Geometry of Quantum Information Systems

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Quantum information geometry is extended to manifolds of not necessarily faithful quantum states. A principal fiber bundle structure is introduced over such manifolds. The connection is defined in a natural way from an information geometrical viewpoint. Uhlmann's connection for faithful states and Pancharatnam-Berry's connection for pure states can be regarded as special cases of this geometry. It is also pointed out that an analogous geometrical consideration offers an important viewpoint in quantum error correcting codes.

1 Introduction

The main purpose of this paper is to clarify a close relation between quantum information geometry and Berry-Uhlmann's geometry of principal fiber bundles. We first present a dualistic geometry for a manifold of quantum states which are not necessarily faithful but have the same (finite) rank. We next introduce a principal fiber bundle over such a manifold endowed with a connection that is closely related to quantum information geometry. It is shown that Pancharatnam-Berry's connection for pure states [1-5] and Uhlmann's connection for faithful states [6-8] are extreme cases of this geometry.

We further mention that an analogous geometrical argument plays an important role in showing that the quantum error correcting codes advocated first by Shor [9] can be formulated in terms of completely positive maps. Since every dynamical change of a physical system is represented by the dual of a completely positive map, such a viewpoint is quite important.

This article is a summary of our recent works: Sections 3-4 are based on a joint work with Keiji Matsumoto and Section 6 is based on a joint work with Paul Algoet. All the proofs omitted here are found in forthcoming papers, where more detailed discussions are also presented.

2 Information geometry for general quantum states

Let $\mathcal{H}$ be a finite dimensional (say $\dim_{\mathbf{C}} \mathcal{H} = n$) Hilbert space [a] which represents a physical system of interest, and let $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_h(\mathcal{H})$ denote the sets of linear operators and Hermitian operators on $\mathcal{H}$. We are interested in the family of density operators (quantum states) with a given rank r:

$$\{\rho \in \mathcal{B}_h(\mathcal{H}) \; ; \; \rho \geq 0, \; \mathrm{Tr}\, \rho = 1, \; \mathrm{rank}\, \rho = r\}.$$

This family can be naturally regarded as a $(2nr - r^2 - 1)$-dimensional real manifold. Let $\mathcal{S}$ be an open subset of this family. In order to introduce a dualistic geometrical structure on $\mathcal{S}$, the following lemma is useful.

Lemma 1 *For $\rho \in \mathcal{S}$ and $D \in \mathcal{B}_h(\mathcal{H})$, the following conditions are equivalent.*

(a) *There exists a unique tangent vector $X \in T_\rho \mathcal{S}$ that satisfies*

$$D = X\rho.$$

(b) *There exists an operator $L \in \mathcal{B}_h(\mathcal{H})$ that satisfies*

$$D = \frac{1}{2}(\rho L + L\rho), \qquad \mathrm{Tr}\, \rho L = 0.$$

The operator L in (b) is called the *symmetric logarithmic derivative* (SLD) and plays an essential role in quantum estimation theory.[11-14]

It can be shown that when $\rho > 0$, the SLD is unique and (a) (b) are also equivalent to (c) $\mathrm{Tr}\, D = 0$. Therefore if $\mathcal{S}$ is composed of full-rank (faithful) density operators, then there is a pair of standard one-one homomorphisms [15] from $T_\rho \mathcal{S}$ into $\mathcal{B}_h(\mathcal{H})$: one is

$$X \longleftrightarrow D \quad \text{such that} \quad \mathrm{Tr}\, D = 0,$$

and is called the *mixture* representation: the other is

$$X \longleftrightarrow L \quad \text{such that} \quad \mathrm{Tr}\, \rho L = 0,$$

and is called the *exponential* representation. The counterparts of D and L in classical information geometry are Xp and $X \log p$, where p denotes the probability density function.[16]

[a] We assume the finite dimensionality not only for mathematical simplicity but also for clarifying the correspondence between classical and quantum information geometry. Most results in this paper can be extended to the separable Hilbert space case.[10]

For general density operators, the SLD is uniquely determined only up to

$$\mathcal{K}_h(\rho) = \{K \in \mathcal{B}_h(\mathcal{H})\ ;\ K\rho = 0\}. \tag{1}$$

Because of this ambiguity, we must arbitrarily choose a representative of the SLD in order to define a one-one homomorphism

$$\mathcal{L}_\rho : T_\rho\mathcal{S} \longrightarrow \mathcal{B}_h(\mathcal{H})$$

which satisfies

$$d\rho = \frac{1}{2}(\rho\mathcal{L}_\rho + \mathcal{L}_\rho\rho).$$

In addition we assume that $\mathcal{L}_\rho$ is smooth in ρ. Such an operator-valued one-form $\mathcal{L}_\rho$ is called an *SLD representation*. When no confusion is likely to arise, we simply denote $\mathcal{L}_\rho(X)$ as L_X for each $X \in T_\rho\mathcal{S}$.

Let us introduce a dualistic geometrical structure on $\mathcal{S}$. We first define a Riemannian metric. Although there are infinitely many quantum counterparts of the Fisher metric,[17]-[19] we adopt the following one and refer to it as the *SLD Fisher metric*:

$$g(X,Y) := \frac{1}{2}\mathrm{Tr}\,\rho(L_XL_Y + L_YL_X) = \mathrm{Tr}\,(X\rho)L_Y.$$

This metric has important features. First of all, it is invariant under the arbitrariness of SLD representations (1). Actually, it is this fact that enables us to treat degenerate densities of an arbitrary rank in a unified manner. Secondly, it plays an essential role in one-parameter quantum estimation theory; i.e., it gives the achievable quantum Cramér-Rao bound for a one-parameter family of density operators.[14]

We next introduce a pair of affine connections that are mutually dual with respect to the SLD Fisher metric. One is defined by

$$(\nabla_XY)\,\rho := \frac{1}{2}\{\rho(XL_Y - \mathrm{Tr}\,\rho(XL_Y)) + (XL_Y - \mathrm{Tr}\,\rho(XL_Y))\rho\},$$

and is called the *exponential* connection. It is well-defined because the right-hand side uniquely defines a derivative of ρ by Lemma 1. The other connection is defined via duality:

$$g(\nabla^*_XY, Z) := Xg(Y,Z) - g(Y, \nabla_XZ) = \mathrm{Tr}\,(X(Y\rho))L_Z,$$

and is called the *mixture* connection. Note that the mixture connection cannot be defined by $(\nabla^*_XY)\rho = X(Y\rho)$ unless $\rho > 0$, since $X(Y\rho)$ does not correspond to a derivative of ρ in general.

The torsion fields T and T^* which correspond to ∇ and ∇^* are

$$T(X,Y)\rho = \frac{1}{4}[\,[L_X, L_Y], \rho], \qquad T^*(X,Y) = 0.$$

The curvature fields do not vanish in general unless $\rho > 0$. Thus one cannot expect the existence of the divergence on the space $(\mathcal{S}, g, \nabla, \nabla^*)$ in general.

3 Principal fiber bundle over $\mathcal{S}$

Given a density operator $\rho \in \mathcal{S}$, an ordered list of nonzero vectors $W = [\hat{\phi}_1, \ldots, \hat{\phi}_r]$ ($r = \operatorname{rank}\rho$) is called an *ordered ρ-ensemble* if

$$\rho = \sum_{j=1}^{r} |\hat{\phi}_j\rangle\langle\hat{\phi}_j|.$$

Associated with each $\rho \in \mathcal{S}$ is the set

$$\mathcal{W}_\rho := \{W\,;\, W \text{ is an ordered } \rho\text{-ensemble}\}.$$

Letting

$$\mathcal{W} := \bigcup_{\rho \in \mathcal{S}} \mathcal{W}_\rho,$$

we have a canonical projection

$$\pi : \mathcal{W} \longrightarrow \mathcal{S} : [\hat{\phi}_1, \ldots, \hat{\phi}_r] \longmapsto \sum_{j=1}^{r} |\hat{\phi}_j\rangle\langle\hat{\phi}_j|.$$

There is a natural right action of the r-dimensional unitary group $U(r)$ on $\mathcal{W}_\rho$:

$$W = [\hat{\phi}_j]_{1\le j\le r} \longmapsto WU = \left[\sum_{k=1}^{r} \hat{\phi}_k u_{kj}\right]_{1\le j\le r}.$$

Moreover the action of $U(r)$ on $\mathcal{W}_\rho$ is free and transitive. We thus have a principal fiber bundle $(\mathcal{W}, \pi, \mathcal{S}, U(r))$.

In order to introduce the connection, let us consider the projection

$$\begin{array}{rcl} P : T_W\mathcal{W} & \longrightarrow & T_W\mathcal{W} \\ X & \longmapsto & \overline{X}, \end{array}$$

where $\overline{X}$ is defined by

$$\overline{X}W = \frac{1}{2}\mathcal{L}_W(X)W, \qquad \mathcal{L}_W = \pi^*\mathcal{L}_{\pi(W)}.$$

Note that this definition is independent of the arbitrariness of the SLD representation $\mathcal{L}_\rho$ since $K \in \mathcal{K}_h(\rho)$ implies that $KW = 0$ for all $W \in \pi^{-1}(\rho)$. Now we decompose the tangent space $T_W\mathcal{W}$ into the direct sum

$$T_W\mathcal{W} = V_W \oplus H_W,$$

where

$$H_W = P(T_W\mathcal{W}), \qquad V_W = (1-P)(T_W\mathcal{W}) = \mathrm{Ker}(\pi_*)_W.$$

The subspace H_W has the property that $H_{WU} = R_{U*}H_W$. Thus there is a unique Ehresmann connection A in which H_W becomes the horizontal subspace:

$$dW = WA + \frac{1}{2}\mathcal{L}_W W.$$

The curvature form $F(A)(X,Y) := -A([\overline{X},\overline{Y}])$ becomes

$$\begin{aligned} WF(A)(X,Y) &= -[\overline{X},\overline{Y}]W + \frac{1}{2}\mathcal{L}_\rho(\pi_*[\overline{X},\overline{Y}])W \\ &= \frac{1}{4}\left\{[L_X,L_Y] - \frac{1}{2}\mathcal{L}_\rho([\,[L_X,L_Y],\rho])\right\}W. \end{aligned}$$

As is naturally expected from the definition, the above principal fiber bundle is closely related to the dualistic geometry introduced in Section 2. For example, the following theorem holds.

Theorem 2 *Consider the following conditions:*

(a) $[L_X, L_Y] = 0$
(b) $[L_X, L_Y]W = 0$
(c) $W^*[L_X, L_Y]W = 0$
(d) $F(A)(\overline{X}, \overline{Y}) = 0$
(e) $T(\underline{X}, \underline{Y}) = 0$

These conditions are related as follows:

$$\begin{array}{lclcl} \text{(a)} & & & & \\ \Downarrow & & & & \\ \text{(b)} & \Leftrightarrow & \{\text{(c)},\text{(e)}\} & \Leftrightarrow & \{\text{(d)},\text{(e)}\} \\ \Downarrow & & & & \\ \text{(c)} & \Leftrightarrow & \text{(d)} & & \end{array}$$

Further, when $\rho > 0$,

$$\text{(a)} \Leftrightarrow \text{(b)} \Leftrightarrow \text{(c)} \Leftrightarrow \text{(d)} \Rightarrow \text{(e)}$$

It is not yet clear whether the fiber space $\mathcal{W}$ has some physical significance. The following fact suggests that each element of $\mathcal{W}$ corresponds to some "information" about the system.

Theorem 3 *Suppose we are given a pure state $|\psi\rangle\langle\psi|$ and a general state ρ. For an arbitrary ρ-ensemble $W = [\hat{\phi}_1, \ldots, \hat{\phi}_r]$, there exists an instrument $V = [V_1, \ldots, V_r]$ such that W is the post-measurement ensemble when V is applied to the pure state ψ; i.e. $\hat{\phi}_j = V_j \psi$, $(j = 1, \ldots, r)$.*

4 Relation with Uhlmann's formulation

As Dąbrowski and Jadczyk [7] pointed out, Uhlmann's formulation for a generalization of Berry's phase to density operators works only for faithful density operators. In practice, this restriction never reduces the importance of Uhlmann's idea because every state can be approximated by a faithful state. However its relation to Berry's phase seems to have become vague. In this section, we comment that the principal fiber bundle and its connection introduced in Section 3 realizes Uhlmann's original program; i.e., it actually generalizes Berry's phase for an arbitrary family of density operators of the same rank.

Let us introduce an abridged notation

$$W = \Big[|\hat{\phi}_1\rangle, \ldots, |\hat{\phi}_r\rangle\Big], \qquad W^* = \begin{bmatrix} \langle\hat{\phi}_1| \\ \vdots \\ \langle\hat{\phi}_r| \end{bmatrix}.$$

Using this notation, the parallelism between our formulation and Uhlmann's can be exhibited as follows:

$$\begin{aligned}
\rho = \sum_{j=1}^{r} |\hat{\phi}_j\rangle\langle\hat{\phi}_j| \quad &\Longleftrightarrow \quad \rho = WW^* \\
\hat{\psi}_j = \sum_{k=1}^{r} \hat{\phi}_k u_{kj} \quad &\Longleftrightarrow \quad V = WU \\
\pi : \mathcal{W} \longrightarrow \mathcal{S} \quad &\Longleftrightarrow \quad \pi : W \longmapsto WW^* \\
U(r) \text{ preserves each fiber} \quad &\Longleftrightarrow \quad WW^* = (WU)(WU)^*
\end{aligned}$$

As for the connection, one must regard $\mathcal{W}$ as a metric space with metric

$$d(W^{(1)}, W^{(2)}) := \sqrt{\sum_{j=1}^{r} \left(\langle \hat{\phi}_j^{(1)} | - \langle \hat{\phi}_j^{(2)} | \right) \left(| \hat{\phi}_j^{(1)} \rangle - | \hat{\phi}_j^{(2)} \rangle \right)}\,,$$

where $W^{(i)} = \left[|\hat{\phi}_1^{(i)}\rangle, \ldots, |\hat{\phi}_r^{(i)}\rangle \right]$. Given $W \in \pi^{-1}(\rho)$, we define

$$\begin{array}{rcl} f_W : \mathcal{S} & \longrightarrow & \mathcal{W} \\ \sigma & \longmapsto & \displaystyle\mathop{\arg\min}_{W' \in \pi^{-1}(\sigma)} d(W, W'). \end{array}$$

It is easily verified that its differential $df_W : T_\rho \mathcal{S} \to T_W \mathcal{W}$ maps $\underline{X} \in T_\rho \mathcal{S}$ to $\overline{X} \in T_W \mathcal{W}$ such that

$$\overline{X} W = \frac{1}{2} \mathcal{L}_\rho(\underline{X}) W.$$

This relation implies that df_W gives the horizontal lift (in our sense) of tangent vectors on $\mathcal{S}$. This fact shows that Uhlmann's connection is equivalent to ours.

In passing, we note that the so-called Bures distance

$$B(\rho, \sigma) := \min_{\substack{W \in \pi^{-1}(\rho) \\ W' \in \pi^{-1}(\sigma)}} d(W, W')^2 = 2 \left(1 - \mathrm{Tr} \sqrt{\sqrt{\rho}\ \sigma \sqrt{\rho}} \right)$$

is a noncommutative analogue of the Hellinger distance.[b] From the observation above, it is straightforward to show that the Bures metric (the infinitesimal Bures distance) is identical to the SLD Fisher metric up to a constant factor.

We next show that our geometrical structure naturally leads to Berry's phase as a special case. We start with the following lemma.[14]

Lemma 4 *Let ρ be a pure state and let $W \in \pi^{-1}(\rho)$. For $K \in \mathcal{B}(\mathcal{H})$, the following conditions are equivalent.*

(a) $KW = 0$

(b) $K\rho + \rho K^* = 0, \quad \mathrm{Tr}\, \rho K = 0$

Now let $\rho(t)$ be a curve of pure states satisfying the equation

$$\dot{\rho}(t) = i[\rho(t), H(t)], \qquad H(t) \in \mathcal{B}_h(\mathcal{H}).$$

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[b] If $\rho, \sigma > 0$, then there is a simple measurement $E = \{E_j\}$ which satisfies $B(\rho, \sigma) = \sum_j \left(\sqrt{p_j} - \sqrt{q_j} \right)^2$, where $p_j = \mathrm{Tr}\, \rho E_j$, $q_j = \mathrm{Tr}\, \sigma E_j$. Actually, $E = \{E_j\}$ is given by the spectral measure of the positive operator $\left(\sqrt{\rho^{-1}} \sqrt{\sqrt{\rho}\ \sigma \sqrt{\rho}}\ \sqrt{\rho^{-1}} \right)^2$.

Let $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$, and consider the horizontal lift

$$\frac{d}{dt}|\psi(t)\rangle = \frac{1}{2}L(t)|\psi(t)\rangle, \tag{2}$$

where $L(t) := \mathcal{L}_{\rho(t)}(\dot{\rho}(t))$. Then

$$\frac{d\rho(t)}{dt} = \frac{1}{2}(\rho(t)L(t) + L(t)\rho(t)) = i(\rho(t)\hat{H}(t) - \hat{H}(t)\rho(t)),$$

where

$$\hat{H}(t) := H(t) - \operatorname{Tr}\rho(t)H(t).$$

Thus $K(t) := \frac{1}{2}L(t) + i\hat{H}(t)$ satisfies the condition (b) in Lemma 4. Then $K(t)|\psi(t)\rangle = 0$. This shows that (2) admits another form

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle,$$

which is nothing but the Schrödinger equation with the dynamical phase removed.[4,5] Thus we see that Berry's phase is a special example of holonomy in our framework.

This fact can be generalized as follows. Actually the condition (b) in the theorem is always satisfied for pure states.

Theorem 5 *Suppose the state evolves unitarily in $\mathcal{S}$ along a curve $\rho(t)$ with equation*

$$\dot{\rho}(t) = i[\rho(t), H(t)], \qquad H(t) \in \mathcal{B}_h(\mathcal{H}).$$

For this curve, the following conditions are equivalent.

(a) *The horizontal lift $W(t)$ of $\rho(t)$ is also a unitary evolution of the form*

$$i\dot{W}(t) = \hat{H}(t)W(t), \qquad \textit{for some } \hat{H}(t) \in \mathcal{B}_h(\mathcal{H}).$$

(b) $\rho(t)\dot{\rho}(t)\rho(t) = 0$.

5 Example

In this section we present a simple example of holonomy in a two-level quantum system. It is well-known that density operators of a two-level quantum system can be represented as

$$\rho = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix},$$

where the vector (x, y, z) of Stokes parameters [13] satisfies $x^2 + y^2 + z^2 \leq 1$. It can be shown that a geodesic with respect to the exponential connection ∇ forms an ellipse in the Stokes parameter space. Now consider two ellipses in the plane $z = 0$ depending on a parameter $R \in (0, 1)$:

$$x^2 + \left(\frac{y}{R}\right)^2 = 1, \qquad \left(\frac{x}{R}\right)^2 + y^2 = 1.$$

The intersection of the convex hulls of the two ellipses admits a curvilinear quadrilateral boundary $C(R)$ which is oriented counterclockwise. Since each latus of $C(R)$ is a ∇-geodesic, the horizontal lift of $C(R)$ can be easily calculated. The corresponding holonomy becomes

$$\begin{bmatrix} e^{-i\phi(R)} & 0 \\ 0 & e^{i\phi(R)} \end{bmatrix} \in U(2), \qquad \phi(R) = \cos^{-1}(1 - 2R^4).$$

In particular,

$$\lim_{R \to 1} \phi(R) = \pi,$$

which is identical to Berry's phase for a great circle on the Bloch sphere.[4]

6 A related topic

In this section, we observe that an analogous geometrical argument provides a useful viewpoint in quantum error correcting codes.

6.1 Completely positive maps

A linear map $\kappa : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called *completely positive* (CP) if it can be represented in the form

$$\kappa(X) = \sum_j A_j^* X A_j,$$

where $\{A_j\}_j$ is a (finite) collection of operators on $\mathcal{H}$.[20-23] When a CP map κ is represented in this way, the (ordered) collection of operators $\mathcal{A} = \{A_j\}_j$ is called a *generator* of κ and κ is denoted by $\kappa_{\mathcal{A}}$. The dual map κ^* defined by $\operatorname{Tr} \kappa^*(X) Y = \operatorname{Tr} X \kappa(Y)$ is explicitly given by

$$\kappa_{\mathcal{A}}^*(X) = \sum_j A_j X A_j^*.$$

As is well-known, CP maps play an essential role in quantum theory.[21] Indeed, every dynamical change of a physical system is described by the dual of a CP map κ which satisfies $\kappa(I) = I$, or equivalently $\operatorname{Tr} \kappa^*(X) = \operatorname{Tr} X$ for all $X \in \mathcal{B}(\mathcal{H})$.

Let us recall the following fundamental characterization.[23]

Proposition 6 *Two collections of operators $\{A_j\}_{1 \le j \le J}$, $\{B_k\}_{1 \le k \le K}$ ($J \le K$) give the same CP map iff there is a matrix $Q = [Q_{jk}] \in \mathbf{C}^{J \times K}$ such that $QQ^* = I_J$ (I_J denotes the $J \times J$ identity matrix) and $B_k = \sum_j A_j Q_{jk}$.*

Now given a CP map κ, define the set

$$\mathcal{G}(\kappa) := \{\mathcal{A}\ ;\ \mathcal{A} \text{ is a finite collection of operators such that } \kappa_{\mathcal{A}} = \kappa\}.$$

The number J of operators in a list $\mathcal{A} = \{A_1, \ldots, A_J\}$ is denoted by $|\mathcal{A}|$. We define the *rank* of κ as

$$\operatorname{rank} \kappa := \min_{\mathcal{A} \in \mathcal{G}(\kappa)} |\mathcal{A}|,$$

and the set of *minimal* generators of κ as

$$\mathcal{G}_m(\kappa) := \{\mathcal{A} \in \mathcal{G}(\kappa)\ ;\ |\mathcal{A}| = \operatorname{rank} \kappa\}.$$

Proposition 6 immediately leads us to the following.

Corollary 7 *$\mathcal{A} \in \mathcal{G}(\kappa)$ belongs to $\mathcal{G}_m(\kappa)$ iff $\mathcal{A}$ is a linearly independent set of operators.*

Corollary 8 *$\mathcal{A} \in \mathcal{G}_m(\kappa)$ is unique up to an $r \times r$ unitary matrix, where $r = \operatorname{rank} \kappa$.*

Corollary 8 shows that by regarding the set of minimal generators $\mathcal{G}_m(\kappa)$ as the fiber over κ, one can introduce a principal fiber bundle structure over the family of all CP maps with a given rank r. The structure group of this fiber bundle is $U(r)$.

6.2 Application to quantum error correcting codes

Now we show that the quantum error correcting schemes advocated first by Shor[9] can be formulated in terms of CP maps. This gives a slightly different viewpoint from Schumacher and Nielsen.[24] Suppose the error process is described by the CP map κ. A *quantum error correcting code* (QECC) for the error process κ is a pair $(\mathcal{C}, \tau)$, where $\mathcal{C}$ is a subspace of $\mathcal{H}$ and τ a CP map such that

$$\tau^*(\kappa^* |\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|, \qquad \forall \psi \in \mathcal{C}. \tag{3}$$

The physical meaning of (3) is as follows: If all the pure states to be manipulated belong only to the subspace $\mathcal{C}$, then there is a map $\tau^* : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ which inverts the noise $\kappa^* : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ and restores the original state

$\psi \in \mathcal{C}$. ($\mathcal{S}(\mathcal{H})$ is the totality of density operators on $\mathcal{H}$.) The space $\mathcal{C}$ is called the *code subspace*. Note that due to the linearity of (3), the error correcting scheme also works for density operators whose images lie in $\mathcal{C}$.

In terms of generators $\mathcal{A} = \{A_i\}$ and $\mathcal{R} = \{R_\alpha\}$ for κ and τ, (3) can be rewritten as

$$\sum_i \sum_\alpha R_\alpha A_i |\psi\rangle\langle\psi| A_i^* R_\alpha^* = |\psi\rangle\langle\psi|, \qquad \forall\psi \in \mathcal{C}.$$

Since both sides are rank-one operators, the necessary and sufficient condition for the existence of $(\mathcal{C}, \tau)$ is that there is a set of scalars $\{\lambda_{i\alpha}\}_{i,\alpha}$ which satisfies

$$R_\alpha A_i |\psi\rangle = \lambda_{i\alpha} |\psi\rangle, \qquad \forall\psi \in \mathcal{C},\ \forall i,\ \forall\alpha \tag{4}$$

$$\sum_i \sum_\alpha |\lambda_{i\alpha}|^2 = 1. \tag{5}$$

These conditions are essentially identical to those in Theorem III.1 of Knill and Laflamme.[25] However, it should be noted that these conditions are representation free: they are not for a particular choice of generators $\mathcal{A}, \mathcal{R}$ but for the CP maps κ, τ themselves. Actually, let $\mathcal{B} = \{B_j\}$ and $\mathcal{S} = \{S_\beta\}$ be other sets of generators for κ and τ. Then from Proposition 6 there are partial unitaries Q, W such that

$$B_j = \sum_i A_i Q_{ij}, \qquad S_\beta = \sum_\alpha R_\alpha W_{\alpha\beta}.$$

Then under (4) (5), the quantities

$$\mu_{j\beta} := \sum_i \sum_\alpha \lambda_{i\alpha} Q_{ij} W_{\alpha\beta}$$

satisfy

$$S_\beta B_j |\psi\rangle = \mu_{j\beta} |\psi\rangle, \qquad \forall\psi \in \mathcal{C},\ \forall j,\ \forall\beta,$$

$$\sum_j \sum_\beta |\mu_{j\beta}|^2 = 1.$$

This fact could be paraphrased by saying that the conditions (4) (5) are "gauge invariant."

The following theorem is also the "gauge invariant" version of that obtained independently by Knill and Laflamme[25] and by Bennett *et al.*[26]

Theorem 9 *Suppose we are given a CP map κ. The subspace $\mathcal{C}$ of $\mathcal{H}$ can be extended to a QECC $(\mathcal{C},\tau)$ for κ iff*

$$\langle e_\mu | A_i^* A_j | e_\mu \rangle = \langle e_\nu | A_i^* A_j | e_\nu \rangle, \qquad \forall i,j,\mu \neq \nu,$$

$$\langle e_\mu | A_i^* A_j | e_\nu \rangle = 0, \qquad \forall i,j,\mu \neq \nu.$$

Here $\{e_\mu\}$ is an arbitrary complete orthonormal system of $\mathcal{C}$, and $\mathcal{A} = \{A_i\}$ an arbitrary generator of κ.

7 Conclusions

We first extended the quantum information geometry to manifolds of quantum states which are not necessarily faithful but have the same rank. We next introduced a principal fiber bundle structure over such a manifold. The connection was naturally defined via the symmetric logarithmic derivative which plays an essential role in quantum estimation theory. Pancharatnam-Berry's connection for pure states and Uhlmann's connection for faithful states were special cases of this geometry.

We also mentioned a new formulation of quantum error correcting codes by means of completely positive maps, in which a similar principal fiber bundle structure played an important role.

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