d \setminus \mathbb{R}^d)$ to the origin, we have a POVM N over $\mathbb{R}^d$ defined by

$$N(B) := \bar{N}(B) + \delta_0(B)\bar{N}(\bar{\mathbb{R}}^d \setminus \mathbb{R}^d) \qquad (B \in \mathcal{B}(\mathbb{R}^d)).$$

Let us prove that $\phi_h(\bar{N}(B)) = \phi_h(N(B))$ for all $B \in \mathcal{B}(\mathbb{R}^d)$. For each $m \in \mathbb{N}$, let $\hat{N}^{(L_m)}$ be the POVM that corresponds to the resolution of identity $\hat{N}_t^{(L_m)}$, and let

$$\begin{aligned}
\mu_h^{(m)}(B) &:= \phi_h\left(\hat{N}^{(L_m)}(B)\right) \\
&= \frac{1}{(2L_m)^d} \int_{[-L_m,L_m]^d} \phi_h\left(U(k)^* M(B+k) U(k)\right) dk \\
&= \frac{1}{(2L_m)^d} \int_{[-L_m,L_m]^d} \phi_{h+k}\left(M(B+k)\right) dk.
\end{aligned}$$

Then, letting $y := x + k$,

$$\begin{aligned}\text{(E.15)}\quad &\int_{\mathbb{R}^d} G_{ij}(x-h)^i(x-h)^j \mu_h^{(m)}(dx) \\ &= \frac{1}{(2L_m)^d}\int_{[-L_m,L_m]^d} dk \int_{\mathbb{R}^d+k} G_{ij}(y-h-k)^i(y-h-k)^j \phi_{h+k}\left(M(dy)\right) \\ &\le \sup_{\ell\in\mathbb{R}^d}\int_{\mathbb{R}^d} G_{ij}(y-\ell)^i(y-\ell)^j \phi_\ell\left(M(dy)\right).\end{aligned}$$

This shows that the second moments of $\{\mu_h^{(m)}\}_{m\in\mathbb{N}}$ are uniformly bounded. As a consequence, $\{\mu_h^{(m)}\}_{m\in\mathbb{N}}$ is uniformly tight, and by Prohorov's lemma, there exists a subsequence $\{m_s\} \subset \{m\}$ and a probability measure $\check{\mu}_h$ that satisfy $\mu_h^{(m_s)} \rightsquigarrow \check{\mu}_h$. We show that

$$\check{\mu}_h(B) = \phi_h(\bar{N}(B)) = \phi_h(N(B)) \tag{E.16}$$

for all $B \in \mathcal{B}(\mathbb{R}^d)$. Actually, since $\check{\mu}_h$ is a probability measure on $\mathbb{R}^d$, having no positive mass at infinity, it suffices to prove that $\check{\mu}_h(-\infty,t] = \phi_h(\bar{N}_t)$ for all continuity point $t \in \mathbb{R}^d$ of $t \mapsto \check{\mu}_h(-\infty,t]$. For any $\alpha \in \mathbb{Q}^d$ satisfying $\alpha > t$,

$$\check{\mu}_h(-\infty,t] = \lim_{s\to\infty}\phi_h\left(\hat{N}_t^{(m_s)}\right) \le \lim_{s\to\infty}\phi_h\left(\hat{N}_\alpha^{(m_s)}\right) \le \check{\mu}_h(-\infty,\alpha].$$

In the last inequality, the portmanteau lemma is used. Taking the limit $\alpha \downarrow t$, and recalling the definition (E.14) as well as

$$\lim_{s\to\infty}\phi_h\left(\hat{N}_\alpha^{(m_s)}\right) = \phi_h\left(\hat{N}_\alpha\right),$$

which follows from (E.13), we have $\check{\mu}_h(-\infty,t] = \phi_h(\bar{N}_t)$.

Now we proceed to the proof Theorem 5.5. To this end, it suffices to show the following (i) and (ii):

(i) N is equivalent in law.
(ii) N satisfies the following inequality:

$$\begin{aligned}&\sup_{h\in\mathbb{R}^d}\int_{\mathbb{R}^d} G_{ij}(x-h)^i(x-h)^j\phi_h\left(N(dx)\right) \\ &\qquad\le \sup_{h\in\mathbb{R}^d}\int_{\mathbb{R}^d} G_{ij}(x-h)^i(x-h)^j\phi_h(M(dx)).\end{aligned}$$

In fact, suppose that (i) is true. Then Lemma 5.1 tells us that the first line of the inequality in (ii) is further bounded from below by the Holevo bound $c_G^{(H)}$. This is nothing but the desired minimax theorem.

Let us prove (ii) first. From (E.15), we have

$$\begin{aligned}&\sup_{h\in\mathbb{R}^d}\int_{\mathbb{R}^d} G_{ij}(x-h)^i(x-h)^j\phi_h(M(dx)) \\ &\quad\ge \liminf_{s\to\infty}\int_{\mathbb{R}^d} G_{ij}(x-h)^i(x-h)^j\mu_h^{(m_s)}(dx) \\ &\quad\ge \int_{\mathbb{R}^d} G_{ij}(x-h)^i(x-h)^j\check{\mu}_h(dx) \\ &\quad= \int_{\mathbb{R}^d} G_{ij}(x-h)^i(x-h)^j\phi_h\left(N(dx)\right).\end{aligned}$$

Here, the second inequality follows from the portmanteau lemma, and the last equality from (E.16). Since the first line is independent of h, we have (ii).

We next prove (i), that is,

$$\phi_h\left(N_{t+h}\right)=\phi_0\left(N_t\right) \tag{E.17}$$

for all $t\in\mathbb{R}^d$ and $h\in\mathbb{R}^d$. Since

$$\phi_h\left(N_{t+h}\right)=\phi_h\left(\bar{N}_{t+h}\right)=\inf_{\alpha>t,\alpha\in\mathbb{Q}^d}\phi_h\left(\hat{N}_{\alpha+h}\right),$$

it suffice to prove

$$\phi_h\left(\hat{N}_{\alpha+h}\right)=\phi_0\left(\hat{N}_\alpha\right) \tag{E.18}$$

for all $\alpha\in\mathbb{Q}^d$ and $h\in\mathbb{R}^d$. The left-hand side is rewritten as

$$\begin{aligned}\phi_h\left(\hat{N}_{\alpha+h}\right)&=\lim_{m\to\infty}\phi_h\left(\hat{N}^{(L_m)}_{\alpha+h}\right)\\&=\lim_{m\to\infty}\frac{1}{(2L_m)^d}\int_{[-L_m,L_m]^d}\phi_h\left(U(k)^*M_{\alpha+h+k}U(k)\right)dk\\&=\lim_{m\to\infty}\frac{1}{(2L_m)^d}\int_{[-L_m,L_m]^d}\phi_{h+k}\left(M_{\alpha+h+k}\right)dk\\&=\lim_{m\to\infty}\frac{1}{(2L_m)^d}\int_{[-L_m,L_m]^d+h}\phi_\ell\left(M_{\alpha+\ell}\right)d\ell.\end{aligned}$$

Since $|h^i|<2L_m$ $(i=1,\dots,d)$ for sufficiently large m,

$$\begin{aligned}\left|\phi_h\left(\hat{N}_{\alpha+h}\right)-\phi_0\left(\hat{N}_\alpha\right)\right|&\le\lim_{m\to\infty}\frac{1}{(2L_m)^d}\int_{\mathbb{R}^d}\mathbb{1}_{([-L_m,L_m]^d+h)\triangle([-L_m,L_m]^d)}(k)dk\\&=\lim_{m\to\infty}\frac{1}{(2L_m)^d}\times2\left\{(2L_m)^d-\prod_{i=1}^d(2L_m-|h^i|)\right\}\\&=\lim_{m\to\infty}2\left\{1-\prod_{i=1}^d(1-\frac{|h^i|}{2L_m})\right\}=0.\end{aligned}$$

Here, $\triangle$ denotes the symmetric difference. This proves (E.18). □

E.6. Proof of Theorem 5.6.

PROOF. For notational simplicity, we denote by $\mu_h^{(n)}$ the probability measure of outcomes of POVM $M^{(n)}$ applied to $\rho^{(n)}_{\theta_0+h/\sqrt{n}}$. The first inequality immediately follows from the fact that for any $\delta>0$ and finite subset H of $\mathbb{R}^d$, there exist $N\in\mathbb{N}$ so that $n\ge N$ implies

$$\sup_{\|h\|\le\delta\sqrt{n}}\int_{\mathbb{R}^d}G_{ij}(x-h)^i(x-h)^j\mu_h^{(n)}(dx)\ge\sup_{h\in H}\int_{\mathbb{R}^d}G_{ij}(x-h)^i(x-h)^j\mu_h^{(n)}(dx).$$

The second inequality is obvious. We prove the last inequality.

Following the proof of [1, Theorem 8.11], place the elements of $\mathbb{Q}^d$ in an arbitrary order, and let H_k consist of the first k elements in this sequence. Let

$$c_k^n:=\sup_{h\in H_k}\int_{\mathbb{R}^d}k\wedge\left\{G_{ij}(x-h)^i(x-h)^j\right\}\mu_h^{(n)}(dx),$$

and let

$$c_k := \liminf_{n\to\infty} c_k^n \quad \text{and} \quad c := \lim_{k\to\infty} c_k.$$

Since c is not greater than the third line of (27), it suffices to show that $c \geq c_{\theta_0}^{(rep)}$. Since the inequality is trivial when $c = \infty$, we assume that $c < \infty$.

Take a subsequence $\{n_k\} \subset \{n\}$ that satisfies

$$\lim_{k\to\infty} c_k^{n_k} = c.$$

In fact, just choose n_k so that $n_k > n_{k-1}$ and

$$\left| c_k^{n_k} - c_k \right| < 1/k$$

hold for all $k \in \mathbb{N}$. Let us prove that $\{\mu_h^{(n_k)}\}_k$ is uniformly tight for all $h \in \mathbb{Q}^d$.

Suppose that $\{\mu_h^{(n_k)}\}_k$ is not uniformly tight for some $h \in \mathbb{Q}^d$. For this h, let

$$K_L := \left\{ x \in \mathbb{R}^d : G_{ij}(x-h)^i (x-h)^j \leq L \right\}, \qquad (L > 0).$$

Then there exists an $\varepsilon > 0$ such that

$$\limsup_{k\to\infty} \mu^{(n_k)}(K_L^c) \geq \varepsilon$$

for all $L > 0$. Since $h \in H_k$ for sufficiently large k, it holds that

$$\begin{aligned} c &= \lim_{k\to\infty} c_k^{n_k} \\ &\geq \limsup_{k\to\infty} \int_{\mathbb{R}^d} k \wedge \left\{ G_{ij}(x-h)^i (x-h)^j \right\} \mu_h^{(n_k)}(dx) \\ &\geq L \cdot \limsup_{k\to\infty} \mu_h^{(n_k)}(K_L^c) \\ &\geq L \cdot \varepsilon. \end{aligned}$$

Since $L > 0$ is arbitrary, this contradicts the assumption that $c < \infty$.

Now that $\{\mu_h^{(n_k)}\}_k$ is proved uniformly tight for all $h \in \mathbb{Q}^d$, by the Prohorov lemma and the diagonal sequence trick, we can take a further subsequence $\{k_s\} \subset \{k\}$ that satisfies

$$\mu_h^{(n_{k_s})} \rightsquigarrow \exists \mu_h$$

for all $h \in \mathbb{Q}^d$. It then follows from the asymptotic representation theorem for $h \in \mathbb{Q}^d$ that there is a POVM $M^{(\infty)}$ on $N((\operatorname{Re}\tau)\, h, \Sigma) \sim (X, \phi_h)$ such that

$$\phi_h\left(M^{(\infty)}(B) \right) = \mu_h(B), \qquad (\forall B \in \mathcal{B}(\mathbb{R}^d),\ \forall h \in \mathbb{Q}^d).$$

Now, for any $h \in \mathbb{Q}^d$ and $L > 0$,

$$\begin{aligned} c &= \lim_{s\to\infty} c_{k_s}^{n_{k_s}} \\ &\geq \liminf_{s\to\infty} \int_{\mathbb{R}^d} L \wedge \left\{ G_{ij}(x-h)^i (x-h)^j \right\} \mu_h^{(n_{k_s})}(dx) \\ &= \int_{\mathbb{R}^d} L \wedge \left\{ G_{ij}(x-h)^i (x-h)^j \right\} \mu_h(dx). \end{aligned}$$

In the last equality, we used the portmanteau lemma. Thus

$$\begin{aligned} c &\geq \sup_{L>0} \sup_{h\in\mathbb{Q}^d} \int_{\mathbb{R}^d} L \wedge \left\{G_{ij}(x-h)^i(x-h)^j\right\} \phi_h\left(M^{(\infty)}(dx)\right) \\ &= \sup_{L>0} \sup_{h\in\mathbb{R}^d} \int_{\mathbb{R}^d} L \wedge \left\{G_{ij}(x-h)^i(x-h)^j\right\} \phi_h\left(M^{(\infty)}(dx)\right) \\ &= \sup_{h\in\mathbb{R}^d} \int_{\mathbb{R}^d} G_{ij}(x-h)^i(x-h)^j \phi_h\left(M^{(\infty)}(dx)\right) \\ &\geq c_G^{(rep)}. \end{aligned}$$

Here, the second line follows from the fact that $h \mapsto \phi_h(A)$ is continuous for all $A \in \mathrm{CCR}(\mathrm{Im}\Sigma)$ satisfying $\|A\| \leq 1$, the third line is due to the monotone convergence theorem, and the last line follows from the minimax Theorem 5.5 for a quantum Gaussian shift model, as well as the fact that the asymptotic representation bound $c_G^{(rep)}$ is nothing but the Holevo bound for the quantum Gaussian shift model $\{N((\mathrm{Re}\,\tau)h, \Sigma) : h \in \mathbb{R}^d\}$.

Finally, we prove that the last inequality of (27) is tight. Recall that the sequence $M_\star^{(n)}$ of POVMs constructed in the proof of Theorem 5.3 satisfies

$$\left(M_\star^{(n)}, \rho^{(n)}_{\theta_0+h/\sqrt{n}}\right) \rightsquigarrow N(h, V_\star) \qquad (\forall h \in \mathbb{R}^d)$$

and

$$\mathrm{Tr}\, GV_\star = c_G^{(rep)}.$$

We show that this sequence $M_\star^{(n)}$ saturates the last inequality in (27). Let p_h be the probability density of the classical Gaussian shift model $N(h, V_\star)$. Then

$$\begin{aligned} &\sup_{L>0} \sup_{H} \liminf_{n\to\infty} \sup_{h\in H} \int_{\mathbb{R}^d} L \wedge \left\{G_{ij}(x-h)^i(x-h)^j\right\} \mathrm{Tr}\, \rho^{(n)}_{\theta_0+h/\sqrt{n}} M_\star^{(n)}(dx) \\ &= \sup_{L>0} \sup_{H} \sup_{h\in H} \int_{\mathbb{R}^d} L \wedge \left\{G_{ij}(x-h)^i(x-h)^j\right\} p_h(x)dx \\ &= \mathrm{Tr}\, GV_\star. \end{aligned}$$

Here, the first equality follows from the portmanteau lemma and the fact that H is a finite set. The proof is complete. □

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