SUPPLEMENTARY MATERIAL TO "QUANTUM LOCAL ASYMPTOTIC NORMALITY BASED ON A NEW QUANTUM LIKELIHOOD RATIO"

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APPENDIX A: PROOFS OF MAIN THEOREMS

This section is devoted to proofs of main results presented in Sections 2-3 of [4].

A.1. Proof of Lemma 2.6. We shall prove (2.3) in [4] for $\{\xi_t\}_{t=1}^r \subset \mathbb{C}^d$ and $\{\eta_t\}_{t=1}^r \subset \mathbb{C}$.

$$\begin{aligned}
&\mathrm{Tr}\,\rho^{\otimes n}\left(\prod_{t=1}^{r} \mathrm{e}^{\sqrt{-1}\left(\xi_t^i X_i^{(n)}+\eta_t R^{(n)}\right)}\right)\\
&\quad= \mathrm{Tr}\,\rho^{\otimes n}\left[\prod_{t=1}^{r}\exp\left\{\frac{\sqrt{-1}}{\sqrt{n}}\sum_{k=1}^{n} I^{\otimes(k-1)}\otimes\left(\xi_t^i A_i+\eta_t P(n)\right)\otimes I^{\otimes(n-k)}\right\}\right]\\
&\quad= \mathrm{Tr}\,\rho^{\otimes n}\left[\prod_{t=1}^{r}\left\{\exp\left(\frac{\sqrt{-1}}{\sqrt{n}}\left(\xi_t^i A_i+\eta_t P(n)\right)\right)\right\}^{\otimes n}\right]\\
&\quad= \mathrm{Tr}\,\rho^{\otimes n}\left[\left\{\prod_{t=1}^{r}\exp\left(\frac{\sqrt{-1}}{\sqrt{n}}\left(\xi_t^i A_i+\eta_t P(n)\right)\right)\right\}^{\otimes n}\right]\\
&\quad= \left[\mathrm{Tr}\,\rho\left\{\prod_{t=1}^{r}\exp\left(\frac{\sqrt{-1}}{\sqrt{n}}\left(\xi_t^i A_i+\eta_t P(n)\right)\right)\right\}\right]^{n}\\
&\quad= \left[\mathrm{Tr}\,\rho\left\{\sum_{k_1,\dots,k_r\in\mathbb{Z}_+}\left(\frac{\sqrt{-1}}{\sqrt{n}}\right)^{k_1+\cdots+k_r}\prod_{t=1}^{r}\frac{1}{k_t!}\left(\xi_t^i A_i+\eta_t P(n)\right)^{k_t}\right\}\right]^{n},
\end{aligned}$$

where $\mathbb{Z}_+ = \{0,1,2,\dots\}$. The terms corresponding to $k_1+\cdots+k_r = 1$ in the summand are

$$\mathrm{Tr}\,\rho\left(\frac{\sqrt{-1}}{\sqrt{n}}\sum_{t=1}^{r}\left(\xi_t^i A_i+\eta_t P(n)\right)\right) = \left(\sum_{t=1}^{r}\eta_t\right)\frac{\sqrt{-1}}{\sqrt{n}}\mathrm{Tr}\,\rho P(n) = o\left(\frac{1}{n}\right)$$

because $\mathrm{Tr}\,\rho A_i = 0$ and $\mathrm{Tr}\,\rho P(n) = o(\frac{1}{\sqrt{n}})$. The terms corresponding to $k_1+\cdots+k_r = 2$ are

$$\begin{aligned}
&-\frac{1}{n}\mathrm{Tr}\,\rho\left\{\sum_{k_1+\cdots+k_r=2}\left(\prod_{t=1}^{r}\frac{1}{k_t!}\left(\xi_t^i A_i+\eta_t P(n)\right)^{k_t}\right)\right\}\\
&\quad= -\frac{1}{2n}\sum_{t=1}^{r}\mathrm{Tr}\,\rho\left(\xi_t^i A_i+\eta_t P(n)\right)^2 - \frac{1}{n}\sum_{t=1}^{r}\sum_{s=t+1}^{r}\mathrm{Tr}\,\rho\left(\xi_t^i A_i+\eta_t P(n)\right)\left(\xi_s^j A_j+\eta_s P(n)\right)\\
&\quad= -\frac{1}{2n}\sum_{t=1}^{r}\xi_t^i\xi_t^j\,\mathrm{Tr}\,\rho A_i A_j - \frac{1}{n}\sum_{t=1}^{r}\sum_{s=t+1}^{r}\xi_t^i\xi_s^j\,\mathrm{Tr}\,\rho A_i A_j + o\left(\frac{1}{n}\right)\\
&\quad= -\frac{1}{2n}\sum_{t=1}^{r}\xi_t^i\xi_t^j J_{ji} - \frac{1}{n}\sum_{t=1}^{r}\sum_{s=t+1}^{r}\xi_t^i\xi_s^j J_{ji} + o\left(\frac{1}{n}\right).
\end{aligned}$$

In the third line, we used the fact that $P(n) = o(1)$. Let us denote the terms corresponding to $k_1 + \cdots + k_r \geq 3$ by

$$r_n := \operatorname{Tr} \rho \left\{ \sum_{k_1+\cdots+k_r \geq 3} \left(\frac{\sqrt{-1}}{\sqrt{n}} \right)^{(k_1+\cdots+k_r)} \prod_{t=1}^{r} \frac{1}{k_t!} \left(\xi_t^i A_i + \eta_t P(n) \right)^{k_t} \right\}.$$

Then

$$\begin{aligned} |r_n| &\leq \sum_{k_1+\cdots+k_r \geq 3} \left\| \left(\frac{1}{\sqrt{n}} \right)^{(k_1+\cdots+k_r)} \prod_{t=1}^{r} \frac{1}{k_t!} \left(\xi_t^i A_i + \eta_t P(n) \right)^{k_t} \right\| \\ &\leq \frac{1}{n\sqrt{n}} \sum_{k_1+\cdots+k_r \geq 3} \left\| \prod_{t=1}^{r} \frac{1}{k_t!} \left(\xi_t^i A_i + \eta_t P(n) \right)^{k_t} \right\| \\ &\leq \frac{1}{n\sqrt{n}} \sum_{k_1+\cdots+k_r \geq 3} \prod_{t=1}^{r} \frac{1}{k_t!} \left\| \xi_t^i A_i + \eta_t P(n) \right\|^{k_t} \\ &\leq \frac{1}{n\sqrt{n}} \sum_{k_1,\ldots,k_r \in \mathbb{Z}_+} \prod_{t=1}^{r} \frac{1}{k_t!} \left\| \xi_t^i A_i + \eta_t P(n) \right\|^{k_t} \\ &= \frac{1}{n\sqrt{n}} \prod_{t=1}^{r} \exp \left\| \xi_t^i A_i + \eta_t P(n) \right\| \\ &\leq \frac{1}{n\sqrt{n}} \exp \left(\sum_{t=1}^{r} \left(\left\| \xi_t^i A_i \right\| + \| \eta_t P(n) \| \right) \right). \end{aligned}$$

Since $\lim_{n\to\infty} P(n) = 0$, the operators $P(n)$ are uniformly bounded. As a consequence, $\lim_{n\to\infty} n\,|r_n| = 0$, so that $r_n = o\left(\frac{1}{n}\right)$. Thus we conclude that

$$\begin{aligned} \lim_{n\to\infty} \operatorname{Tr} \rho^{\otimes n} \left(\prod_{t=1}^{r} \mathrm{e}^{\sqrt{-1}\left(\xi_t^i X_i^{(n)} + \eta_t R^{(n)}\right)} \right) &= \lim_{n\to\infty} \left(1 - \frac{1}{2n} \sum_{t=1}^{r} \xi_t^i \xi_t^j J_{ji} - \frac{1}{n} \sum_{t=1}^{r} \sum_{s=t+1}^{r} \xi_t^i \xi_s^j J_{ji} + o\left(\frac{1}{n}\right) \right)^n \\ &= \exp \left(-\frac{1}{2} \sum_{t=1}^{r} \xi_t^i \xi_t^j J_{ji} - \sum_{t=1}^{r} \sum_{s=t+1}^{r} \xi_t^i \xi_s^j J_{ji} \right) \\ &= \phi \left(\prod_{t=1}^{r} \mathrm{e}^{\sqrt{-1}\xi_t^i X_i} \right). \end{aligned}$$

The last equation is due to (2.2) in [4] with $h = 0$.

A.2. Proof of Theorem 2.9. Let $X_1, \ldots, X_r, \Delta_1, \ldots, \Delta_d$ be the basic canonical observables of $\mathrm{CCR}\left(\operatorname{Im}\begin{pmatrix} \Sigma & \tau \\ \tau^* & J \end{pmatrix}\right)$, and $\tilde{\phi}$ the quantum Gaussian state $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^* & J \end{pmatrix}\right)$ on that CCR. Assumption (2.6) in [4] guarantees that the quantities

$$R_h^{(n)} := \mathcal{L}_h^{(n)} - \left\{ h^i \Delta_i^{(n)} - \frac{1}{2} J_{ij} h^i h^j I^{(n)} \right\}$$

enjoy $R_h^{(n)} = o\left(\begin{pmatrix} X^{(n)} \\ \Delta^{(n)} \end{pmatrix}, \rho_{\theta_0}^{(n)}\right)$ for each $h \in \mathbb{R}^d$. Consequently, for a finite subset $\{\xi_t\}_{t=1}^r$ of $\mathbb{C}^d$,

$$\begin{aligned}
&\operatorname{Tr} \rho_{\theta_0+h/\sqrt{n}}^{(n)} \left(\prod_{t=1}^{r} \mathrm{e}^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right) \\
&= \operatorname{Tr} \left(\mathrm{e}^{\frac{1}{2}\mathcal{L}_h^{(n)}} \rho_{\theta_0}^{(n)} \mathrm{e}^{\frac{1}{2}\mathcal{L}_h^{(n)}} \right) \left(\prod_{t=1}^{r} \mathrm{e}^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right) \\
&= \mathrm{e}^{-\frac{1}{2}h^i h^j J_{ij}} \operatorname{Tr} \rho_{\theta_0}^{(n)} \mathrm{e}^{\frac{1}{2}\left(h^i \Delta_i^{(n)} + R_h^{(n)}\right)} \left(\prod_{t=1}^{r} \mathrm{e}^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right) \mathrm{e}^{\frac{1}{2}\left(h^i \Delta_i^{(n)} + R_h^{(n)}\right)} \\
&= \mathrm{e}^{-\frac{1}{2}h^i h^j J_{ij}} \operatorname{Tr} \rho_{\theta_0}^{(n)} \left(\mathrm{e}^{-\sqrt{-1}\left(\frac{\sqrt{-1}}{2}h^i \Delta_i^{(n)} + \frac{\sqrt{-1}}{2}R_h^{(n)}\right)} \right) \left(\prod_{t=1}^{r} \mathrm{e}^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right) \left(\mathrm{e}^{-\sqrt{-1}\left(\frac{\sqrt{-1}}{2}h^i \Delta_i^{(n)} + \frac{\sqrt{-1}}{2}R_h^{(n)}\right)} \right).
\end{aligned}$$

Since $R_h^{(n)}$ is infinitesimal relative to the convergence (2.5) in [4], we see from (2.3) in [4] that

$$\begin{aligned}
&\lim_{n\to\infty} \operatorname{Tr} \rho_{\theta_0+h/\sqrt{n}}^{(n)} \left(\prod_{t=1}^{r} \mathrm{e}^{\sqrt{-1}\xi_t^i X_i^{(n)}} \right) \\
&= \mathrm{e}^{-\frac{1}{2}h^i h^j J_{ij}} \tilde{\phi} \left(\mathrm{e}^{-\sqrt{-1}\frac{\sqrt{-1}}{2}h^i \Delta_i} \left(\prod_{t=1}^{r} \mathrm{e}^{\sqrt{-1}\xi_t^i X_i} \right) \mathrm{e}^{-\sqrt{-1}\frac{\sqrt{-1}}{2}h^i \Delta_i} \right) \\
&= \mathrm{e}^{-\frac{1}{2}h^i h^j J_{ij}} \exp\left(-\frac{1}{2}\sum_{t=0}^{r+1} \tilde{\xi}_t^i \tilde{\xi}_t^j \tilde{\Sigma}_{ji} - \sum_{t=0}^{r+1}\sum_{s=t+1}^{r+1} \tilde{\xi}_t^i \tilde{\xi}_s^j \tilde{\Sigma}_{ji} \right) \\
&= \mathrm{e}^{-\frac{1}{2}h^i h^j J_{ij}} \exp\left(-\frac{1}{2}\left\{ -\frac{1}{4}h^j h^j J_{ji} + \sum_{t=1}^{r} \xi_t^i \xi_t^j \Sigma_{ji} - \frac{1}{4}h^j h^j J_{ji} \right\} \right) \\
&\qquad \times \exp\left(\frac{\sqrt{-1}}{2}\sum_{t=1}^{r} h^i \xi_t^j \tau_{ji} + \frac{\sqrt{-1}}{2}\sum_{t=1}^{r} \xi_t^i h^j \tau_{ji} + \frac{1}{4}h^i h^j J_{ji} - \sum_{t=1}^{r}\sum_{s=t+1}^{r} \xi_t^i \xi_s^j \Sigma_{ji} \right) \\
&= \exp\left(\sum_{t=1}^{r} \left(\sqrt{-1}\xi_t^i h^j \left(\operatorname{Re}\tau\right)_{ij} - \frac{1}{2}\xi_t^i \xi_t^j \Sigma_{ji} \right) - \sum_{t=1}^{r}\sum_{s=t+1}^{r} \xi_t^i \xi_s^j \Sigma_{ji} \right),
\end{aligned}$$

where $\tilde{\Sigma} := \begin{pmatrix} \Sigma & \tau \\ \tau^* & J \end{pmatrix}$ and $(\tilde{\xi}_0, \tilde{\xi}_1, \ldots, \tilde{\xi}_r, \tilde{\xi}_{r+1}) := (-\frac{\sqrt{-1}}{2}h, \xi_1, \ldots, \xi_r, -\frac{\sqrt{-1}}{2}h)$, and (2.2) of [4] was used at the second equation. This is the quasi-characteristic function of $N((\operatorname{Re}\tau)\,h, \Sigma)$.

A.3. Proof of Theorem 2.10. Since

$$\begin{aligned}
\rho_\theta^{\otimes n} &= \left(\mathrm{e}^{\frac{1}{2}\mathcal{L}\left(\rho_\theta | \rho_{\theta_0}\right)} \rho_{\theta_0} \mathrm{e}^{\frac{1}{2}\mathcal{L}\left(\rho_\theta | \rho_{\theta_0}\right)} \right)^{\otimes n} \\
&= \left(\mathrm{e}^{\frac{1}{2}\sum_{k=1}^{n} I^{\otimes(k-1)} \otimes \mathcal{L}\left(\rho_\theta | \rho_{\theta_0}\right) \otimes I^{\otimes(n-k)}} \right) \rho_{\theta_0}^{\otimes n} \left(\mathrm{e}^{\frac{1}{2}\sum_{k=1}^{n} I^{\otimes(k-1)} \otimes \mathcal{L}\left(\rho_\theta | \rho_{\theta_0}\right) \otimes I^{\otimes(n-k)}} \right),
\end{aligned}$$

we see that

$$\mathcal{L}\left(\rho_\theta^{\otimes n} \,\middle|\, \rho_{\theta_0}^{\otimes n} \right) = \sum_{k=1}^{n} I^{\otimes(k-1)} \otimes \mathcal{L}\left(\rho_\theta | \rho_{\theta_0}\right) \otimes I^{\otimes(n-k)}. \tag{A.1}$$

This proves $\rho_\theta^{\otimes n} \sim \rho_{\theta_0}^{\otimes n}$ for all $\theta \in \Theta$ and $n \in \mathbb{N}$.

Before proceeding to the proof of (ii) and (iii) in Definition 2.8, we give some preliminary consideration. Let the quantum log-likelihood ratio $\mathcal{L}_h := \mathcal{L}\left(\rho_{\theta_0+h}|\rho_{\theta_0}\right)$ be expanded into

$$\mathcal{L}_h = h^i A_i + B_{ij} h^i h^j + o(h^2),$$

where $A_i\,(1 \le i \le d)$ and $B_{ij}\,(1 \le i, j \le d)$ are Hermitian operators on $\mathcal{H}$. Observe that A_i is the SLD in the ith direction. In fact,

$$\begin{aligned}\rho_{\theta_0+h} &= \exp\left[\frac{1}{2}\left(h^i A_i + o(h)\right)\right]\rho_{\theta_0}\exp\left[\frac{1}{2}\left(h^i A_i + o(h)\right)\right]\\ &= \rho_{\theta_0} + \frac{1}{2}h^i\left(\rho_{\theta_0}A_i + A_i\rho_{\theta_0}\right) + o(h),\end{aligned}$$

so that

$$\partial_i \rho_{\theta_0} = \frac{1}{2}\left(\rho_{\theta_0}A_i + A_i\rho_{\theta_0}\right).$$

This observation also shows that $\operatorname{Tr}\rho_{\theta_0}A_i = 0$ for all i. On the other hand,

$$\begin{aligned}\operatorname{Tr}\rho_{\theta_0+h} &= \operatorname{Tr}\rho_{\theta_0}\exp\left(h^i A_i + B_{ij}h^i h^j + o(h^2)\right)\\ &= \operatorname{Tr}\rho_{\theta_0}\left(I + \left(h^i A_i + B_{ij}h^i h^j\right) + \frac{1}{2}\left(h^i A_i + B_{ij}h^i h^j\right)^2 + o(h^2)\right)\\ &= 1 + h^i\left(\operatorname{Tr}\rho_{\theta_0}A_i\right) + h^i h^j \operatorname{Tr}\rho_{\theta_0}\left(B_{ij} + \frac{1}{2}A_i A_j\right) + o(h^2)\\ &= 1 + h^i h^j \operatorname{Tr}\rho_{\theta_0}\left(B_{ij} + \frac{1}{2}A_i A_j\right) + o(h^2).\end{aligned}$$

Since $\operatorname{Tr}\rho_{\theta_0+h} = 1$ for all h, the above equation leads to

$$\operatorname{Tr}\rho_{\theta_0}\left(B_{ij} + \frac{1}{2}A_i A_j\right) = 0. \tag{A.2}$$

Now we prove (ii). Let $J_{ij} := \operatorname{Tr}\rho_{\theta_0}A_j A_i$, and let

$$\Delta_i^{(n)} := \frac{1}{\sqrt{n}}\sum_{k=1}^{n} I^{\otimes(k-1)} \otimes A_i \otimes I^{\otimes(n-k)}.$$

It then follows from the quantum central limit theorem (Proposition 2.5) that $\left(\Delta^{(n)}, \rho_{\theta_0}^{\otimes n}\right) \underset{q}{\rightsquigarrow} N(0, J)$.

Finally, we prove (iii). It follows from (A.1) that

$$\mathcal{L}_h^{(n)} = \sum_{k=1}^{n} I^{\otimes(k-1)} \otimes \mathcal{L}_{h/\sqrt{n}} \otimes I^{\otimes(n-k)}.$$

Let us show that

$$R_h^{(n)} := \mathcal{L}_h^{(n)} - \left(h^i \Delta_i^{(n)} - \frac{1}{2}\left(J_{ij}h^i h^j\right) I^{\otimes n}\right)$$

is infinitesimal relative to the convergence $\left(\Delta^{(n)}, \rho_{\theta_0}^{\otimes n}\right) \underset{q}{\rightsquigarrow} N(0,J)$. It is rewritten as

$$\begin{aligned}
R_h^{(n)} &= \sum_{k=1}^{n} I^{\otimes(k-1)} \otimes \left[\mathcal{L}_{h/\sqrt{n}} - \frac{1}{\sqrt{n}} h^i A_i + \frac{1}{2n}\left(J_{ij} h^i h^j\right) I\right] \otimes I^{\otimes(n-k)} \\
&= \sum_{k=1}^{n} I^{\otimes(k-1)} \otimes \left[\frac{1}{\sqrt{n}} h^i A_i + \frac{1}{n} B_{ij} h^i h^j + o(\frac{1}{n}) - \frac{1}{\sqrt{n}} h^i A_i + \frac{1}{2n}\left(J_{ij} h^i h^j\right) I\right] \otimes I^{\otimes(n-k)} \\
&= \sum_{k=1}^{n} I^{\otimes(k-1)} \otimes \left[\frac{1}{n} B_{ij} h^i h^j + \frac{1}{2n}\left(J_{ij} h^i h^j\right) I + o(\frac{1}{n})\right] \otimes I^{\otimes(n-k)} \\
&= \sum_{k=1}^{n} I^{\otimes k-1} \otimes \frac{1}{\sqrt{n}} P(n) \otimes I^{\otimes(n-k)},
\end{aligned}$$

where

$$P(n) := \sqrt{n}\left(\frac{1}{n}\left(B_{ij} + \frac{1}{2} J_{ij} I\right) h^i h^j + o(\frac{1}{n})\right).$$

Note that $\lim_{n\to\infty} P(n) = 0$, and that

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n}\, \mathrm{Tr}\, \rho_{\theta_0} P(n) &= \mathrm{Tr}\, \rho_{\theta_0}\left(B_{ij} + \frac{1}{2} J_{ij} I\right) h^i h^j \\
&= \mathrm{Tr}\, \rho_{\theta_0}\left(B_{ij} + \frac{1}{2} J_{ji} I\right) h^i h^j \\
&= \mathrm{Tr}\, \rho_{\theta_0}\left(B_{ij} + \frac{1}{2} A_i A_j\right) h^i h^j \\
&= 0
\end{aligned}$$

because of (A.2). It then follows from Lemma 2.6 that $R_h^{(n)} = o(\Delta^{(n)}, \rho_{\theta_0}^{\otimes n})$ for each $h \in \mathbb{R}^d$. This completes the proof.

A.4. Proof of Corollary 2.11. That $\rho_\theta^{\otimes n} \sim \rho_{\theta_0}^{\otimes n}$ was proven in the proof of Theorem 2.10. Let $\Delta_1^{(n)}, \dots, \Delta_d^{(n)}$ be as in the proof of Theorem 2.10. It then follows from the quantum central limit theorem that

$$\left(\begin{pmatrix} X^{(n)} \\ \Delta^{(n)} \end{pmatrix}, \rho_{\theta_0}^{\otimes n}\right) \underset{q}{\rightsquigarrow} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \tau \\ \tau^* & J \end{pmatrix}\right). \tag{A.3}$$

Further, because of Lemma 2.6, the sequence $R_h^{(n)}$ of observables given in the proof of Theorem 2.10 is also infinitesimal relative to the convergence (A.3). Now that $(\rho_\theta^{\otimes n},\, X^{(n)})$ are jointly QLAN at θ_0, the property $\left(X^{(n)}, \rho_{\theta_0+h/\sqrt{n}}^{\otimes n}\right) \underset{q}{\rightsquigarrow} N((\mathrm{Re}\,\tau)h, \Sigma)$ is an immediate consequence of Theorem 2.9. This completes the proof.

A.5. Proof of Theorem 3.1. Let $\mathcal{D} := \mathcal{D}_{\rho_{\theta_0}}$ be the commutation operator with respect to the state ρ_{θ_0} (see Section B.1), and let $\mathcal{T}$ be the minimal $\mathcal{D}$ invariant extension of the SLD tangent space $\mathrm{span}_{\mathbb{R}}\, \{L_i\}_{i=1}^d$ of the model $\{\rho_\theta\}$ at $\theta = \theta_0$, i.e., the smallest $\mathcal{D}$ invariant real linear subspace of Hermitian operators on $\mathcal{H}$ containing all the SLDs $\{L_i\}_{i=1}^d$ of ρ_θ at θ_0. The minimality ensures that $\mathrm{Tr}\, \rho_{\theta_0} A = 0$ for all $A \in \mathcal{T}$ because $\mathcal{T}' = \{A \in \mathcal{T}\,;\ \mathrm{Tr}\, \rho_{\theta_0} A = 0\}$ is also $\mathcal{D}$ invariant.

Let $\{D_j\}_{j=1}^r$ be a basis of $\mathcal{T}$, thus $d \le r$. Let Σ be an $r \times r$ matrix whose (i,j)th entry is given by $\Sigma_{ij} = \mathrm{Tr}\, \rho_{\theta_0} D_j D_i$, and let τ be an $r \times d$ matrix whose (i,j)th entry is given by $\tau_{ij} = \mathrm{Tr}\, \rho_{\theta_0} L_j D_i$.

It can be shown (see Theorem B.1) that the Holevo bound for a weight $G>0$ is expressed as

$$C_{\theta_0}(\rho_\theta, G) = \min_F \{\operatorname{Tr} GZ + \operatorname{Tr}\left|\sqrt{G}\operatorname{Im} Z\sqrt{G}\right| \; ; \; Z = {}^tF\Sigma F,$$

$$F \text{ is an } r\times d \text{ real matrix satisfying } {}^tF\operatorname{Re}(\tau) = I\}. \tag{A.4}$$

Letting

$$X_i^{(n)} := \frac{1}{\sqrt{n}}\sum_{k=1}^n I^{\otimes(k-1)}\otimes D_i \otimes I^{\otimes(n-k)} \quad (1\le i\le r),$$

Corollary 2.11 asserts that $\left(\left\{\rho_\theta^{\otimes n}\right\}, X^{(n)}\right)$ is jointly QLAN at θ_0, and that

$$\left(X^{(n)}, \rho_{\theta_0+h/\sqrt{n}}^{\otimes n}\right) \underset{q}{\rightsquigarrow} N((\operatorname{Re}\tau)h, \Sigma). \tag{A.5}$$

Let F be the matrix that attains the minimum in (A.4), and let $Z := {}^tF\Sigma F$, $\tilde{V} := \operatorname{Re} Z$, $\tilde{S} := \operatorname{Im} Z$, $\hat{V} = \sqrt{G^{-1}}\left|\sqrt{G}\operatorname{Im} Z\sqrt{G}\right|\sqrt{G^{-1}}$, and $\hat{Z} = \hat{V} - \sqrt{-1}\tilde{S}$. It is shown (see Corollary B.6 and Theorem B.7) that

$$C_{\theta_0}(\rho_\theta, G) = \operatorname{Tr} G\left(\tilde{V}+\hat{V}\right).$$

Further, Lemma B.9 assures that there exist a finite dimensional Hilbert space $\hat{\mathcal{H}}$ and a state σ and observables B_i $(1\le i\le d)$ on $\hat{\mathcal{H}}$ such that $\operatorname{Tr}\sigma B_i = 0$ and $\operatorname{Tr}\sigma B_j B_i = \hat{Z}_{ij}$. Let

$$\overline{X}_i^{(n)} := \tilde{X}_i^{(n)}\otimes \hat{I}^{\otimes n} + I^{\otimes n}\otimes Y_i^{(n)} \quad (1\le i\le d),$$

where $\tilde{X}^{(n)} := F_i^k X_k^{(n)}$ $(1\le i\le d)$,

$$Y_i^{(n)} := \frac{1}{\sqrt{n}}\sum_{k=1}^n \hat{I}^{\otimes(k-1)}\otimes B_i\otimes \hat{I}^{\otimes(n-k)} \quad (1\le i\le d),$$

and $\hat{I}$ is the identity on $\hat{\mathcal{H}}$. A crucial observation is that $\left(\overline{X}^{(n)}, \overline{\rho}_h^{(n)}\right)$, where $\overline{\rho}_h^{(n)} := \rho_{\theta_0+h/\sqrt{n}}^{\otimes n}\otimes\sigma^{\otimes n}$, converges to a classical Gaussian state:

$$\left(\overline{X}^{(n)}, \overline{\rho}_h^{(n)}\right) \underset{q}{\rightsquigarrow} N(h, \tilde{V}+\hat{V}), \tag{A.6}$$

for all $h\in\mathbb{R}^d$. In fact,

$$\begin{aligned}
\lim_{n\to\infty}\operatorname{Tr}\overline{\rho}_h^{(n)}\left(\prod_{t=1}^s e^{\sqrt{-1}\xi_t^i\overline{X}_i^{(n)}}\right) &= \lim_{n\to\infty}\operatorname{Tr}\overline{\rho}_h^{(n)}\left\{\left(\prod_{t=1}^s e^{\sqrt{-1}\xi_t^i\tilde{X}_i^{(n)}}\right)\otimes\left(\prod_{t=1}^s e^{\sqrt{-1}\xi_t^i Y_i^{(n)}}\right)\right\}\\
&= \lim_{n\to\infty}\left[\operatorname{Tr}\rho_{\theta_0+h/\sqrt{n}}^{\otimes n}\left(\prod_{t=1}^s e^{\sqrt{-1}\xi_t^i\tilde{X}_i^{(n)}}\right)\right]\left[\operatorname{Tr}\sigma^{\otimes n}\left(\prod_{t=1}^s e^{\sqrt{-1}\xi_t^i Y_i^{(n)}}\right)\right]\\
&= \phi_h\left(\prod_{t=1}^s e^{\sqrt{-1}\xi_t^i\tilde{X}_i}\right)\psi\left(\prod_{t=1}^s e^{\sqrt{-1}\xi_t^i Y_i}\right),
\end{aligned} \tag{A.7}$$

where $\tilde{X}_i := F_i^k X_k$ $(1\le i\le d)$ are canonical observables with $X_1,\ldots,X_r$ being the basic canonical observables of $\operatorname{CCR}(\operatorname{Im}\Sigma)$ and $(X,\phi_h)\sim N((\operatorname{Re}\tau)h,\Sigma)$, and $Y_1,\ldots,Y_d$ are the basic canonical observables of $\operatorname{CCR}\left(\operatorname{Im}\hat{Z}\right)$ with $(Y,\psi)\sim N(0,\hat{Z})$. In the last line in (A.7), we used (A.5) as well as the quantum central limit theorem for $Y^{(n)}$. By using the explicit form (2.2) of the quasi-characteristic function for the quantum Gaussian state, (A.7) is rewritten as

$$\exp\left(\sum_{t=1}^{r}\left(\sqrt{-1}\xi_t^i h_i - \frac{1}{2}\xi_t^i\xi_t^j Z_{ji}\right) - \sum_{t=1}^{r}\sum_{s=t+1}^{r}\xi_t^i\xi_s^j Z_{ji}\right)\exp\left(-\frac{1}{2}\sum_{t=1}^{r}\xi_t^i\xi_t^j\hat{Z}_{ji} - \sum_{t=1}^{r}\sum_{s=t+1}^{r}\xi_t^i\xi_s^j\hat{Z}_{ji}\right)$$
$$= \exp\left(\sum_{t=1}^{r}\left(\sqrt{-1}\xi_t^i h_i - \frac{1}{2}\xi_t^i\xi_t^j(\tilde{V}+\hat{V})_{ji}\right) - \sum_{t=1}^{r}\sum_{s=t+1}^{r}\xi_t^i\xi_s^j(\tilde{V}+\hat{V})_{ji}\right).$$

This proves (A.6).

Now according to Lemma A.1 below, there exist a quintuple sequence

$$M^{(n,m,\ell,q,p)} = \left\{M_\omega^{(n,m,\ell,q,p)}\,;\ \omega\in\Omega^{(n,m,l,p,q)}\right\}$$

of POVMs on $\left(\mathcal{H}\otimes\hat{\mathcal{H}}\right)^{\otimes n}$, taking values in a certain finite subset $\Omega^{(n,m,l,p,q)}$ of $\mathbb{R}^d$, that enjoys the properties

$$\lim_{p\to\infty}\lim_{q\to\infty}\lim_{\ell\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\overline{E}_h^{(n)}[M^{(n,m,\ell,q,p)}] = h,$$

and

$$\lim_{p\to\infty}\lim_{q\to\infty}\lim_{\ell\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\overline{V}_h^{(n)}[M^{(n,m,\ell,q,p)}] = \tilde{V}+\hat{V},$$

for all $h\in\mathbb{R}^d$, where $\overline{E}_h^{(n)}[\,\cdot\,]$ and $\overline{V}_h^{(n)}[\,\cdot\,]$ denote the expectation and the covariance with respect to $\overline{\rho}_h^{(n)}$. It then follows from Lemma A.2 below that for any countable dense subset D of $\mathbb{R}^d$ and any $h\in D$, there exist a subsequence $\{(n,m(n),\ell(n),q(n),p(n)\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\overline{E}_h^{(n)}[M^{(n,m(n),\ell(n),q(n),p(n))}] = h,$$

and

$$\lim_{n\to\infty}\overline{V}_h^{(n)}[M^{(n,m(n),\ell(n),q(n),p(n))}] = \tilde{V}+\hat{V}.$$

This implies that the POVM $M^{(n)}$ on $\mathcal{H}^{\otimes n}$ that is uniquely defined by the requirement

$$\mathrm{Tr}\,\rho^{(n)}M_\omega^{(n)} = \mathrm{Tr}\,\left(\rho^{(n)}\otimes\sigma^{\otimes n}\right)M_\omega^{(n,m(n),\ell(n),q(n),p(n))}$$

for all density operator $\rho^{(n)}$ on $\mathcal{H}^{\otimes n}$ and $\omega\in\Omega^{(n,m(n),l(n),p(n),q(n))}$ enjoys

$$\lim_{n\to\infty}E_h^{(n)}[M^{(n)}] = h,$$

$$\lim_{n\to\infty}V_h^{(n)}[M^{(n)}] = \tilde{V}+\hat{V}.$$

for all $h\in D$. Recalling that $\mathrm{Tr}\,G(\tilde{V}+\hat{V}) = C_{\theta_0}(\rho_\theta, G)$, the proof is complete.

Lemma A.1. *Given a sequence $\mathcal{H}^{(n)}$ of finite dimensional Hilbert spaces, let $X^{(n)} = \left(X_1^{(n)},\dots,X_d^{(n)}\right)$ be a list of observables on $\mathcal{H}^{(n)}$, and let $\left\{\rho_h^{(n)}\right\}_h$ be a family of density operators on $\mathcal{H}^{(n)}$ parametrized by $h\in\mathbb{R}^d$. If there is a real $d\times d$ positive definite matrix V such that*

$$\left(X^{(n)},\rho_h^{(n)}\right)\underset{q}{\rightsquigarrow} N(h,V) \tag{A.8}$$

holds for all $h \in \mathbb{R}^d$, *then there exist a quintuple sequence* $\left\{M^{(n,m,\ell,q,p)};(n,m,\ell,q,p)\in\mathbb{N}^5\right\}$ *of POVMs on* $\mathcal{H}^{(n)}$ *that enjoy the properties*

$$\lim_{p\to\infty}\lim_{q\to\infty}\lim_{\ell\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} E_h^{(n)}[M^{(n,m,\ell,q,p)}] = h,$$

and

$$\lim_{p\to\infty}\lim_{q\to\infty}\lim_{\ell\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} V_h^{(n)}[M^{(n,m,\ell,q,p)}] = V.$$

Proof. Let

$$\Omega^{(m,\ell)} := \left\{\frac{\ell}{m}\overrightarrow{k} + \frac{\ell}{2m}(1,\,\dots,\,1)\,;\ \overrightarrow{k}\in\mathbb{Z}^d\right\}\cap[-l,l]^d$$

be a finite subset of $\mathbb{R}^d$, comprising $(2m)^d$ lattice points in the hypercube $[-l,l]^d$, and let $\Omega^{(m,\ell,p)} := \Omega^{(m,\ell)}\cap[-p,p]^d$ and $\Omega_0^{(m,\ell,p)} := \Omega^{(m,\ell,p)}\cup\{0\}$. We introduce a Gaussian density function $f_\omega^{(q)}(x)$ on $\mathbb{R}^d$ centered at $\omega=(\omega_1,\,\dots,\,\omega_d)\in\mathbb{R}^d$ by

$$f_\omega^{(q)}(x) := \left\{\prod_{i=1}^d g_{\omega_{d+1-i}}^{(q)}(x_{d+1-i})\right\}\left\{\prod_{i=1}^d g_{\omega_i}^{(q)}(x_i)\right\},$$

where $x=(x_1,\,\dots,\,x_d)\in\mathbb{R}^d$ and

$$g_s^{(q)}(t) := \left(\frac{q}{2\pi}\right)^{\frac14}\exp\left(-\frac{q}{4}(t-s)^2\right),\qquad (s,t\in\mathbb{R}).$$

By using this function, we define a POVM $M^{(n,m,l,q,p)} = \left\{M_\omega^{(n,m,l,q,p)};\,\omega\in\Omega_0^{(m,\ell,p)}\right\}$ on $\mathcal{H}^{(n)}$ that takes values in the finite subset $\Omega_0^{(m,\ell,p)}$ by

$$M_\omega^{(n,m,\ell,q,p)} := R^{(m,\ell,q)}(X^{(n)})\left[f_\omega^{(q)}(X^{(n)}) + \frac{I^{(n)}}{(2m)^d}\right]R^{(m,\ell,q)}(X^{(n)})$$

for $\omega\in\Omega^{(m,\ell,p)}$, and

$$M_0^{(n,m,\ell,q,p)} := \sum_{\omega\in\Omega^{(m,\ell)}\setminus\Omega^{(m,\ell,p)}}\left\{R^{(m,\ell,q)}(X^{(n)})\left[\left(f_\omega^{(q)}(X^{(n)}) + \frac{I^{(n)}}{(2m)^d}\right)\right]R^{(m,\ell,q)}(X^{(n)})\right\}.$$

Here

$$R^{(m,\ell,q)}(x) := g\left(\sum_{\omega\in\Omega^{(m,\ell)}} f_\omega^{(q)}(x)\right)$$

is the normalization with

$$g(t) := \frac{1}{\sqrt{t+1}}.$$

Intuitively speaking, the difference set $\Omega^{(m,\ell)}\setminus\Omega^{(m,\ell,p)}$ works as a "buffer" zone that gives the default outcome $\omega=0$. This device is meaningful only when $p<\ell$.

We shall prove that

$$\lim_{p\to\infty}\lim_{q\to\infty}\lim_{\ell\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\sum_{\omega\in\Omega_0^{(m,\ell,p)}} P(\omega)\,\mathrm{Tr}\,\rho_h^{(n)}M_\omega^{(n,m,\ell,q,p)} = \int_{\mathbb{R}^d}P(\omega)p_h(\omega)\mathrm{d}\omega, \tag{A.9}$$

where $P(\omega)$ is an arbitrary polynomial of ω such that $P(0)=0$ and $p_h(\omega)$ is a probability density function of the classical normal distribution $N(h,V)$. Once (A.9) has been proved, we can verify

$$\lim_{p\to\infty}\lim_{q\to\infty}\lim_{\ell\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} E_h^{(n)}[M^{(n,m,\ell,q,p)}]=h$$

and

$$\lim_{p\to\infty}\lim_{q\to\infty}\lim_{\ell\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} V_h^{(n)}[M^{(n,m,\ell,q,p)}]=V$$

just by letting $P(\omega)=\omega_i$ or $P(\omega)=\omega_i\omega_j$ $(1\le i,j\le d)$ in (A.9).

The first limit $n\to\infty$ in (A.9) yields

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{\omega\in\Omega_0^{(m,\ell,p)}} P(\omega)\operatorname{Tr}\rho_h^{(n)}M_\omega^{(n,m,\ell,q,p)}\\
&=\lim_{n\to\infty}\sum_{\omega\in\Omega^{(m,\ell,p)}} P(\omega)\operatorname{Tr}\rho_h^{(n)}M_\omega^{(n,m,\ell,q,p)}\\
&=\lim_{n\to\infty}\sum_{\omega\in\Omega^{(m,\ell,p)}} P(\omega)\operatorname{Tr}\rho_h^{(n)}R^{(m,\ell,q)}(X^{(n)})\left[f_\omega^{(q)}(X^{(n)})+\frac{I^{(n)}}{(2m)^d}\right]R^{(m,\ell,q)}(X^{(n)})\\
&=\sum_{\omega\in\Omega^{(m,\ell,p)}} P(\omega)E_h\left[R^{(m,\ell,q)}(X)^2\left(f_\omega^{(q)}(X)+\frac{I}{(2m)^d}\right)\right]\\
&=\int_{\mathbb{R}^d}\frac{\sum_{\omega\in\Omega^{(m,\ell,p)}}P(\omega)\left(f_\omega^{(q)}(x)+\frac{1}{(2m)^d}\right)}{\sum_{\omega\in\Omega^{(m,\ell)}}\left(f_\omega^{(q)}(x)+\frac{1}{(2m)^d}\right)}p_h(x)\mathrm{d}x. \qquad \text{(A.10)}
\end{aligned}$$

In the fourth line, we used the assumption (A.8) and Corollary A.4 in Section A.6, as well as the fact that functions $g_s^{(q)}(t)$ on $\mathbb{R}$ and $g(t)$ on $t\ge 0$ are both bounded and continuous. Further, $X=(X_1,\ldots,X_d)$ is a classical random vector that follow the normal distribution $N(h,V)$, and $E_h[\,\cdot\,]$ denotes the expectation with respect to $N(h,V)$. As for the second limit $m\to\infty$, due to

$$\left|\frac{\sum_{\omega\in\Omega^{(m,\ell,p)}}P(\omega)\left(f_\omega^{(q)}(x)+\frac{1}{(2m)^d}\right)}{\sum_{\omega\in\Omega^{(m,\ell)}}\left(f_\omega^{(q)}(x)+\frac{1}{(2m)^d}\right)}\right|\le\max_{\omega\in[-p,p]^d}|P(\omega)|<\infty,$$

the bounded convergence theorem yields

$$\begin{aligned}
\lim_{m\to\infty}\text{(A.10)} &= \int_{\mathbb{R}^d}\lim_{m\to\infty}\frac{\left(\frac{\ell}{m}\right)^d\sum_{\omega\in\Omega^{(m,\ell,p)}}P(\omega)\left(f_\omega^{(q)}(x)+\frac{1}{(2m)^d}\right)}{\left(\frac{\ell}{m}\right)^d\sum_{\omega\in\Omega^{(m,\ell)}}\left(f_\omega^{(q)}(x)+\frac{1}{(2m)^d}\right)}p_h(x)\mathrm{d}x\\
&= \int_{\mathbb{R}^d}\frac{\int_{\omega\in[-p,p]^d}P(\omega)p^{(q)}(\omega,x)\mathrm{d}\omega}{\int_{\omega\in[-\ell,\ell]^d}p^{(q)}(\omega,x)\mathrm{d}\omega}p_h(x)\mathrm{d}x, \qquad \text{(A.11)}
\end{aligned}$$

where $p^{(q)}(\omega,x)=\left(\frac{q}{2\pi}\right)^{\frac{d}{2}}\exp\left(-\frac{q}{2}\sum_{i=1}^d(x_i-\omega_i)^2\right)$, and Darboux's theorem for the Riemann integral was used in the second line. Finally, the dominated convergence theorem and Fubini's theorem yield

$$\begin{aligned}
\lim_{p\to\infty}\lim_{q\to\infty}\lim_{\ell\to\infty}(\text{A.11}) &= \lim_{p\to\infty}\lim_{q\to\infty}\int_{\mathbb{R}^d}\frac{\int_{\omega\in[-p,p]^d}P(\omega)p^{(q)}(\omega,x)\mathrm{d}\omega}{\int_{\mathbb{R}^d}p^{(q)}(\omega,x)\mathrm{d}\omega}p_h(x)\mathrm{d}x\\
&= \lim_{p\to\infty}\lim_{q\to\infty}\int_{\mathbb{R}^d}\left(\int_{\omega\in[-p,p]^d}P(\omega)p^{(q)}(\omega,x)\mathrm{d}\omega\right)p_h(x)\mathrm{d}x\\
&= \lim_{p\to\infty}\lim_{q\to\infty}\int_{\omega\in[-p,p]^d}\left(\int_{\mathbb{R}^d}p^{(q)}(\omega,x)p_h(x)\mathrm{d}x\right)P(\omega)\mathrm{d}\omega\\
&= \lim_{p\to\infty}\lim_{q\to\infty}\int_{\omega\in[-p,p]^d}p_h^{(q)}(\omega)P(\omega)\mathrm{d}\omega\\
&= \lim_{p\to\infty}\int_{\omega\in[-p,p]^d}p_h(\omega)P(\omega)\mathrm{d}\omega\\
&= \int_{\mathbb{R}^d}p_h(\omega)P(\omega)\mathrm{d}\omega, \qquad \text{(A.12)}
\end{aligned}$$

where $p_h^{(q)}(\omega)$ is the density function of $N(h, V+\frac{1}{q}I)$. This completes the proof. □

LEMMA A.2. *For each $i\in\mathbb{N}$, let $\left\{a^i_{n_1n_2\cdots n_rn};\,(n_1,n_2,\ldots,n_r,n)\in\mathbb{N}^{(r+1)}\right\}$ be an $(r+1)$-tuple sequence on a normed space V. If, for each $i\in\mathbb{N}$, there exists an $\alpha^i\in V$ such that*

$$\lim_{n_1\to\infty}\lim_{n_2\to\infty}\cdots\lim_{n_r\to\infty}\lim_{n\to\infty}a^i_{n_1n_2\cdots n_rn}=\alpha^i,$$

then there exist a subsequence $\{(n_1(n),n_2(n),\ldots,n_r(n),n)\}_{n\in\mathbb{N}}$ that satisfies

$$\lim_{n\to\infty}a^i_{n_1(n)n_2(n)\cdots n_r(n)n}=\alpha^i$$

for all $i\in\mathbb{N}$.

PROOF. We first prove the case when $r=1$. Let $a^i_{n_1}:=\lim_{n\to\infty}a^i_{n_1n}$. We construct a subsequence $\{(n_1(k),n(k))\}_{k\in\mathbb{N}}$ in a recursive manner as follows. We set $n_1(1)=n(1)=1$. For $k\geq 2$, it follows from $\lim_{n_1\to\infty}a^i_{n_1}=\alpha^i$ that there exist an $N_1(k)\in\mathbb{N}$ such that $n_1\geq N_1(k)$ implies

$$\max_{1\leq i\leq k}\left|a^i_{n_1}-\alpha^i\right|<\frac{1}{k}.$$

Thus the number $n_1(k):=\max\{N_1(k),n_1(k-1)+1\}$ satisfies

$$\max_{1\leq i\leq k}\left|a^i_{n_1(k)}-\alpha^i\right|<\frac{1}{k}. \qquad \text{(A.13)}$$

For this $n_1(k)$, it follows from $\lim_{n\to\infty}a^i_{n_1(k)n}=a^i_{n_1(k)}$ that there exist an $N(k)\in\mathbb{N}$ such that $n\geq N(k)$ implies

$$\max_{1\leq i\leq k}\left|a^i_{n_1(k)n}-a^i_{n_1(k)}\right|<\frac{1}{k}. \qquad \text{(A.14)}$$

Thus we set $n(k):=\max\{N(k),n(k-1)+1\}$.

Now let $k(n):=\max\{k;\,n(k)\leq n\}$, which is non-decreasing in n and $\lim_{n\to\infty}k(n)=\infty$. We show that the subsequence $\{n_1(k(n)),n);\,n\in\mathbb{N}\}$ enjoys the required property: for all $i\in\mathbb{N}$,

$$\lim_{n\to\infty}a^i_{n_1(k(n))n}=\alpha^i.$$

Given $i \in \mathbb{N}$ and $\varepsilon > 0$ arbitrarily, there exist an $N \in \mathbb{N}$ such that $n \geq N$ implies $k(n) \geq \max\left\{i, \left\lceil \frac{2}{\varepsilon} \right\rceil\right\}$. Then for all $n \geq N$, we have

$$\begin{aligned}
\left|a^i_{n_1(k(n))n} - \alpha^i\right| &\leq \left|a^i_{n_1(k(n))n} - a^i_{n_1(k(n))}\right| + \left|a^i_{n_1(k(n))} - \alpha^i\right| \\
&\leq \max_{1\leq j\leq k(n)} \left|a^j_{n_1(k(n))n} - a^j_{n_1(k(n))}\right| + \max_{1\leq j\leq k(n)} \left|a^j_{n_1(k(n))} - \alpha^j\right| \\
&< \frac{2}{k(n)} \leq \varepsilon.
\end{aligned}$$

In the third inequality, we used (A.13) and (A.14), as well as its premise $n \geq n(k(n)) \geq N(k(n))$.

The proof for a generic r is similar. □

A.6. Quantum central limit theorem. Jakšić, Pautrat, and Pillet [3] proved the following strong version of a quantum central limit theorem.

PROPOSITION A.3. *Given a sequence $\mathcal{H}^{(n)}$ of Hilbert space, let $\rho^{(n)}$ and $A^{(n)} = (A^{(n)}_1, \ldots, A^{(n)}_d)$ be a state and a list of observables on $\mathcal{H}^{(n)}$ that enjoy the quantum central limit theorem in the sense of convergence of the quasi-characteristic function:*

$$\left(A^{(n)}, \rho^{(n)}\right) \underset{q}{\rightsquigarrow} N(h, J) \sim (X, \phi),$$

where J is a $d\times d$ positive semidefinite matrix. Then for any bounded continuous functions $f_1, \ldots, f_m$ and a noncommutative polynomial P, it follows that

$$\lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} P\left(\overrightarrow{f(A^{(n)})}\right) = \phi\left(P\left(\overrightarrow{f(X)}\right)\right),$$

where $\overrightarrow{f(B)} := (f_1(B_1), \ldots, f_1(B_d), \ldots, f_m(B_1), \ldots, f_m(B_d))$ for a given list $B = (B_1, \ldots, B_d)$ of observables, and $P\left(\overrightarrow{f(B)}\right) := P(f_1(B_1), \ldots, f_1(B_d), \ldots, f_m(B_1), \ldots, f_m(B_d))$.

Proposition A.3 is strong enough to prove the following, which is essential in constructing a sequence of POVMs that asymptotically achieves the Holevo bound (Section 3 in [4]).

COROLLARY A.4. *Under the same assumption as in Proposition A.3, for any bounded continuous functions $g, f_1, \ldots, f_m$, and noncommutative polynomials P, Q, with P being Hermitian operator-valued, it follows that*

$$\begin{aligned}
&\lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} g\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) Q\left(\overrightarrow{f(A^{(n)})}\right) g\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) \\
&\quad = \phi\left(g\left(P\left(\overrightarrow{f(X)}\right)\right) Q\left(\overrightarrow{f(X)}\right) g\left(P\left(\overrightarrow{f(X)}\right)\right)\right).
\end{aligned}$$

PROOF. Let $l := \max_{1\leq i\leq m} \sup_x |f_i(x)|$. There exist $l_P > 0$ and $l_Q > 0$ such that $l_P > \left\|P(\overrightarrow{B})\right\|$ and $l_Q > \left\|Q(\overrightarrow{B})\right\|$ for any list $\overrightarrow{B} = (B_1, \ldots, B_{dm})$ of observables such that $\|B_i\| \leq l$. Let $l_g := \sup\{|g(x)|\ ;\ x \in [-l_P, l_P]\}$. There exist a sequence $R^{(k)}(x)$ of polynomials that uniformly converges to $g(x)$ on $[-l_P, l_P]$.

Let

$$a_{kn} := \operatorname{Tr} \rho^{(n)} R^{(k)}\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) Q\left(\overrightarrow{f(A^{(n)})}\right) R^{(k)}\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right),$$

and let

$$a_n := \operatorname{Tr} \rho^{(n)} g\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) Q\left(\overrightarrow{f(A^{(n)})}\right) g\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right).$$

We show that a_{kn} uniformly converges to a_n as $k \to \infty$. In fact, letting $l_R := \sup\{R^{(k)}(x);\ k \in \mathbb{N}, x \in [-l_P, l_P]\}$,

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} |a_n - a_{kn}| \\
&= \sup_{n\in\mathbb{N}} \Big| \operatorname{Tr} \rho^{(n)} g\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) Q\left(\overrightarrow{f(A^{(n)})}\right) g\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) \\
&= -\operatorname{Tr} \rho^{(n)} R^{(k)}\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) Q\left(\overrightarrow{f(A^{(n)})}\right) R^{(k)}\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) \Big| \\
&\le \sup_{n\in\mathbb{N}} \left| \operatorname{Tr} \rho^{(n)} g\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) Q\left(\overrightarrow{f(A^{(n)})}\right) \left[g\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) - R^{(k)}\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right)\right]\right| \\
&\quad + \sup_{n\in\mathbb{N}} \left| \operatorname{Tr} \rho^{(n)} \left[g\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) - R^{(k)}\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right)\right] Q\left(\overrightarrow{f(A^{(n)})}\right) R^{(k)}\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right)\right| \\
&\le l_g l_Q \sup_{n\in\mathbb{N}} \left\| g\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) - R^{(k)}\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right)\right\| \\
&\quad + l_Q l_R \sup_{n\in\mathbb{N}} \left\| g\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) - R^{(k)}\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right)\right\| \\
&\le l_Q(l_g + l_R) \sup_{x\in[-l_P, l_P]} \left| g(x) - R^{(k)}(x)\right|,
\end{aligned}$$

which converges to zero as $k \to \infty$.

The uniform convergence $a_{kn} \rightrightarrows a_n$ as well as the existence of $\lim_{k\to\infty} \lim_{n\to\infty} a_{kn}$, which follows from Proposition A.3, ensure that

$$\begin{aligned}
&\lim_{n\to\infty} \operatorname{Tr} \rho^{(n)} g\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) Q\left(\overrightarrow{f(A^{(n)})}\right) g\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) \\
&\quad = \lim_{n\to\infty} \lim_{k\to\infty} \operatorname{Tr} \rho^{(n)} R^{(k)}\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) Q\left(\overrightarrow{f(A^{(n)})}\right) R^{(k)}\left(P\left(\overrightarrow{f(A^{(n)})}\right)\right) \\
&\quad = \lim_{n\to\infty} \lim_{k\to\infty} a_{kn} \\
&\quad = \lim_{k\to\infty} \lim_{n\to\infty} a_{kn} \\
&\quad = \lim_{k\to\infty} \phi\left(R^{(k)}\left(P\left(\overrightarrow{f(X)}\right)\right) Q\left(\overrightarrow{f(X)}\right) R^{(k)}\left(P\left(\overrightarrow{f(X)}\right)\right)\right) \\
&\quad = \phi\left(g\left(P\left(\overrightarrow{f(X)}\right)\right) Q\left(\overrightarrow{f(X)}\right) g\left(P\left(\overrightarrow{f(X)}\right)\right)\right).
\end{aligned}$$

This proves the claim. □

APPENDIX B: ELEMENTS OF QUANTUM ESTIMATION THEORY

This section is devoted to a brief account of quantum estimation theory.

B.1. Commutation operator and the Holevo bound. In the study of quantum statistics, Holevo [2] introduced useful mathematical tools called the square summable operators and the commutation operators associated with quantum states. Let $\mathcal{H}$ be a separable Hilbert space and let ρ be a density operator. We define a real Hilbert space $\mathcal{L}_h^2(\rho)$ associated with ρ by the completion of the set $\mathcal{B}_h(\mathcal{H})$ of bounded Hermitian operators with respect to the pre-inner product $\langle X, Y\rangle_\rho :=$

$\mathrm{Re}\,\mathrm{Tr}\,\rho XY$. Letting $\rho = \sum_j s_j |\psi_j\rangle\langle\psi_j|$ be the spectral representation, an element $X \in \mathcal{L}_h^2(\rho)$ can be regarded as an equivalence class of those Hermitian operators, called the square summable operators, which satisfy $\sum_j s_j \|X\psi_j\|^2 < \infty$ (so that $\psi_j \in \mathrm{Dom}(X)$ if $s_j \neq 0$) under the identification $X_1 \sim X_2$ if $X_1\psi_j = X_2\psi_j$ for $s_j \neq 0$. The space $\mathcal{L}_h^2(\rho)$ thus provides a convenient tool to cope with unbounded observables. Note that when $\mathcal{H}$ is finite dimensional, the setup is considerably simplified to be $\mathcal{L}_h^2(\rho) = \mathcal{B}_h(\mathcal{H})/\ker\rho$.

Let $\mathcal{L}^2(\rho)$ be the complexification of $\mathcal{L}_h^2(\rho)$, which is also regarded as the completion of $\mathcal{B}(\mathcal{H})$ with respect to the pre-inner product

$$\langle X, Y\rangle_\rho := \frac{1}{2}\mathrm{Tr}\,\rho(YX^* + X^*Y).$$

Thus $\mathcal{L}^2(\rho)$ is a complex Hilbert space with this inner product. Let us further introduce two sesquilinear forms on $\mathcal{B}(\mathcal{H})$ by

$$(X, Y)_\rho := \mathrm{Tr}\,\rho Y X^*, \qquad [X, Y]_\rho := \frac{1}{2\sqrt{-1}}\mathrm{Tr}\,\rho(YX^* - X^*Y).$$

and extend them to $\mathcal{L}^2(\rho)$ by continuity. (Note that $(X, X)_\rho \le 2\langle X, X\rangle_\rho$ and $(X, Y)_\rho = \langle X, Y\rangle_\rho + \sqrt{-1}[X, Y]_\rho$.)

The *commutation operator* $\mathcal{D}_\rho : \mathcal{L}^2(\rho) \to \mathcal{L}^2(\rho)$ with respect to ρ is defined by

$$[X, Y]_\rho = \langle X, \mathcal{D}_\rho Y\rangle_\rho,$$

which is formally represented by the operator equation

$$\mathcal{D}_\rho(X)\rho + \rho\mathcal{D}_\rho(X) = \sqrt{-1}\,(X\rho - \rho X).$$

(To be precise, Holevo's original definition is different from the above one by a factor of 2.) The operator $\mathcal{D}_\rho$ is a $\mathbb{C}$-linear bounded skew-adjoint operator. Moreover, since the forms $[\,\cdot\,,\,\cdot\,]_\rho$ and $\langle\,\cdot\,,\,\cdot\,\rangle_\rho$ are real on the real subspace $\mathcal{L}_h^2(\rho)$, this subspace is invariant under the operation of $\mathcal{D}_\rho$. Thus $\mathcal{D}_\rho$ can be regarded as an $\mathbb{R}$-linear bounded skew-adjoint operator when restricted to $\mathcal{L}_h^2(\rho)$ as $\mathcal{D}_\rho : \mathcal{L}_h^2(\rho) \to \mathcal{L}_h^2(\rho)$. When no confusion is likely to arise, we drop the subscript ρ of $\mathcal{D}_\rho$ and simply denote it as $\mathcal{D}$.

Let $\mathcal{S} = \{\rho_\theta\,;\,\theta \in \Theta \in \mathbb{R}^d\}$ be a quantum statistical model satisfying the conditions: 1) the parametrization $\theta \mapsto \rho_\theta$ is smooth and nondegenerate so that the derivatives $\{\partial\rho_\theta/\partial\theta^i\}_{1\le i\le d}$ exist in trace class and form a linearly independent set at each point $\theta \in \Theta$, and 2) there exists a constant c such that

$$\left|\frac{\partial}{\partial\theta^i}\,\mathrm{Tr}\,\rho_\theta X\right|^2 \le c\,\langle X, X\rangle_{\rho_\theta}$$

for all $X \in \mathcal{B}(\mathcal{H})$ and i. The second condition assures that the linear functionals $X \mapsto (\partial/\partial\theta^i)\mathrm{Tr}\,\rho_\theta X$ can be extended to continuous linear functionals on $\mathcal{L}^2(\rho_\theta)$. Given a quantum statistical model satisfying the above conditions, the *symmetric logarithmic derivative* (SLD) $L_{\theta,i}$ in the ith direction is defined as the operator in $\mathcal{L}^2(\rho_{\rho_\theta})$ satisfying

$$\frac{\partial}{\partial\theta^i}\mathrm{Tr}\,\rho_\theta X = \langle L_{\theta,i}, X\rangle_{\rho_\theta}.$$

It is easily verified that $L_{\theta,i} \in \mathcal{L}_h^2(\rho_\theta)$; so the definition is formally written as

$$\frac{\partial\rho_\theta}{\partial\theta^i} = \frac{1}{2}(L_{\theta,i}\,\rho_\theta + \rho_\theta\,L_{\theta,i}). \tag{B.1}$$

When no confusion occurs, we simply denote $L_{\theta,i}$ as L_i. Since L_i is a faithful operator representation of the tangent vector $\partial/\partial\theta^i$, we shall call the $\mathbb{R}$-linear space $\mathrm{span}_{\mathbb{R}}\{L_i\}_{i=1}^d$ the *SLD tangent space* of the model ρ_θ at θ. Incidentally the $d\times d$ real symmetric matrix $J_\theta := [\mathrm{Re}\,\mathrm{Tr}\,\rho_\theta L_i L_j]_{1\le i,j\le d}$ is called the *SLD Fisher information matrix* of the model $\mathcal{S}$ at θ.

An estimator $\hat{M}$ for the parameter θ of the model $\mathcal{S}$ is called *unbiased* if

$$E_\theta[\hat{M}] = \theta \tag{B.2}$$

for all $\theta\in\Theta$, where $E_\theta[\,\cdot\,]$ denotes the expectation with respect to ρ_θ. An estimator $\hat{M}$ is called *locally unbiased* at $\theta_0\in\Theta$ if the condition (B.2) is satisfied around $\theta=\theta_0$ up to the first order of the Taylor expansion. It is well known that an estimator $\hat{M}$ that is locally unbiased at θ_0 satisfies the quantum (SLD) Cramér-Rao inequality, $V_{\theta_0}[\hat{M}]\ge J_{\theta_0}^{-1}$, where $V_{\theta_0}[\,\cdot\,]$ denotes the covariance matrix with respect to ρ_{θ_0}. The lower bound $J_{\theta_0}^{-1}$ cannot be attained in general due to the non-commutativity of the SLDs. Because of this fact, we often switch the problem to minimizing the weighted sum of covariances, $\mathrm{Tr}\,GV_{\theta_0}[\hat{M}]$, given a $d\times d$ real positive definite matrix G. It is known that this quantity also has a variety of Cramér-Rao type lower bounds [2]:

$$\mathrm{Tr}\,GV_{\theta_0}[\hat{M}] \ge C_{\theta_0}(\rho_\theta, G).$$

Among others, we concentrate our attention to the *Holevo bound* [2]:

$$\begin{aligned} C_{\theta_0}(\rho_\theta,G) \;:=\; & \min_{V,B}\{\mathrm{Tr}\,GV\,;\ V \text{ is a real matrix such that } V\ge Z(B),\ Z_{ij}(B)=\mathrm{Tr}\,\rho_{\theta_0}B_jB_i,\\ & B_1,\dots,B_d \text{ are Hermitian operators on } \mathcal{H} \text{ such that } \mathrm{Re}\,\mathrm{Tr}\,\rho_{\theta_0}L_iB_j=\delta_{ij}\}. \end{aligned} \tag{B.3}$$

The minimization problem over V is explicitly solved, to obtain

$$\begin{aligned} C_{\theta_0}(\rho_\theta,G) \;=\; & \min_{B}\{\mathrm{Tr}\,GZ(B)+\mathrm{Tr}\left|\sqrt{G}\,\mathrm{Im}\,Z(B)\sqrt{G}\right|\,;\ Z_{ij}(B)=\mathrm{Tr}\,\rho_{\theta_0}B_jB_i,\\ & B_1,\dots,B_d \text{ are Hermitian operators on } \mathcal{H} \text{ such that } \mathrm{Re}\,\mathrm{Tr}\,\rho_{\theta_0}L_iB_j=\delta_{ij}\}. \end{aligned}$$

Our aim here is to derive a further concise expression for it in terms of a $\mathcal{D}$ invariant extension of the SLD tangent space, a subspace of $\{X\in\mathcal{L}_h^2(\rho_{\theta_0});\ \mathrm{Tr}\,\rho_{\theta_0}X=0\}$ including the SLD tangent space such that $\mathcal{D}(\mathcal{T})\subset\mathcal{T}$.

THEOREM B.1. *Suppose that a quantum statistical model $\mathcal{S}=\left\{\rho_\theta\,;\ \theta\in\Theta\subset\mathbb{R}^d\right\}$ on $\mathcal{H}$ has a finite dimensional $\mathcal{D}$ invariant extension $\mathcal{T}$ of the SLD tangent space of $\mathcal{S}$ at $\theta=\theta_0$. Letting $\{D_j\}_{j=1}^r$ be a basis of $\mathcal{T}$, the Holevo bound defined by (B.3) is rewritten as*

$$\begin{aligned} C_{\theta_0}(\rho_\theta,G) \;=\; & \min_{F}\{\mathrm{Tr}\,GZ+\mathrm{Tr}\left|\sqrt{G}\,\mathrm{Im}\,Z\sqrt{G}\right|\,;\ Z={}^tF\Sigma F,\\ & F \text{ is an } r\times d \text{ real matrix satisfying } {}^tF\,\mathrm{Re}\,(\tau)=I\}, \end{aligned} \tag{B.4}$$

where Σ and τ are $r\times r$ and $r\times d$ complex matrices whose (i,j)th entries are given by $\Sigma_{ij}=\mathrm{Tr}\,\rho_{\theta_0}D_jD_i$ and $\tau_{ij}=\mathrm{Tr}\,\rho_{\theta_0}L_jD_i$.

PROOF. Let $\mathcal{T}^\perp$ be the orthogonal complement of $\mathcal{T}$ in $\mathcal{L}_h^2(\rho_{\theta_0})$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\rho_{\theta_0}}$, and let $\mathcal{P}:\mathcal{L}_h^2(\rho_{\theta_0})\to\mathcal{T}$ and $\mathcal{P}^\perp:\mathcal{L}_h^2(\rho_{\theta_0})\to\mathcal{T}^\perp$ be the projections associated with the decomposition $\mathcal{L}_h^2(\rho_{\theta_0})=\mathcal{T}\oplus\mathcal{T}^\perp$. Note that if $X\in\mathcal{T}^\perp$ and $Y\in\mathcal{T}$, then

$$(X,Y)_{\rho_{\theta_0}}=\langle X,Y\rangle_{\rho_{\theta_0}}+\sqrt{-1}\,\langle X,\mathcal{D}Y\rangle_{\rho_{\theta_0}}=0.$$

We show that the operators $\{B_j\}_{j=1}^d$ that achieve the minimum in (B.3) can be taken from $\mathcal{T}$. Let $\{B_j\}_{j=1}^d \subset \mathcal{L}_h^2(\rho_{\theta_0})$ satisfies the local unbiasedness condition $\mathrm{Re}\,\mathrm{Tr}\,\rho_{\theta_0} L_i B_j = \delta_{ij}$, which is rewritten as

$$\langle L_i, B_j \rangle_{\rho_{\theta_0}} = \delta_{ij}.$$

Then $\{\mathcal{P}(B_j)\}_{j=1}^d$ also satisfies the local unbiasedness

$$\langle L_i, \mathcal{P}(B_j) \rangle_{\rho_{\theta_0}} = \langle L_i, B_j \rangle_{\rho_{\theta_0}} = \delta_{ij}.$$

Further,

$$\begin{aligned} Z_{ij}(B) &= (B_i, B_j)_{\rho_{\theta_0}} = \left(\mathcal{P}(B_i) + \mathcal{P}^\perp(B_i), \mathcal{P}(B_j) + \mathcal{P}^\perp(B_j)\right)_{\rho_{\theta_0}} \\ &= (\mathcal{P}(B_i), \mathcal{P}(B_j))_{\rho_{\theta_0}} + \left(\mathcal{P}^\perp(B_i), \mathcal{P}^\perp(B_j)\right)_{\rho_{\theta_0}} = Z_{ij}(\mathcal{P}(B)) + Z_{ij}(\mathcal{P}^\perp(B)). \end{aligned}$$

Since $Z(\,\cdot\,)$ is a Gram matrix and is positive semidefinite, this decomposition implies that $Z(B) \geq Z(\mathcal{P}(B))$. Thus the observables B that minimize (B.3) can be taken from $\mathcal{T}$.

Let $B_j \in \mathcal{T}$ be expanded as $B_j = F_j^k D_k$, where F is an $r \times d$ real matrix. Then the local unbiasedness condition is rewritten as

$$\langle L_i, B_j \rangle_{\rho_{\theta_0}} = F_j^k \langle L_i, D_k \rangle_{\rho_{\theta_0}} = \delta_{ij},$$

or in a matrix form,

$${}^t F\,(\mathrm{Re}\,\tau) = I.$$

Further, the Gram matrix $Z(B)$ is rewritten as

$$Z_{ij}(B) = (B_i, B_j)_{\rho_{\theta_0}} = F_i^k F_j^\ell\,(D_k, D_\ell)_{\rho_{\theta_0}},$$

or,

$$Z(B) = {}^t F \Sigma F.$$

This proves the claim. □

When the SLD tangent space itself is $\mathcal{D}$ invariant, the Holevo bound can be represented in terms of the RLD Fisher information matrix as follows.

Corollary B.2. *Let $\left\{\rho_\theta\,;\ \theta \in \Theta \subset \mathbb{R}^d\right\}$ be a quantum statistical model, and let L_i $(1 \leq i \leq d)$ be the SLDs at θ_0. If the SLD tangent space $\mathrm{span}_{\mathbb{R}}\,\{L_i\}_{i=1}^d$ at θ_0 is $\mathcal{D}$ invariant, then*

$$C_{\theta_0}(\rho_\theta, G) = \mathrm{Tr}\,G(J^{(R)})^{-1} + \mathrm{Tr}\left|\sqrt{G}\,\mathrm{Im}\,(J^{(R)})^{-1}\sqrt{G}\right|,$$

where $(J^{(R)})^{-1} := (\mathrm{Re}\,J)^{-1}\,J\,(\mathrm{Re}\,J)^{-1}$ with $J_{ij} = \mathrm{Tr}\,\rho_{\theta_0} L_j L_i$.

Proof. Let us set $D_i := L_i$ for $1 \leq i \leq d$ in Theorem B.1. Then $\Sigma = \tau$, and the local unbiasedness condition ${}^t F\,(\mathrm{Re}\,\tau) = I$ has a unique solution $F = (\mathrm{Re}\,\Sigma)^{-1}$, whereby $Z = (\mathrm{Re}\,J)^{-1}\,J\,(\mathrm{Re}\,J)^{-1}$. □

Note that RLDs may not exist if the model is degenerate (i.e., non-faithful). This means that $J^{(R)}$ may not be well-defined for such a model. Nevertheless we use the notation $(J^{(R)})^{-1}$ even for a degenerate model, and call it the inverse of the RLD Fisher information matrix, as long as the SLD tangent space is $\mathcal{D}$ invariant. For an idea behind this nomenclature, consult [1].

Finally, we show that the Holevo bound for the nth i.i.d. extension model is precisely $\frac{1}{n}$ times that for the base model.

COROLLARY B.3. *Given a quantum statistical model* $\mathcal{S} = \left\{\rho_\theta\,;\ \theta \in \Theta \subset \mathbb{R}^d\right\}$ *that has a finite dimensional* $\mathcal{D}$ *invariant extension of the SLD tangent space, let* $\mathcal{S}^{(n)} = \left\{\rho_\theta^{\otimes n}\,;\ \theta \in \Theta \subset \mathbb{R}^d\right\}$ *be the nth i.i.d. extension model. Then*

$$C_{\theta_0}\left(\rho_\theta^{\otimes n}, G\right) = \frac{1}{n} C_{\theta_0}\left(\rho_\theta, G\right).$$

PROOF. Let us distinguish quantities that belong to models of different extension by specifying the degree n of extension in the superscript. Letting $\{L_i\}_{i=1}^d$ and $\{D_j\}_{j=1}^r$ be SLDs and a basis of $\mathcal{T}$ in Theorem B.1, the corresponding quantities for $\mathcal{S}^{(n)}$ are given by

$$L_i^{(n)} = \sum_{k=1}^n I^{\otimes k-1} \otimes L_i \otimes I^{\otimes n-k}$$

and

$$D_j^{(n)} = \sum_{k=1}^n I^{\otimes k-1} \otimes D_j \otimes I^{\otimes n-k}.$$

Thus

$$\Sigma^{(n)} = n\Sigma^{(1)}, \quad \tau^{(n)} = n\tau^{(1)}, \quad F^{(n)} = \frac{1}{n}F^{(1)},$$

so that

$$Z^{(n)} = {}^tF^{(n)}\Sigma^{(n)}F^{(n)} = \frac{1}{n}Z^{(1)},$$

and

$$C_{\theta_0}\left(\rho_\theta^{\otimes n}, G\right) = \frac{1}{n} C_{\theta_0}\left(\rho_\theta, G\right)$$

doe to Theorem B.1. □

B.2. Estimation of quantum Gaussian shift model. In this section, we briefly overview the estimation theory for a quantum Gaussian shift model. For a mathematically rigorous treatment, consult [2].

LEMMA B.4. *Let* $(X, \phi_h) \sim N(h, J)$*, where* J *is a* $d \times d$ *positive semidefinite complex matrix. Then*

$$\phi_h(X_i) = h_i \tag{B.5}$$

and

$$\phi_h((X_j - h_j)(X_i - h_i)) = J_{ij} \tag{B.6}$$

hold.

PROOF. Letting $U(\xi) := \mathrm{e}^{\sqrt{-1}\xi^i X_i}$,

$$\begin{aligned}
\phi_h(U(\xi)) &= 1 + \sqrt{-1}\phi_h(\xi^i X_i) - \frac{1}{2}\phi_h(\left(\xi^i X_i\right)^2) + o(\xi^2) \\
&= 1 + \sqrt{-1}\phi_h(X_i)\xi^i - \frac{1}{2}\phi_h(X_i X_j)\xi^i\xi^j + o(\xi^2) \\
&= 1 + \sqrt{-1}\phi_h(X_i)\xi^i - \frac{1}{2}\phi_h(X_i \circ X_j)\xi^i\xi^j + o(\xi^2),
\end{aligned}$$

where $X_i \circ X_j = \frac{1}{2}(X_iX_j + X_JX_i)$. Further, letting $V = \operatorname{Re} J$ and $S = \operatorname{Im} J$,

$$\begin{aligned} e^{\sqrt{-1}\xi^i h_i - \frac{1}{2}V_{ij}\xi^i\xi^j} &= 1 + \left(\sqrt{-1}\xi^i h_i - \frac{1}{2}V_{ij}\xi^i\xi^j\right) + \frac{1}{2}\left(\sqrt{-1}\xi^i h_i - \frac{1}{2}V_{ij}\xi^i\xi^j\right)^2 + o(\xi^2) \\ &= 1 + \sqrt{-1}\xi^i h_i - \frac{1}{2}(V_{ij} + h_ih_j)\xi^i\xi^j + o(\xi^2). \end{aligned}$$

A comparison immediately leads to (B.5) and the identity $\phi_h(X_i \circ X_j) = V_{ij} + h_ih_j$. Thus

$$\begin{aligned} \phi_h((X_j - h_j)(X_i - h_i)) &= \phi_h(X_jX_i - h_jX_i - h_iX_j + h_ih_j) \\ &= \phi_h(X_jX_i) - h_ih_j \\ &= \phi_h\left(X_i \circ X_j - \frac{1}{2}[X_i, X_j]\right) - h_ih_j = J_{ij}. \end{aligned}$$

□

In what follows, we treat the quantum Gaussian shift model $\left\{N(\tau h, \Sigma)\,;\ h \in \mathbb{R}^d\right\}$ on $\mathrm{CCR}(\operatorname{Im}\Sigma)$, where Σ is an $r \times r$ complex matrix such that $\Sigma \geq 0$ and $\operatorname{Re}\Sigma > 0$, and τ is an $r \times d$ real matrix with $d \leq r$ such that $\operatorname{rank}\tau = d$. Let $X = (X_1, \ldots, X_r)$ be the basic canonical observables of $\mathrm{CCR}(\operatorname{Im}\Sigma)$, and $(X, \phi_h) \sim N(\tau h, \Sigma)$.

Lemma B.5. *Let $U(\xi) := e^{\sqrt{-1}\xi^i X_i}$. The SLD L_i $(1 \leq i \leq d)$ at h defined by*

$$\frac{\partial}{\partial h_k}\phi_h(U(\xi)) = \frac{1}{2}\phi_h\left(U(\xi)L_k + L_kU(\xi)\right) \tag{B.7}$$

is given by

$$L_k = \sum_{\ell=1}^{r}\left[(\operatorname{Re}\Sigma)^{-1}\tau\right]_{\ell k}(X_\ell - (\tau h)_\ell I). \tag{B.8}$$

Proof. In this proof we lift Einstein's summation convention. Let $V = \operatorname{Re}\Sigma$ and $S = \operatorname{Im}\Sigma$, and fix a $k \in \{1, \ldots, d\}$ arbitrarily. Due to the Baker-Hausdorff formula,

$$U(\xi) = e^{\sqrt{-1}\sum_{i=1}^r \xi^i X_i} = \exp\left(-\sqrt{-1}\sum_{i=1}^{r} S_{ki}\xi^k\xi^i\right)\exp\left(\sqrt{-1}\xi^k X_k\right)\exp\left(\sqrt{-1}\sum_{i\neq k}\xi^i X_i\right).$$

By differentiating in ξ^k, we have

$$\frac{\partial}{\partial \xi^k}U(\xi) = -\sqrt{-1}\left(\sum_{i=1}^{r} S_{ki}\xi^i - X_k\right)U(\xi).$$

Thus

$$\begin{aligned}\phi_h((X_k-(\tau h)_k I)U(\xi)) &= \phi_h\left(\left(\sum_{i=1}^r S_{ki}\xi^i - \sqrt{-1}\frac{\partial}{\partial \xi^k} - (\tau h)_k I\right)U(\xi)\right)\\ &= \left(\sum_{i=1}^r S_{ki}\xi^i - \sqrt{-1}\frac{\partial}{\partial \xi^k} - (\tau h)_k\right)\phi_h(U(\xi))\\ &= \left(\sum_{i=1}^r S_{ki}\xi^i - \sqrt{-1}\frac{\partial}{\partial \xi^k} - (\tau h)_k\right)\mathrm{e}^{\sqrt{-1}\,{}^t\xi\tau h - \frac{1}{2}\,{}^t\xi V\xi}\\ &= \left(\sum_{i=1}^r S_{ki}\xi^i - (\tau h)_k\right)\phi_h(U(\xi)) - \sqrt{-1}\left(\sqrt{-1}(\tau h)_k - (V\xi)_k\right)\phi_h(U(\xi))\\ &= \left(S\xi + \sqrt{-1}V\xi\right)_k \phi_h(U(\xi))\\ &= \sqrt{-1}\,(\bar{J}\xi)_k\,\phi_h(U(\xi)). \qquad \text{(B.9)}\end{aligned}$$

Similarly, we obtain

$$\phi_h(U(\xi)(X_k-(\tau h)_k I)) = \sqrt{-1}\,(J\xi)_k\,\phi_h(U(\xi)). \tag{B.10}$$

By combining (B.9) and (B.10),

$$\phi_h\left(\,(X_k-(\tau h)_k I)U(\xi) + U(\xi)(X_k-(\tau h)_k I)\,\right) = 2\sqrt{-1}\,(V\xi)_k\,\phi_h(U(\xi)). \tag{B.11}$$

On the other hand, by a direct calculation

$$\frac{\partial}{\partial h_k}\phi_h(U(\xi)) = \frac{\partial}{\partial h_k}\mathrm{e}^{\sqrt{-1}\,{}^t\xi\tau h - \frac{1}{2}\,{}^t\xi V\xi} = \sqrt{-1}({}^t\xi\tau)_k\phi_h(U(\xi)). \tag{B.12}$$

A comparison between (B.11) and (B.12) yields

$$L_k = \sum_{\ell=1}^r \left[V^{-1}\tau\right]_{\ell k}(X_\ell - (\tau h)_\ell I).$$

□

Let $\tilde{L}_k := X_k - (\tau h)_k I$. It follows from (B.9) and (B.10) that $\mathcal{D}_{\phi_h}(\tilde{L}_i) = \sum_{i=1}^r (V^{-1}S)_{ki}\tilde{L}_k$, where $\mathcal{D}_{\phi_h}$ is the commutation operator with respect to ϕ_h defined by

$$\phi_h\left(U(\xi)\mathcal{D}_{\phi_h}(X) + \mathcal{D}_{\phi_h}(X)U(\xi)\right) = \sqrt{-1}\phi_h\left(U(\xi)X - XU(\xi)\right).$$

This means $\mathcal{T} = \mathrm{span}\left\{\tilde{L}_k\right\}_{k=1}^r$ is $\mathcal{D}_{\phi_h}$ invariant. Further, we can check from (B.8) that $\mathrm{span}\,\{L_i\}_{i=1}^d \subset \mathcal{T}$ and

$$\phi_h(\tilde{L}_j\tilde{L}_i) = \Sigma_{ij} \tag{B.13}$$

and

$$\mathrm{Re}\,\phi_h(L_j\tilde{L}_i) = \tau_{ij}. \tag{B.14}$$

These relations play a fundamental role in connecting a general quantum statistical model $\mathcal{S} = \left\{\rho_\theta\,;\ \theta\in\Theta\subset\mathbb{R}^d\right\}$ on $\mathcal{H}$ with a quantum Gaussian shift model $\mathcal{G} = \left\{N(\tau h,\Sigma)\,;\ h\in\mathbb{R}^d\right\}$ as follows.

Let $\{L_i^{\mathcal{S}}\}_{i=1}^d$ be the SLDs of the model $\mathcal{S}$ at $\theta=\theta_0$, and let $\mathcal{T}^{\mathcal{S}}$ be a $\mathcal{D}^{\mathcal{S}}$ invariant extension of the SLD tangent space $\operatorname{span}\{L_i^{\mathcal{S}}\}_{i=1}^d$. Further let $\{D_j^{\mathcal{S}}\}_{j=1}^r$ be a basis of $\mathcal{T}^{\mathcal{S}}$ and let Σ and τ are $r\times r$ and $r\times d$ matrices whose (i,j)th entries are given by $\Sigma_{ij}=\operatorname{Tr}\rho_{\theta_0}D_jD_i$ and $\tau_{ij}=\operatorname{Re}\operatorname{Tr}\rho_{\theta_0}L_jD_i$. Based on those information, we introduce a quantum Gaussian shift model $\mathcal{G}=\left\{N(\tau h,\Sigma)\,;\ h\in\mathbb{R}^d\right\}$ on $\mathrm{CCR}\,(\operatorname{Im}\Sigma)$, which exhibits relations (B.13) and (B.14). Recall that the Holevo bound of a quantum statistical model is completely determined by the information Σ and τ (Theorem B.1). We thus obtain the following important consequence.

Corollary B.6. *The Holevo bound $C_{\theta_0}(\rho_\theta,G)$ for the model $\mathcal{S}$ at $\theta=\theta_0$ is identical to the Holevo bound $C_h\left(N(\tau h,\Sigma),G\right)$ for the Gaussian shift model $\mathcal{G}$.*

As to the achievability of the Holevo bound $C_h\left(N(\tau h,\Sigma),G\right)$ for the Gaussian shift model $\mathcal{G}$, we have the following.

Theorem B.7. *Given a weight $G>0$, there exist an unbiased estimator $\hat{M}$ that achieves the Holevo bound for the model $\{N(\tau h,\Sigma)\,;\,h\in\mathbb{R}^d\}$, i.e.,*

$$\operatorname{Tr}GV_h[\hat{M}]=C_h\left(N(\tau h,\Sigma),G\right).$$

Proof. Let F be the matrix that achieve the minimum of (B.4) for the model $\{N(\tau h,\Sigma)\}_h$, and let $Z={}^tF\Sigma F$. Further, let $\tilde{V}=\operatorname{Re}Z$, $\tilde{S}=\operatorname{Im}Z$. $\hat{V}=\sqrt{G^{-1}}\left|\sqrt{G}\operatorname{Im}Z\sqrt{G}\right|\sqrt{G^{-1}}$, and $\hat{Z}=\hat{V}-\sqrt{-1}\tilde{S}$. We introduce an ancillary quantum Gaussian state $(Y,\psi)\sim N(0,\hat{Z})$ on another $\mathrm{CCR}\left(-\tilde{S}\right)$, and a set of canonical observables

$$\overline{X}_i:=\tilde{X}_i\otimes I+I\otimes Y_i\quad(1\le i\le d),$$

on $\mathrm{CCR}\left(\tilde{S}\right)\otimes\mathrm{CCR}\left(-\tilde{S}\right)$, where $\tilde{X}_i=F_i^kX_k$. It is important to notice that the CCR subalgebra $\mathcal{A}[\overline{X}]$ generated by $\{\overline{X}_i\}_{1\le i\le d}$ is a commutative one because

$$\frac{\sqrt{-1}}{2}[\overline{X}_i,\overline{X}_j]=\tilde{S}_{ij}-\tilde{S}_{ij}=0$$

for $1\le i,j\le d$. Moreover

$$(\phi_h\otimes\psi)(\mathrm{e}^{\sqrt{-1}\xi^i\overline{X}_i})=\left[\phi_h\left(\mathrm{e}^{\sqrt{-1}\xi^i\tilde{X}_i}\right)\right]\left[\psi\left(\mathrm{e}^{\sqrt{-1}\xi^iY_i}\right)\right]=\mathrm{e}^{\sqrt{-1}\xi^ih_i-\frac{1}{2}\xi^i\xi^j(\tilde{V}+\hat{V})_{ij}}.$$

This means that the observables $\overline{X}_i$ $(1\le i\le d)$ follow the classical Gaussian distribution $N(h,\tilde{V}+\hat{V})$. In particular,

$$E_h[\overline{X}]=h$$

for all $h\in\mathbb{R}^d$, and

$$\operatorname{Tr}GV_h[\overline{X}]=\operatorname{Tr}G(\tilde{V}+\hat{V})=C_h\left(N(\tau h,\Sigma),G\right).$$

The claim was verified. $\square$

B.3. Estimation theory for pure state models.

LEMMA B.8. *Let ρ be a pure state and $A_1, \ldots, A_d$ observables on a finite dimensional Hilbert space $\mathcal{H}$. If $J_{ij} := \operatorname{Tr} \rho A_j A_i$ are all real for $1 \le i, j \le d$, there exist observables $K_1, \ldots, K_d$ such that*

$$[A_i + K_i, A_j + K_j] = 0,$$

for $1 \le i, j \le d$ and

$$K_i \rho = 0$$

for $1 \le i \le d$.

PROOF. Let $\rho := |\psi\rangle \langle\psi|$, and let $|l_i\rangle := A_i |\psi\rangle$ for $1 \le i \le d$. Because $\langle\psi|l_i\rangle$ and $\langle l_i|l_j\rangle$ $(= J_{ji})$ are all real, there exist a CONS $\{|e_k\rangle\}_{k=1}^{\dim \mathcal{H}}$ of $\mathcal{H}$ such that $\langle e_k|\psi\rangle$ and $\langle e_k|l_i\rangle$ are all real, and that $\langle e_k|\psi\rangle \ne 0$ for all k. Let

$$\tilde{A}_i := \sum_{k=1}^{\dim \mathcal{H}} \frac{\langle e_k|l_i\rangle}{\langle e_k|\psi\rangle} |e_k\rangle \langle e_k| ,$$

and $K_i := \tilde{A}_i - A_i$. Obviously $[A_i + K_i, A_j + K_j] = [\tilde{A}_i, \tilde{A}_j] = 0$, and

$$K_i |\psi\rangle = \left(\tilde{A}_i - A_i\right) |\psi\rangle = |l_i\rangle - |l_i\rangle = 0.$$

This means $K_i \rho = 0$. □

LEMMA B.9. *Given a $d \times d$ positive semidefinite Hermitian matrix J, there exist a finite dimensional Hilbert space $\mathcal{H}$ and a pure state ρ and observables A_i $(1 \le i \le d)$ on $\mathcal{H}$ such that $\operatorname{Tr} \rho A_i = 0$ and $\operatorname{Tr} \rho A_j A_i = J_{ij}$.*

PROOF. Let $\mathcal{H} = \mathbb{C}^{d+1}$, and let $\{|i\rangle\}_{i=0}^{d}$ be a CONS of $\mathcal{H}$. We set $|\psi\rangle := |0\rangle$ and $|\ell_i\rangle := \sum_{k=1}^{d} \left[\sqrt{J}\right]_{ik} |k\rangle$ for $i = 1, \ldots, d$. Then $\rho := |\psi\rangle \langle\psi|$ and $A_i := |\ell_i\rangle \langle\psi| + |\psi\rangle \langle\ell_i|$ satisfy $\operatorname{Tr} \rho A_i = 0$ and $\operatorname{Tr} \rho A_j A_i = J_{ij}$. □

THEOREM B.10. *Let $\left\{\rho_\theta\, ;\ \theta \in \Theta \subset \mathbb{R}^d\right\}$ be a quantum statistical model comprising pure states on a finite dimensional Hilbert space $\mathcal{H}$, and let $C_{\theta_0}(\rho_\theta, G)$ be the Holevo bound at $\theta_0 \in \Theta$ for a given weight $G > 0$. There exist a locally unbiased estimator $\hat{M}$ at $\theta_0 \in \Theta$ such that $\operatorname{Tr} G V[\hat{M}] = C_{\theta_0}(\rho_\theta, G)$.*

PROOF. Let $\mathcal{T}$ be a $\mathcal{D}$ invariant extension of the SLD tangent space span $\{L_i\}_{i=1}^{d}$ of the model $\{\rho_\theta\}$ at $\theta = \theta_0$, i.e., containing all the SLDs $\{L_i\}_{i=1}^{d}$ of $\{\rho_\theta\}$ at θ_0, let $\{D_j\}_{j=1}^{r}$ be a basis of $\mathcal{T}$. Let Σ, τ be $r \times r$, $r \times d$ complex matrices defined by $\Sigma_{ij} = \operatorname{Tr} \rho_{\theta_0} D_j D_i$, $\tau_{ij} = \operatorname{Tr} \rho_{\theta_0} L_j D_i$. According to Theorem B.1, the Holevo bound for a weight $G > 0$ can be expressed

$$\begin{aligned} C_{\theta_0}(\rho_\theta, G) \quad &= \quad \min_F \{\operatorname{Tr} G Z + \operatorname{Tr} \left|\sqrt{G} \operatorname{Im} Z \sqrt{G}\right|\ ;\ Z = {}^t F \Sigma F, \\ &\qquad F \text{ is an } r \times d \text{ real matrix satisfying } {}^t F \operatorname{Re}(\tau) = I\}. \end{aligned} \tag{B.15}$$

Let F be the matrix that attains the minimum in (B.15), and let $Z := {}^t F \Sigma F$, $\tilde{V} := \operatorname{Re} Z$, $\tilde{S} := \operatorname{Im} Z$, $\hat{V} = \sqrt{G^{-1}} \left|\sqrt{G} \operatorname{Im} Z \sqrt{G}\right| \sqrt{G^{-1}}$, and $\hat{Z} = \hat{V} - \sqrt{-1}\tilde{S}$. Lemma B.9 assures that there exist a Hilbert space $\hat{\mathcal{H}}$ and a pure state σ and observables B_i $(1 \le i \le d)$ on $\hat{\mathcal{H}}$ such that $\mathrm{Tr}\,\sigma B_i = 0$ and $\mathrm{Tr}\,\sigma B_j B_i = \hat{Z}_{ij}$. Further, let

$$\overline{X}_i := \tilde{X}_i \otimes \hat{I} + I \otimes B_i \quad (1 \le i \le d),$$

where $\tilde{X}_i := F_i^k D_k$ $(1 \le i \le d)$, and $\hat{I}$ is the identity matrix on $\hat{\mathcal{H}}$. It then follows that

$$\mathrm{Tr}\,(\rho_{\theta_0} \otimes \sigma)\,\overline{X}_j \overline{X}_i = \left(\tilde{V} + \hat{V}\right)_{ij}. \tag{B.16}$$

According to Lemma B.8, there exist observables $K_1, \ldots, K_d$ on $\mathcal{H} \otimes \hat{\mathcal{H}}$ such that $[\overline{X}_i + K_i, \overline{X}_j + K_j] = 0$ and $K_i\,(\rho_{\theta_0} \otimes \sigma) = 0$. Let $\hat{T}_i := \theta_0^i I \otimes \hat{I} + \left(\overline{X}_i + K_i\right)$. Then $\hat{T}_1, \ldots, \hat{T}_d$ are simultaneously measurable, and satisfy the local unbiasedness condition:

$$\mathrm{Tr}\,(\rho_{\theta_0} \otimes \sigma)\,\hat{T}_j = \theta_0^j$$

and

$$\begin{aligned}
\mathrm{Tr}\,(\partial_i \rho_{\theta_0} \otimes \sigma)\,\hat{T}_j &= \mathrm{Tr}\,\partial_i \rho_{\theta_0} \tilde{X}_j \\
&= F_j^k \mathrm{Tr}\,\partial_i \rho_{\theta_0} D_k \\
&= F_j^k \mathrm{Re}\,\mathrm{Tr}\,\rho_{\theta_0} L_i D_k \\
&= \{F\,(\mathrm{Re}\,\tau)\}_{ji} = \delta_{ij}.
\end{aligned}$$

Further

$$V_{\theta_0}[\hat{T}]_{ij} = \mathrm{Tr}\,(\rho_{\theta_0} \otimes \sigma)\left(\overline{X}_i + K_i\right)\left(\overline{X}_i + K_i\right) = \left(\tilde{V} + \hat{V}\right)_{ij}.$$

This completes the proof. □

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