Fibre bundle over manifolds of quantum channels and its application to quantum statistics

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Abstract

A fibre bundle structure is introduced over manifolds of quantum channels. This structure has a close connection with the problem of estimating an unknown quantum channel Γ_θ specified by a parameter θ. It is shown that the quantum Fisher information of the family of output states $(\mathrm{id}\otimes\Gamma_\theta)(\tilde\sigma)$ maximized over all input states $\tilde\sigma$, which quantifies the ultimate statistical distinguishability of the parameter θ, is expressed in terms of a geometrical quantity on the fibre bundle. Using this formula, a criterion for the maximum quantum Fisher information of the nth extended channel $(\mathrm{id}\otimes\Gamma_\theta)^{\otimes n}$ to be $O(n)$ is derived. This criterion further proves that for almost all quantum channels, the maximum quantum Fisher information increases in the order of $O(n)$.

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1 Introduction

Let $\mathcal{H}$ be a finite dimensional (say $\dim_{\mathbb{C}}\mathcal{H}=d$) complex Hilbert space that represents the physical system of interest, and let $\mathcal{B}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H})$ denote the sets of linear operators and density operators on $\mathcal{H}$. A dynamical change $\Gamma:\mathcal{S}(\mathcal{H})\to\mathcal{S}(\mathcal{H})$ of the physical system, called a *quantum channel*, is represented by a trace-preserving completely positive map [1] [2] [3] [4]. It is known that $\Gamma:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$ is completely positive if and only if $(\mathrm{id}\otimes\Gamma)(\tilde\sigma_{ME})$ is a positive operator, where $\tilde\sigma_{ME}\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$ is a maximally entangled pure state and id denotes the identity map. Furthermore, the correspondence

$$\Gamma\longmapsto(\mathrm{id}\otimes\Gamma)(\tilde\sigma_{ME}) \tag{1}$$

establishes an affine isomorphism between the set of quantum channels on $\mathcal{S}(\mathcal{H})$ and the convex subset

$$\mathcal{S}_1(\mathcal{H}\otimes\mathcal{H}):=\left\{\tilde\rho\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})\,;\ \mathrm{Tr}_2\,\tilde\rho=\frac{1}{d}I\right\}$$

of the extended state space $\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$, where Tr_2 denotes the partial trace on the second Hilbert space. In this way, one obtains a one-to-one affine parametrization of quantum channels [5] [6].

While the map (1) defines a faithful embedding of quantum channels into the extended state space $\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$, it does not always give an optimal embedding in view of statistical estimation. To put it more precisely, given a one-dimensional parametric family $\{\Gamma_\theta\,;\,\theta\in\Theta\subset\mathbb{R}\}$ of quantum channels, the symmetric logarithmic derivative (SLD) Fisher information [7] [8] [9] of the family

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$(\mathrm{id} \otimes \Gamma_\theta)(\tilde{\sigma})$ does not always take the maximum at a maximally entangled input $\tilde{\sigma} = \tilde{\sigma}_{ME}$ [6] [10]. The problem of finding an optimal estimation scheme for a given family of quantum channels is called a quantum channel identification problem [11]. Among others, evaluating the SLD Fisher information of the output family $(\mathrm{id} \otimes \Gamma_\theta)^{\otimes n}(\tilde{\sigma})$ of the nth extended channel maximized over all inputs $\tilde{\sigma} \in \mathcal{S}((\mathcal{H} \otimes \mathcal{H})^{\otimes n})$, given $n \in \mathbb{N}$, is of fundamental importance, because it quantifies the ultimate statistical distinguishability of the parameter θ. This problem has been studied in two special classes of quantum channels, i.e., the generalized Pauli channels [12] and the $SU(d)$ channels [13], and antithetical asymptotic behavior has been obtained. To be exact, the maximum SLD Fisher information is $O(n)$ in the former, whereas it is $O(n^2)$ in the latter, which is in a striking contrast to the classical statistics. What about other quantum channels? Is there a class of channels that exhibits $O(n^\alpha)$ with $\alpha \neq 1, 2$?

The purpose of this paper is to give a partial answer to this question. We show that the maximum SLD Fisher information increases at most in the order of $O(n^2)$ for any class of quantum channels. We further derive a simple criterion for the order to be $O(n)$, and prove that "almost all" families of quantum channels exhibit $O(n)$. Here, the geometry of fibre bundle over the manifold of quantum channels plays an essential role.

The paper is organized as follows. Section 2 is devoted to a brief review of differential geometry of quantum statistical manifold. In Section 3, we introduce a fibre bundle structure over manifolds of quantum channels. It is shown that the maximal SLD Fisher information is expressed by means of the operator norm of the "horizontal lift" of the tangent vector on the base manifold (Theorem 4). In Section 4, we proceed to the problem of evaluating the maximal SLD Fisher information. We prove that it increases at most in the order of $O(n^2)$, and derive a criterion for the order to be $O(n)$ (Theorem 5). We further prove that any full-rank quantum channels exhibit the order $O(n)$ (Theorem 8). Since the closure of the set of full-rank quantum channels is identical to the totality of quantum channels, this result could be paraphrased by saying that almost all quantum channels exhibit $O(n)$. In section 5, we present several illustrative examples to demonstrate these results.

2 Geometry of quantum statistical manifold

Let us start with a brief review of differential geometry of quantum statistical manifold [14]. We consider the set of density operators on $\mathcal{H}$ of rank r:

$$\mathcal{S} := \{\rho \in \mathcal{S}(\mathcal{H}) \; ; \; \mathrm{rank}\, \rho = r\}.$$

This can be regarded as a $(2dr - r^2 - 1)$-dimensional real manifold. Given a $\rho \in \mathcal{S}$ and a natural number $q \, (\geq r)$, an ordered list of vectors $W = [\hat{\phi}_1, \dots, \hat{\phi}_q]$ is called an *ordered ρ-ensemble of size q* if

$$\rho = \sum_{j=1}^{q} |\hat{\phi}_j\rangle\langle\hat{\phi}_j|.$$

Associated with each $\rho \in \mathcal{S}$ is the set

$$\mathcal{W}^q[\rho] := \{W \; ; \; W \text{ is an ordered } \rho\text{-ensemble of size } q\}.$$

Letting

$$\mathcal{W}^q := \bigcup_{\rho \in \mathcal{S}} \mathcal{W}^q[\rho],$$

we have a canonical projection

$$\pi : \mathcal{W}^q \longrightarrow \mathcal{S} : [\hat{\phi}_1, \dots, \hat{\phi}_q] \longmapsto \sum_{j=1}^{q} |\hat{\phi}_j\rangle\langle\hat{\phi}_j|.$$

There is a natural right action of the q-dimensional unitary group $U(q)$ on $\mathcal{W}^q[\rho]$:

$$W = [\hat{\phi}_j]_{1\le j\le q} \longmapsto WU = \left[\sum_{k=1}^{q} \hat{\phi}_k u_{kj}\right]_{1\le j\le q}.$$

This $U(q)$-action on $\mathcal{W}^q[\rho]$ is transitive. Moreover, when $q = r$, each $W \in \mathcal{W}^r[\rho]$ comprises linearly independent vectors, and the $U(r)$ action on $\mathcal{W}^r[\rho]$ is free. Therefore the quadruple $(\mathcal{W}^r, \pi, \mathcal{S}, U(r))$ is a principal fibre bundle [15]. For later applications, however, it is essential to treat ordered ρ-ensembles of a general size $q\,(\ge r)$.

The readers may be warned not to confuse the right action $W \mapsto WU$ of unitary matrices $U \in U(q)$ with the left action $W \mapsto LW := [L\hat{\phi}_1, \dots, L\hat{\phi}_q]$ of linear operators $L \in \mathcal{B}(\mathcal{H})$. Also it should be noted that by using an abridged notation

$$W = \left[|\hat{\phi}_1\rangle, \dots, |\hat{\phi}_q\rangle\right], \qquad W^* = \begin{bmatrix} \langle\hat{\phi}_1| \\ \vdots \\ \langle\hat{\phi}_q| \end{bmatrix},$$

the above mentioned properties can be exhibited as follows:

$$\begin{aligned} \rho = \sum_{j=1}^{q} |\hat{\phi}_j\rangle\langle\hat{\phi}_j| \quad &\Longleftrightarrow \quad \rho = WW^* \\ \pi : \mathcal{W}^q \longrightarrow \mathcal{S} \quad &\Longleftrightarrow \quad \pi : W \longmapsto WW^* \\ U(q) \text{ preserves each fibre} \quad &\Longleftrightarrow \quad WW^* = (WU)(WU)^* \end{aligned}$$

This correspondence clarifies that our formulation unifies and extends the geometry of Berry's phase [16] [17] [18] [19] [20] and that of Uhlmann's [21] [22] [23].

One of the most fundamental quantity on the quantum statistical manifold $\mathcal{S}$ is the SLD Fisher metric. A Hermitian operator-valued continuous one-form $\mathcal{L}_\rho : T_\rho\mathcal{S} \longrightarrow \mathcal{B}(\mathcal{H})$ satisfying

$$d\rho = \frac{1}{2}(\rho\mathcal{L}_\rho + \mathcal{L}_\rho\rho)$$

is called the *SLD representation*, and the bilinear form $g\,(= g_\rho) : T_\rho\mathcal{S} \times T_\rho\mathcal{S} \to \mathbb{R}$ defined by

$$g_\rho(X, Y) := \frac{1}{2}\operatorname{Tr}\rho\,(\mathcal{L}_\rho(X)\mathcal{L}_\rho(Y) + \mathcal{L}_\rho(Y)\mathcal{L}_\rho(X))$$

is called the *SLD Fisher metric*. Although the SLD representation is not unique unless $r = d$, the SLD Fisher metric is invariant under the arbitrariness of the SLD representation [24] [25].

The following theorem provides an interpretation of the SLD Fisher metric of $\mathcal{S}$ in terms of the fibre bundle $(\mathcal{W}^q, \pi, \mathcal{S}, U(q))$ [25].

Theorem 1. *Let $\{\rho_\theta\,;\,\theta \in \Theta \subset \mathbb{R}\}$ be a smooth curve on $\mathcal{S}$ and let $q\,(\ge r)$ be an arbitrary natural number. Then the SLD Fisher information $J(\rho_\theta) := g_{\rho_\theta}(\partial_\theta, \partial_\theta)$ at $\theta = \theta_0 \in \Theta$ is given by*

$$J(\rho_{\theta_0}) = 4 \min_{W_\theta} \operatorname{Tr} \dot{W}_\theta \dot{W}_\theta^* \Big|_{\theta=\theta_0}$$

where the dot denotes the differentiation with respect to θ, and the minimum is taken over all smooth families of ordered ρ_θ-ensembles W_θ of size q that are locally defined around $\theta = \theta_0$. The minimum is attained if and only if

$$\dot{W}_\theta = \frac{1}{2}\mathcal{L}_{\rho_\theta}(\partial_\theta) W_\theta \tag{2}$$

holds at $\theta = \theta_0$.

Proof. Let us fix a local smooth family of ρ_θ-ensembles $W_\theta^{(0)} = [\hat{\phi}_1(\theta), \dots, \hat{\phi}_r(\theta)]$ of size r around $\theta = \theta_0$. We extend it to a family of ρ_θ-ensembles of size q by adding $(q-r)$ zero vectors as $[\hat{\phi}_1(\theta), \dots, \hat{\phi}_r(\theta), 0, \dots, 0]$, and regard it as a reference family. Letting $V^{(0)} := [\, I_r \mid O \,]$, where I_r and O denote the $r \times r$ identity matrix and the $r \times (q-r)$ zero matrix, the reference family is written as $W_\theta^{(0)} V^{(0)}$. Given a smooth family of ρ_θ-ensembles W_θ of size q, there is a smooth family U_θ of $q \times q$ unitary matrices that satisfies

$$W_\theta = W_\theta^{(0)} V^{(0)} U_\theta.$$

Then

$$\dot{W}_\theta = \dot{W}_\theta^{(0)} V^{(0)} U_\theta + W_\theta^{(0)} V^{(0)} \dot{U}_\theta.$$

Since $\dot{W}_\theta^{(0)}$ describes the change of linearly independent vectors $[\hat{\phi}_j(\theta)]_{1 \le j \le r}$ in $\mathcal{H}$, there is an operator $L_\theta \in B(\mathcal{H})$ that satisfies

$$\dot{W}_\theta^{(0)} = \frac{1}{2} L_\theta^* W_\theta^{(0)}.$$

As a consequence

$$\dot{W}_\theta = \frac{1}{2} L_\theta^* W_\theta + W_\theta U_\theta^* \dot{U}_\theta.$$

In order for this equation to be consistent with the change of ρ_θ, we claim

$$\dot{\rho}_\theta = \dot{W}_\theta W_\theta^* + W_\theta \dot{W}_\theta^* = \frac{1}{2}(\rho_\theta L_\theta + L_\theta^* \rho_\theta).$$

This shows that the operator L_θ is a logarithmic derivative.

Let $K_\theta := L_\theta - L_\theta^S$ where $L_\theta^S := \mathcal{L}_{\rho_\theta}(\partial_\theta)$. Then $\rho_\theta K_\theta + K_\theta^* \rho_\theta = 0$, and we have

$$\mathrm{Tr}\, \dot{W}_\theta \dot{W}_\theta^* = \frac{1}{4} \mathrm{Tr}\, \rho_\theta (L_\theta^S)^2 + \mathrm{Tr}\, \left(\frac{1}{2} K_\theta^* W_\theta + W_\theta U_\theta^* \dot{U}_\theta \right) \left(\frac{1}{2} K_\theta^* W_\theta + W_\theta U_\theta^* \dot{U}_\theta \right)^* .$$

Since the second term in the right-hand side is nonnegative, we conclude that

$$\min_{\{W_\theta\}} \mathrm{Tr}\, \dot{W}_\theta \dot{W}_\theta^* = \frac{1}{4} \mathrm{Tr}\, \rho_\theta (L_\theta^S)^2 = \frac{1}{4} J(\rho_\theta).$$

The minimum is attained if and only if

$$\frac{1}{2} K_\theta^* W_\theta + W_\theta U_\theta^* \dot{U}_\theta = 0$$

or equivalently

$$\dot{W}_\theta = \frac{1}{2} L_\theta^S W_\theta.$$

□

In order to obtain an intuitive geometrical insight into Theorem 1, let us regard $\mathcal{W}^q$ as a metric space with metric

$$d(W^{(1)}, W^{(2)}) := \sqrt{\sum_{j=1}^{q} \left(\langle \hat{\phi}_j^{(1)} | - \langle \hat{\phi}_j^{(2)} | \right) \left(| \hat{\phi}_j^{(1)} \rangle - | \hat{\phi}_j^{(2)} \rangle \right)},$$

where $W^{(i)} = \left[|\hat{\phi}_1^{(i)}\rangle, \ldots, |\hat{\phi}_q^{(i)}\rangle\right]$. Given $W \in \pi^{-1}(\rho)$, we define

$$\begin{array}{rcl} f_W : \mathcal{S} & \longrightarrow & \mathcal{W}^q \\ \sigma & \longmapsto & \underset{W' \in \pi^{-1}(\sigma)}{\arg\min}\; d(W, W'). \end{array}$$

Theorem 1 implies that the minimal squared-distance between two near-by fibres is given by a quarter of the SLD Fisher information, and the differential df_W maps $X \in T_\rho \mathcal{S}$ to $\overline{X} \in T_W \mathcal{W}^q$ that satisfies

$$\overline{X} W = \frac{1}{2} \mathcal{L}_\rho(X) W.$$

An integral curve of the differential equation (2) is called a horizontal lift of the curve ρ_θ.

It would be worth mentioning that when $q = r$, the above observation leads to a connection of the principal fibre bundle $(\mathcal{W}^r, \pi, \mathcal{S}, U(r))$. Let us introduce the projection

$$\begin{array}{rcl} P : T_W \mathcal{W}^r & \longrightarrow & T_W \mathcal{W}^r \\ X & \longmapsto & \overline{X}, \end{array}$$

where $\overline{X}$ is defined by

$$\overline{X} W = \frac{1}{2} \mathcal{L}_W(X) W, \qquad \mathcal{L}_W = \pi^* \mathcal{L}_{\pi(W)}.$$

Now we decompose the tangent space $T_W \mathcal{W}^r$ into the direct sum

$$T_W \mathcal{W}^r = V_W \oplus H_W,$$

where

$$H_W = P(T_W \mathcal{W}^r), \qquad V_W = (1 - P)(T_W \mathcal{W}^r) = \mathrm{Ker}(\pi_*)_W.$$

The subspace H_W has the property that $H_{WU} = R_{U*} H_W$, where R_U denotes the right action of $U \in U(r)$. Thus there is a unique Ehresmann connection A in which H_W becomes the horizontal subspace:

$$dW = WA + \frac{1}{2} \mathcal{L}_W W.$$

The curvature form $F(A)(X, Y) := -A([\overline{X}, \overline{Y}])$ becomes

$$\begin{aligned} W F(A)(X, Y) &= -[\overline{X}, \overline{Y}] W + \frac{1}{2} \mathcal{L}_\rho(\pi_* [\overline{X}, \overline{Y}]) W \\ &= \frac{1}{4} \left\{ [L_X, L_Y] - \frac{1}{2} \mathcal{L}_\rho([\,[L_X, L_Y], \rho]) \right\} W, \end{aligned}$$

where $\rho := \pi(W)$ and $L_X := \mathcal{L}_W(X)$. It is shown that the curvature $F(A)$ is closely related to the torsion of the exponential connection $\nabla^{(e)}$ of the base manifold $\mathcal{S}$. For more information, see [14].

3 Fibre bundle over quantum channels

Let us proceed to geometry of manifolds of quantum channels. It is well known [3] that a quantum channel $\Gamma : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ is represented in the form

$$\Gamma(\rho) = \sum_j A_j \rho A_j^*$$

where $\mathcal{A} = \{A_j\}_j$ is a finite collection of operators satisfying

$$\sum_j A_j^* A_j = I.$$

This is sometimes referred to as the operator sum representation. When a quantum channel Γ is represented in this way, the ordered collection of operators $\mathcal{A} = \{A_j\}_j$ is called a *generator* of Γ. The number J of operators in a list $\mathcal{A} = \{A_1, \dots, A_J\}$ is called the size of $\mathcal{A}$. Given a quantum channel Γ, let $\mathcal{G}[\Gamma]$ be the set of generators of Γ. It is further decomposed as

$$\mathcal{G}[\Gamma] = \bigcup_{q \geq r} \mathcal{G}^q[\Gamma],$$

where $\mathcal{G}^q[\Gamma]$ denotes the set of generators of size q, and the minimal size r is called the *rank* of Γ.

Let us recall the following fundamental characterization [4].

Proposition 2. *Two collections of operators $\{A_j\}_{1 \leq j \leq J}$, $\{B_k\}_{1 \leq k \leq K}$ $(J \leq K)$ give the same quantum channel if and only if there is a matrix $Q = [Q_{jk}] \in \mathbb{C}^{J \times K}$ such that $QQ^* = I_J$ (I_J denotes the $J \times J$ identity matrix) and $B_k = \sum_j A_j Q_{jk}$.*

Corollary 3. *Let $r = \operatorname{rank} \Gamma$. Then $\mathcal{A} \in \mathcal{G}[\Gamma]$ belongs to $\mathcal{G}^r[\Gamma]$ if and only if $\mathcal{A}$ is a linearly independent set of operators. Further, $\mathcal{A} \in \mathcal{G}^r[\Gamma]$ is unique up to an $r \times r$ unitary matrix.*

Proposition 2 is easily seen by recalling that a generator $\mathcal{A}$ of Γ of size q is obtained by rearranging the components of columns of a $d^2 \times q$ matrix $\boldsymbol{A}$ that satisfies

$$\frac{1}{d} \boldsymbol{A}\boldsymbol{A}^* = (\mathrm{id} \otimes \Gamma)(\tilde{\sigma}_{ME}).$$

Here the operator $(\mathrm{id} \otimes \Gamma)(\tilde{\sigma}_{ME})$ is identified with its $d^2 \times d^2$ matrix representation. This fact also implies that $\operatorname{rank} \Gamma = \operatorname{rank} (\mathrm{id} \otimes \Gamma)(\tilde{\sigma}_{ME})$. See [5] [6] for details.

Corollary 3 shows that by regarding the set of minimal generators $\mathcal{G}^r[\Gamma]$ as the fibre over Γ, one can define a principal fibre bundle over the set of all quantum channels of rank r. The structure group is $U(r)$. This principal fibre bundle was first introduced in [14]. In view of applications to quantum channel identification problems, however, it is useful to treat fibre spaces $\mathcal{G}^q[\Gamma]$ of general size $q\, (\geq r)$, as demonstrated below.

Let us recall the quantum channel identification problem which allows extensions of the channel [11]. Suppose that we have an unknown quantum channel that belongs to a parametric family $\{\Gamma_\theta \,;\, \theta \in \Theta\}$ of quantum channels, and that we wish to estimate the true value of the parameter θ as accurate as possible. Our task is to find an optimal input $\tilde{\sigma} \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$ to the extended channel $\mathrm{id} \otimes \Gamma_\theta$ and an optimal measurement for estimating the parameter θ of the family $(\mathrm{id} \otimes \Gamma_\theta)(\tilde{\sigma})$ of output states. In what follows, we restrict ourselves, for the sake of simplicity, to the case when θ is one-dimensional. In this case, the problem amounts to finding an input $\tilde{\sigma}$ that maximizes the SLD Fisher information $J((\mathrm{id} \otimes \Gamma_\theta)(\tilde{\sigma}))$ of the output family. When the optimal input depends on the true value of θ, we make use of an adaptive estimation scheme [26].

Now there is a delicate problem concerning the existence of the SLD of $(\mathrm{id} \otimes \Gamma_\theta)(\tilde{\sigma})$. First of all, the family $\{\Gamma_\theta\}_\theta$ must be differentiable in some sense. Here we assume the following.

(RC1) Γ_θ has a generator $\mathcal{A}(\theta) = \{A_j(\theta)\}_{1 \leq j \leq r_0} \in \mathcal{G}^{r_0}[\Gamma_\theta]$ with $r_0 := \max\{\operatorname{rank} \Gamma_\theta \,;\, \theta \in \Theta\}$ such that each component $A_j(\theta)$, $(1 \leq j \leq r_0)$, is continuously differentiable in θ.

If this condition is satisfied, we simply call the family $\{\Gamma_\theta\}_\theta$ *smooth.* Note that (RC1) is stronger than the requirement that $\Gamma_\theta(\sigma)$ is continuously differentiable for all $\sigma \in \mathcal{S}(\mathcal{H})$. In fact, (RC1) is much closer in spirit to the condition that $\sqrt{\Gamma_\theta(\sigma)}$ is continuously differentiable.

We next observe the following fact: the rank of the output state $(\mathrm{id}\otimes\Gamma_\theta)(\tilde\sigma)$ may vary as θ changes, even if $\operatorname{rank}\Gamma_\theta$ is constant, and the family $(\mathrm{id}\otimes\Gamma_\theta)(\tilde\sigma)$ may not have an SLD at a point where the rank changes. Let us call such a point *singular*, and denote the set of singular points of $(\mathrm{id}\otimes\Gamma_\theta)(\tilde\sigma)$ by $\Theta_{\mathrm{sing}}(\tilde\sigma)\,(\subset\Theta)$. To surmount this difficulty, we first assume the following.

(RC2) The set $\Theta_{\mathrm{sing}}(\tilde\sigma)$ is a finite set for all $\tilde\sigma\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$.

This condition ensures that Theorem 1 is applicable to the evaluation of the SLD Fisher information $J((\mathrm{id}\otimes\Gamma_\theta)(\tilde\sigma))$ at $\theta\notin\Theta_{\mathrm{sing}}(\tilde\sigma)$. Moreover, since the function $\theta\mapsto J((\mathrm{id}\otimes\Gamma_\theta)(\tilde\sigma))$ is continuous at $\theta\notin\Theta_{\mathrm{sing}}(\tilde\sigma)$, one would expect that the SLD Fisher information at a singular point $\theta_0\in\Theta_{\mathrm{sing}}(\tilde\sigma)$ might be *defined* by

$$J((\mathrm{id}\otimes\Gamma_{\theta_0})(\tilde\sigma)) := \lim_{\theta\to\theta_0} J((\mathrm{id}\otimes\Gamma_\theta)(\tilde\sigma)). \tag{3}$$

In order to put this idea into practice, we further assume the following.

(RC3) For each $\sigma\in\mathcal{S}(\mathcal{H})$, the function

$$\theta_0\longmapsto \min_{\mathcal{A}(\theta)} \operatorname{Tr}\left[\sigma\left(\sum_j \dot A_j(\theta)^*\dot A_j(\theta)\right)\right]\Bigg|_{\theta=\theta_0} \tag{4}$$

is continuous at all $\theta_0\in\Theta$, where the minimum is taken over all smooth families of generators $\mathcal{A}(\theta)\in\mathcal{G}[\Gamma_\theta]$ that are locally defined around $\theta=\theta_0$.

Note that under (RC1), the function (4) is always upper semicontinuous. Moreover, given a pure state $\tilde\sigma\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$, the right-hand side of (4) with $\sigma:=\mathrm{Tr}_1\tilde\sigma$ gives the SLD Fisher information $J((\mathrm{id}\otimes\Gamma_{\theta_0})(\tilde\sigma))$ at each $\theta_0\notin\Theta_{\mathrm{sing}}(\tilde\sigma)$; see (5) below. The condition (RC3) is, therefore, essential only at $\theta_0\in\Theta_{\mathrm{sing}}(\tilde\sigma)$. We shall call a family $\{\Gamma_\theta\}_\theta$ *piecewise regular* if it satisfies (RC2) and (RC3).

Let us now proceed to the problem of maximizing the SLD Fisher information $J((\mathrm{id}\otimes\Gamma_\theta)(\tilde\sigma))$ over all input state $\tilde\sigma\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$. A direct evaluation of the SLD Fisher information $J((\mathrm{id}\otimes\Gamma_\theta)(\tilde\sigma))$ as a function of input state $\tilde\sigma$ is often infeasible because of computational difficulty. The following theorem gives an alternative way of evaluating the maximal SLD Fisher information.

Theorem 4. *Let $\{\Gamma_\theta\,;\,\theta\in\Theta\subset\mathbb{R}\}$ be a smooth, piecewise regular one-parameter family of quantum channels, and let $q\,(\geq\max\{\operatorname{rank}\Gamma_\theta\}_\theta)$ be an arbitrary natural number. Then*

$$\max_{\tilde\sigma\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})} J((\mathrm{id}\otimes\Gamma_{\theta_0})(\tilde\sigma)) = 4\min_{\mathcal{A}(\theta)}\left\|\sum_{j=1}^q \dot A_j(\theta)^*\dot A_j(\theta)\right\|\Bigg|_{\theta=\theta_0}$$

for all $\theta_0\in\Theta$, where $\|\cdot\|$ denotes the operator norm of $\mathcal{H}$, and the minimum is taken over all smooth families of generators $\mathcal{A}(\theta)=\{A_j(\theta)\}_{1\leq j\leq q}\in\mathcal{G}^q[\Gamma_\theta]$ that are locally defined around $\theta=\theta_0$.

Proof. Since the SLD Fisher information $J((\mathrm{id}\otimes\Gamma_\theta)(\tilde\sigma))$ takes the maximum at the extreme boundary $\partial_e\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$ [11], we can restrict ourselves, without loss of generality, to pure state inputs $\tilde\sigma=|\psi\rangle\langle\psi|$, where ψ is a unit vector in $\mathcal{H}\otimes\mathcal{H}$. Letting $\mathcal{A}(\theta)=\{A_j(\theta)\}_{1\leq j\leq q}$ be a smooth family of generators of Γ_θ, we have

$$\tilde\rho_\theta := (\mathrm{id}\otimes\Gamma_\theta)(\tilde\sigma) = \sum_{j=1}^q (I\otimes A_j(\theta))\,|\psi\rangle\langle\psi|\,(I\otimes A_j(\theta))^*.$$

This shows that $[(I \otimes A_j(\theta))\psi]_{1\le j\le q}$ is an ordered $\tilde\rho_\theta$-ensemble of size q. Moreover, the transitive right action of unitary group $U(q)$ on the fibre $\mathcal{G}^q[\Gamma_\theta]$:

$$\{A_j(\theta)\}_{1\le j\le q} \longmapsto \left\{\sum_{k=1}^q A_k(\theta)\, u_{kj}\right\}_{1\le j\le q}$$

naturally induces a transitive right action on the fibre $\mathcal{W}^q[\tilde\rho_\theta]$:

$$[(I\otimes A_j(\theta))\psi]_{1\le j\le q} \longmapsto \left[\sum_{k=1}^q ((I\otimes A_k(\theta))\psi)\, u_{kj}\right]_{1\le j\le q}.$$

According to Theorem 1, therefore, the SLD Fisher information $J(\tilde\rho_{\theta_0})$ at $\theta_0 \notin \Theta_{\rm sing}(\tilde\sigma)$ is given by

$$\begin{aligned}
\frac14 J(\tilde\rho_{\theta_0}) &= \min_{\mathcal{A}(\theta)} \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}} \sum_j \left| \frac{d}{d\theta}(I\otimes A_j(\theta))\,\psi\right\rangle \left\langle \frac{d}{d\theta}(I\otimes A_j(\theta))\,\psi\right| \Bigg|_{\theta=\theta_0} \\
&= \min_{\mathcal{A}(\theta)} \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}}\, |\psi\rangle\langle\psi| \left(I\otimes \sum_j \dot A_j(\theta)^* \dot A_j(\theta)\right)\Bigg|_{\theta=\theta_0} \\
&= \min_{\mathcal{A}(\theta)} \mathrm{Tr}_{\mathcal{H}}\, \sigma \left(\sum_j \dot A_j(\theta)^*\dot A_j(\theta)\right)\Bigg|_{\theta=\theta_0}, \qquad (5)
\end{aligned}$$

where $\sigma := \mathrm{Tr}_1 |\psi\rangle\langle\psi| \in \mathcal{S}(\mathcal{H})$, and the minimum is taken over all smooth families of generators $\mathcal{A}(\theta) = \{A_j(\theta)\}_j \in \mathcal{G}_q(\Gamma_\theta)$ that are locally defined around $\theta = \theta_0$. Now, because of (RC3), we see that (3) is actually well-defined for any pure state input $\tilde\sigma = |\psi\rangle\langle\psi|$. This observation further ensures that, under the definition (3), the formula (5) holds for all $\theta_0 \in \Theta$.

In order to evaluate the minimum in (5), let us introduce a smooth reference generator $\mathcal{B}(\theta) = \{B_j(\theta)\}_{1\le j\le q}$. Then there is a smooth family of $q\times q$ unitary matrices $U(\theta) = [u_{kj}(\theta)]$ such that

$$A_j(\theta) = \sum_{k=1}^q B_k(\theta) u_{kj}(\theta).$$

Let $x_{k\ell} := (1/\sqrt{-1})\sum_j \overline{\dot u_{kj}} u_{\ell j} = (1/\sqrt{-1})(U\dot U^*)_{\ell k}$. Then $\overline{x_{k\ell}} = x_{\ell k}$, so that the matrix $X = [x_{k\ell}]$ is Hermitian and $\sqrt{-1}X$ belongs to Lie algebra $u(q)$. Now

$$\begin{aligned}
\sum_j \dot A_j^* \dot A_j &= \sum_j \left(\sum_k \dot B_k u_{kj} + B_k \dot u_{kj}\right)^* \left(\sum_\ell \dot B_\ell u_{\ell j} + B_\ell \dot u_{\ell j}\right) \\
&= \left(\sum_k \dot B_k^* \dot B_k\right) + \left(\sum_{k\ell} \frac{\dot B_k^* B_\ell - B_k^*\dot B_\ell}{\sqrt{-1}} x_{k\ell}\right) + \left(\sum_{k\ell} B_k^* B_\ell \sum_m x_{km}\overline{x_{\ell m}}\right). \qquad (6)
\end{aligned}$$

As a consequence, (5) is rewritten as

$$\frac14 J(\tilde\rho_{\theta_0}) = \min_{\sqrt{-1}X\in u(q)} f_{\theta_0}(\sigma, X) \qquad (7)$$

where

$$f_\theta(\sigma,X) := \left(\sum_k \mathrm{Tr}\,\sigma \dot B_k^* \dot B_k\right) + \sum_{k\ell}\left(\mathrm{Tr}\,\sigma \frac{\dot B_k^* B_\ell - B_k^* \dot B_\ell}{\sqrt{-1}}\right) x_{k\ell} + \sum_m \left(\sum_{k\ell} (\mathrm{Tr}\,\sigma B_k^* B_\ell)\, x_{km}\overline{x_{\ell m}}\right). \qquad (8)$$

Now we are ready to prove Theorem 4. The function $f_\theta(\sigma, X)$ is linear (affine) in σ, and is convex in X because the coefficient matrix $[\mathrm{Tr}\,\sigma B_k^* B_\ell]_{1\le k,\ell\le q}$ is positive semidefinite. Consequently, the maximal SLD Fisher information can be evaluated as follows.

$$\begin{aligned}
\frac{1}{4}\max_{\tilde\sigma\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})} J((\mathrm{id}\otimes\Gamma_{\theta_0})(\tilde\sigma)) &= \frac{1}{4}\max_{\tilde\sigma\in\partial_e\mathcal{S}(\mathcal{H}\otimes\mathcal{H})} J((\mathrm{id}\otimes\Gamma_{\theta_0})(\tilde\sigma)) \\
&= \max_{\sigma\in\mathcal{S}(\mathcal{H})}\ \min_{\sqrt{-1}X\in u(q)} f_{\theta_0}(\sigma, X) \\
&= \min_{\sqrt{-1}X\in u(q)}\ \max_{\sigma\in\mathcal{S}(\mathcal{H})} f_{\theta_0}(\sigma, X) \\
&= \min_{\mathcal{A}(\theta)} \left\| \sum_j \dot A_j(\theta)^*\dot A_j(\theta)\right\| \Bigg|_{\theta=\theta_0}. \qquad (9)
\end{aligned}$$

In the second equality, the relation (7) and the surjectivity of the partial trace $\mathrm{Tr}_1 : \partial_e\mathcal{S}(\mathcal{H}\otimes\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ are used. In the third equality, on the other hand, a version of minimax theorem [27, Corollary 37.3.2] is used, which asserts the exchangeability of min and max when either of the domains of the arguments is compact. □

In what follows, we abbreviate the formula (9) as

$$\max_{\tilde\sigma\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})} J((\mathrm{id}\otimes\Gamma_\theta)(\tilde\sigma)) = 4\min_{\mathcal{A}(\theta)}\left\|\sum_{j=1}^q \dot A_j(\theta)^*\dot A_j(\theta)\right\|. \qquad (10)$$

Several remarks are in order. First, Theorem 4 gives a complete answer to the question raised by Sarovar and Milburn [28] of how to express the maximal SLD Fisher information in terms of generators. Originally they intended to find an expression for the maximal SLD Fisher information $\max_{\sigma\in\mathcal{S}(\mathcal{H})} J(\Gamma_\theta(\sigma))$ of unextended channels. Actually the formula (10) only gives its upper bound:

$$\max_{\sigma\in\mathcal{S}(\mathcal{H})} J(\Gamma_\theta(\sigma)) \le 4\min_{\mathcal{A}(\theta)}\left\|\sum_j \dot A_j(\theta)^*\dot A_j(\theta)\right\|. \qquad (11)$$

In fact, an argument similar to the proof of (7) leads to

$$\frac{1}{4}J(\Gamma_\theta(|\psi\rangle\langle\psi|)) = \min_{\sqrt{-1}X\in u(q)} f_\theta(|\psi\rangle\langle\psi|, X) \qquad (12)$$

for any $|\psi\rangle\langle\psi| \in \partial_e\mathcal{S}(\mathcal{H})$. Therefore, the obvious inequality

$$\max_{\sigma\in\partial_e\mathcal{S}(\mathcal{H})}\ \min_{\sqrt{-1}X\in u(q)} f_\theta(\sigma, X) \le \max_{\sigma\in\mathcal{S}(\mathcal{H})}\ \min_{\sqrt{-1}X\in u(q)} f_\theta(\sigma, X) \qquad (13)$$

leads to (11). It is important to notice that the inequality (13), and hence (11), is not always saturated. This is because the function $\sigma \mapsto \min_X f_\theta(\sigma, X)$ does not in general take the maximum at the extreme boundary $\partial_e\mathcal{S}(\mathcal{H})$, although the function $\sigma \mapsto f_\theta(\sigma, X)$ always does for any X. (A simple example which may help intuition: the function $f : [-1, 1]\times\mathbb{R} \to \mathbb{R} : (a, x) \mapsto x^2 + ax$, which is affine in a and convex in x, has a single saddle point at $(a, x) = (0, 0)$.) These observations clarify the importance of extending the channel into the form $\mathrm{id}\otimes\Gamma_\theta$. In Section 5, we demonstrate these subtleties in more detail.

Second, the similarity between Theorems 1 and 4 clarifies a parallelism between the geometry of optimal estimation scheme for quantum states and that for quantum channels: we need only change the Hilbert-Schmidt norm into the operator norm. In particular, by comparison with the exposition presented after the proof of Theorem 1, Theorem 4 could be interpreted as expressing the maximal SLD Fisher information by means of the operator norm of the "horizontal lift" of the tangent vector ∂_θ on the base manifold. Since the maximal SLD Fisher information quantifies the statistical distinguishability of quantum channels by means of an optimal estimation scheme, we might as well call the quantity (10) the *SLD Fisher information of the quantum channel* Γ_θ, and shall denote it as $\tilde{J}(\mathrm{id}\otimes\Gamma_\theta)$.

4 Application to quantum channel estimation problems

Given a family of quantum channels Γ_θ, how fast does the SLD Fisher information $\tilde{J}((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n})$ of the extended channel $(\mathrm{id}\otimes\Gamma_\theta)^{\otimes n}$ increase as n increases? It has been shown that $\tilde{J}((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n})=O(n)$ for generalized Pauli channels [12], and that $\tilde{J}((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n})=O(n^2)$ for unitary channels [13]. In this section, we prove that "almost all" families of quantum channels Γ_θ exhibit $\tilde{J}((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n})=O(n)$.

We start with the following sufficient condition for $\tilde{J}((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n})=O(n)$.

Theorem 5. *For any smooth, piecewise regular one-parameter family of quantum channels Γ_θ, it holds that $\tilde{J}((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n})\le O(n^2)$. Moreover, if Γ_θ has a generator $\mathcal{A}(\theta)$ that satisfies*

$$\sum_j \dot{A}_j^* A_j = 0,$$

then $\tilde{J}((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n})=O(n)$.

Corollary 6. *A smooth, piecewise regular one-parameter family of quantum channels Γ_θ exhibits the additivity of the SLD Fisher information $\tilde{J}((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n})=n\,\tilde{J}(\mathrm{id}\otimes\Gamma_\theta)$ if Γ_θ has a generator $\mathcal{A}(\theta)$ that satisfies both*

$$\sum_j \dot{A}_j^* A_j = 0 \quad \textit{and} \quad \tilde{J}(\mathrm{id}\otimes\Gamma_\theta)=4\left\|\sum_j \dot{A}_j^*\dot{A}_j\right\|.$$

We next show the following.

Lemma 7. *Let Γ be a full-rank quantum channel. Then*

$$\mathrm{rank}\,(\mathrm{id}\otimes\Gamma)(|\psi\rangle\langle\psi|)=d\cdot(\textit{Schmidt rank of }\psi)$$

for any $\psi\in\mathcal{H}\otimes\mathcal{H}$, where the Schmidt rank of ψ is the number of nonzero components in the Schmidt decomposition of ψ.

Lemma 7 implies that any one-parameter family of full-rank quantum channels is (piecewise) regular, in that $\Theta_{\mathrm{sing}}(\tilde{\sigma})=\emptyset$ for all $\tilde{\sigma}\in\partial_e\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$. Taking account of this fact, we finally reach the following important consequence.

Theorem 8. *Any smooth one-parameter family of full-rank quantum channels Γ_θ exhibits $\tilde{J}((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n})=O(n)$.*

The affine isomorphism $\Gamma\mapsto(\mathrm{id}\otimes\Gamma)(\tilde{\sigma}_{ME})$ mentioned in Section 1 establishes a one-to-one correspondence between the set of full-rank quantum channels on $\mathcal{S}(\mathcal{H})$ and the set of full-rank density operators in $\mathcal{S}_1(\mathcal{H}\otimes\mathcal{H})$. Therefore, the closure of the set of full-rank quantum channels is identical to the totality of quantum channels. This observation prompts us to interpret Theorem 8 as asserting that the SLD Fisher information is of $O(n)$ for "almost all" quantum channels.

4.1 Proof of Theorem 5 and Corollary 6

By a suitable rearrangement of the constituent Hilbert spaces $\mathcal{H}$, we identify $(\mathrm{id} \otimes \Gamma_\theta)^{\otimes n}$ with $\mathrm{id}^{\otimes n} \otimes \Gamma_\theta^{\otimes n}$. Given a smooth family of generators $\mathcal{A}(\theta) = \{A_j(\theta)\}_{1 \le j \le q}$ of Γ_θ, let $A_j^{(1)} := A_j(\theta)$ for $j \in \{1, \dots, q\}$, and let inductively

$$A_\mu^{(n+1)} := A_{\mu_1}^{(n)} \otimes A_{\mu_2}^{(1)} \quad \text{for} \quad \mu := (\mu_1, \mu_2) \in \{1, \dots, q\}^n \times \{1, \dots, q\}.$$

Then $\left\{ A_\mu^{(n)} \,;\, \mu \in \{1, \dots, q\}^n \right\}$ is a generator of $\Gamma_\theta^{\otimes n}$. Let

$$\alpha_n := \sum_{\mu \in \{1, \dots, q\}^n} \dot{A}_\mu^{(n)*} \dot{A}_\mu^{(n)}$$

and

$$\beta_n := \sum_{\mu \in \{1, \dots, q\}^n} \dot{A}_\mu^{(n)*} A_\mu^{(n)}.$$

Since $\beta_n + \beta_n^* = 0$, we see that

$$\begin{aligned} \alpha_{n+1} &= \sum_{\mu_1, \mu_2} \left[\partial_\theta \left(A_{\mu_1}^{(n)} \otimes A_{\mu_2}^{(1)} \right) \right]^* \left[\partial_\theta \left(A_{\mu_1}^{(n)} \otimes A_{\mu_2}^{(1)} \right) \right] \\ &= \alpha_n \otimes I + I^{\otimes n} \otimes \alpha_1 - 2\beta_n \otimes \beta_1 \end{aligned} \tag{14}$$

and that

$$\begin{aligned} \beta_{n+1} &= \sum_{\mu_1, \mu_2} \left[\partial_\theta \left(A_{\mu_1}^{(n)} \otimes A_{\mu_2}^{(1)} \right) \right]^* \left[A_{\mu_1}^{(n)} \otimes A_{\mu_2}^{(1)} \right] \\ &= \beta_n \otimes I + I^{\otimes n} \otimes \beta_1. \end{aligned} \tag{15}$$

Substituting the solution

$$\beta_n = \sum_{i=1}^{n} I^{\otimes (i-1)} \otimes \beta_1 \otimes I^{\otimes (n-i)}$$

of (15) into (14), we have

$$\alpha_n = \sum_{\substack{i, j \ge 0 \\ i+j=n-1}} I^{\otimes i} \otimes \alpha_1 \otimes I^{\otimes j} - 2 \sum_{\substack{i, j, k \ge 0 \\ i+j+k=n-2}} I^{\otimes i} \otimes \beta_1 \otimes I^{\otimes j} \otimes \beta_1 \otimes I^{\otimes k}. \tag{16}$$

As a consequence, the operator norm of α_n is evaluated as

$$\|\alpha_n\| \le n \|\alpha_1\| + n(n-1) \|\beta_1\|^2. \tag{17}$$

Combining inequality (17) with Theorem 4, we have

$$n\tilde{J}(\mathrm{id} \otimes \Gamma_\theta) \le \tilde{J}((\mathrm{id} \otimes \Gamma_\theta)^{\otimes n}) \le 4 \min_{\mathcal{A}(\theta)} \|\alpha_n\| \le 4n \|\alpha_1\| + 4n(n-1) \|\beta_1\|^2,$$

where the last side is evaluated for an arbitrary generator $\mathcal{A}(\theta)$ of Γ_θ. Theorem 5 and Corollary 6 now follows immediately.

4.2 Proof of Lemma 7

Let $\psi \in \mathcal{H} \otimes \mathcal{H}$ be represented in a Schmidt decomposition

$$\psi = \sum_{i=1}^{d} \sqrt{\alpha_i}\, e_i \otimes f_i,$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a probability vector, and $\{e_i\}_{1\le i\le d}$ and $\{f_i\}_{1\le i\le d}$ are orthonormal bases of $\mathcal{H}$. When ψ is represented in this way, we denote it as ψ^α. Thus

$$\begin{aligned}(\mathrm{id} \otimes \Gamma)(|\psi^\alpha\rangle\langle\psi^\alpha|) &= \sum_{ij} \sqrt{\alpha_i \alpha_j}\, |e_i\rangle\langle e_j| \otimes \Gamma(|f_i\rangle\langle f_j|) \\ &= (D^\alpha \otimes I) \cdot ((\mathrm{id} \otimes \Gamma)(|\psi^u\rangle\langle\psi^u|)) \cdot (D^\alpha \otimes I),\end{aligned}$$

where

$$D^\alpha := \sqrt{d} \sum_{i=1}^{d} \sqrt{\alpha_i}\, |e_i\rangle\langle e_i|$$

and $u = (1/d, \ldots, 1/d)$ denotes the uniform distribution. Since Γ is full-rank, the operator $(\mathrm{id} \otimes \Gamma)(|\psi^u\rangle\langle\psi^u|)$ is strictly positive. As a consequence,

$$\mathrm{rank}\,(\mathrm{id} \otimes \Gamma)(|\psi^\alpha\rangle\langle\psi^\alpha|) = \mathrm{rank}\,(D^\alpha \otimes I) = d \cdot \mathrm{rank}\, D^\alpha.$$

4.3 Proof of Theorem 8

We show that any family Γ_θ of full-rank channels has a generator $\mathcal{A}(\theta)$ that satisfies $\beta_1 = 0$. Let $\mathcal{B}(\theta) = \{B_j(\theta)\}_{1\le j\le d^2}$ be an arbitrary smooth reference generator of Γ_θ of size d^2, and let $\mathcal{A}(\theta) = \{A_j(\theta)\}_{1\le j\le d^2}$ be

$$A_j(\theta) = \sum_k B_k(\theta) u_{kj}(\theta),$$

where $U(\theta) = [u_{kj}(\theta)]$ is unitary. Then

$$\begin{aligned}\beta_1 &= \sum_j \dot{A}_j^* A_j \\ &= \sum_j \left(\sum_k \dot{B}_k u_{kj} + B_k \dot{u}_{kj} \right)^* \left(\sum_\ell B_\ell u_{\ell j} \right) \\ &= \sum_k \dot{B}_k^* B_k + \sqrt{-1} \sum_{k\ell} B_k^* B_\ell x_{k\ell},\end{aligned} \tag{18}$$

where $x_{k\ell} = (1/\sqrt{-1}) \sum_j \overline{\dot{u}_{kj}} u_{\ell j}\ (= \overline{x_{\ell k}})$. It suffices to prove that for each θ, there is a Hermitian matrix $X = (x_{k\ell})$ that satisfies

$$\sum_{k\ell} B_k^* B_\ell x_{k\ell} = \sqrt{-1} \sum_m \dot{B}_m^* B_m. \tag{19}$$

Since Γ_θ is full-rank, the generator $\{B_j\}_{1\le j\le d^2}$ forms a basis of the space $\mathcal{B}(\mathcal{H})$ of linear operators on $\mathcal{H}$, and there exist complex numbers $\{\lambda_k\}_{1\le k\le d^2}$ and $\{\mu_k\}_{1\le k\le d^2}$ that satisfy

$$\sum_k \lambda_k B_k = I, \qquad \sum_k \mu_k B_k = \frac{\sqrt{-1}}{2} \sum_m \dot{B}_m^* B_m.$$

Let

$$x_{k\ell} := \overline{\lambda_k}\mu_\ell + \lambda_\ell\overline{\mu_k}.$$

Then $\overline{x_{k\ell}} = x_{\ell k}$ and

$$\begin{aligned}
\sum_{k\ell} B_k^* B_\ell x_{k\ell} &= \left(\sum_k \lambda_k B_k\right)^* \left(\sum_\ell \mu_\ell B_\ell\right) + \left(\sum_k \mu_k B_k\right)^* \left(\sum_\ell \lambda_\ell B_\ell\right) \\
&= \frac{\sqrt{-1}}{2}\sum_m (\dot{B}_m^* B_m - B_m^* \dot{B}_m) \\
&= \sqrt{-1}\sum_m \dot{B}_m^* B_m.
\end{aligned}$$

5 Examples

In this section, we present several examples to demonstrate the results obtained in Sections 3 and 4.

5.1 Depolarizing channel

Let σ_1, σ_2, σ_3 be the standard Pauli matrices. A depolarizing channel $\Gamma_\theta : \mathcal{S}(\mathbb{C}^2) \to \mathcal{S}(\mathbb{C}^2)$ is a full-rank channel defined by the generator $\mathcal{B}(\theta) = \{B_j(\theta)\}_{0\le j\le 3}$ with the parameter space $\Theta = (-1/3, 1)$, where

$$B_0(\theta) = \frac{\sqrt{1+3\theta}}{2}\, I, \qquad B_j(\theta) = \frac{\sqrt{1-\theta}}{2}\,\sigma_j \quad (1 \le j \le 3).$$

It is known [11] that

$$\max_{\sigma\in\mathcal{S}(\mathbb{C}^2)} J(\Gamma_\theta(\sigma)) = \frac{1}{(1-\theta)(1+\theta)} \tag{20}$$

and

$$\max_{\tilde{\sigma}\in\mathcal{S}(\mathbb{C}^2\otimes\mathbb{C}^2)} J((\mathrm{id}\otimes\Gamma_\theta)(\tilde{\sigma})) = \frac{3}{(1-\theta)(1+3\theta)}. \tag{21}$$

Let us investigate these results in the light of the inequality (13).

We make use of the Stokes parametrization for $\sigma \in \mathcal{S}(\mathbb{C}^2)$ as follows:

$$\sigma = \frac{1}{2}(I + a\,\sigma_1 + b\,\sigma_2 + c\,\sigma_3), \qquad (a^2+b^2+c^2 \le 1). \tag{22}$$

By a direct computation, the function (8) is explicitly minimized with respect to X as

$$\min_{\sqrt{-1}X\in u(4)} f_\theta(\sigma, X) = \frac{3(1+\theta) - 2(a^2+b^2+c^2)}{4(1-\theta)(1+\theta)(1+3\theta)}. \tag{23}$$

When σ is restricted to the extreme boundary $\partial_e\mathcal{S}(\mathbb{C}^2)$ where $a^2+b^2+c^2 = 1$, we have

$$\min_{\sqrt{-1}X\in u(4)} f_\theta(\sigma, X)\bigg|_{\sigma\in\partial_e\mathcal{S}(\mathbb{C}^2)} = \frac{1}{4(1-\theta)(1+\theta)}.$$

This relation, combined with (12), reproduces (20) as follows:

$$\max_{\sigma\in\mathcal{S}(\mathbb{C}^2)} J(\Gamma_\theta(\sigma)) = \max_{\sigma\in\partial_e\mathcal{S}(\mathbb{C}^2)} J(\Gamma_\theta(\sigma)) = 4\max_{\sigma\in\partial_e\mathcal{S}(\mathbb{C}^2)} \min_{\sqrt{-1}X\in u(4)} f_\theta(\sigma, X) = \frac{1}{(1-\theta)(1+\theta)}.$$

On the other hand, when no restriction is imposed on $\sigma \in \mathcal{S}(\mathbb{C}^2)$, we have

$$\max_{\sigma\in\mathcal{S}(\mathbb{C}^2)} \min_{\sqrt{-1}X\in u(4)} f_\theta(\sigma, X) = \frac{3}{4(1-\theta)(1+3\theta)}.$$

This leads to (21). Moreover, the maximum is attained if and only if $\sigma = I/2$. Since

$$\mathrm{Tr}_1|\psi^u\rangle\langle\psi^u| = \frac{I}{2}, \qquad \left(\psi^u = \frac{1}{\sqrt{2}}\sum_{i=1}^{2} e_i \otimes f_i\right),$$

we see that the maximum in (21) is attained at a maximally entangled pure state.

Note in passing that

$$\beta_1 = \sum_{j=0}^{3} \dot{B}_j^* B_j = 0$$

and

$$\alpha_1 = \sum_{j=0}^{3} \dot{B}_j^* \dot{B}_j = \frac{3}{4(1-\theta)(1+3\theta)}\, I.$$

We therefore conclude from Corollary 6 that the SLD Fisher information is additive:

$$\tilde{J}((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n}) = n\,\tilde{J}(\mathrm{id}\otimes\Gamma_\theta)).$$

This is in accordance with the result obtained in [12].

5.2 Rank-two quasi-unitary channel

Let $\Gamma_\theta : \mathcal{S}(\mathbb{C}^2) \to \mathcal{S}(\mathbb{C}^2)$ be a one-parameter family of rank-two channels defined by the generator $\mathcal{B}(\theta) = \{B_1(\theta),\ B_2(\theta)\}$ with the parameter space $\Theta = [-\pi/2, \pi/2)$, where

$$\begin{aligned} B_1(\theta) &:= \frac{1}{\sqrt{2}} \exp(\sqrt{-1}\,\theta\,\sigma_1) = \frac{1}{\sqrt{2}}\left(I\cos\theta + \sqrt{-1}\,\sigma_1\sin\theta\right) \\ B_2(\theta) &:= \frac{1}{\sqrt{2}}\,\sigma_2. \end{aligned}$$

We show that

$$\max_{\sigma\in\mathcal{S}(\mathbb{C}^2)} J(\Gamma_\theta(\sigma)) = 2, \qquad (\forall\theta\in\Theta) \tag{24}$$

and

$$\max_{\tilde{\sigma}\in\mathcal{S}(\mathbb{C}^2\otimes\mathbb{C}^2)} J((\mathrm{id}\otimes\Gamma_\theta)(\tilde{\sigma})) = 2, \qquad (\forall\theta\in\Theta). \tag{25}$$

These results imply that the use of entanglement does not help enhance the distinguishability in this channel.

We first prove (25) by a direct application of Theorem 4. Let us set

$$X = \begin{bmatrix} x & y+\sqrt{-1}\,z \\ y-\sqrt{-1}\,z & w \end{bmatrix}. \tag{26}$$

Then (6) becomes

$$\begin{aligned} \sum_j \dot{A}_j^* \dot{A}_j &= \left(\frac{1+x^2+2y^2+2z^2+w^2}{2}\right) I - x\sigma_1 \\ &\quad + ((x+w)y\cos\theta - z\sin\theta)\,\sigma_2 + ((x+w)y\sin\theta + z\cos\theta)\,\sigma_3, \end{aligned}$$

and its maximal eigenvalue is

$$\left\| \sum_j \dot{A}_j^* \dot{A}_j \right\| = \frac{1 + x^2 + 2y^2 + 2z^2 + w^2 + 2\sqrt{x^2 + z^2 + (x+w)^2 y^2}}{2}.$$

Obviously this takes the minimum at $x = y = z = w = 0$, and (25) follows immediately from Theorem 4.

We next prove (24) and (25) in a unified manner based on (13). Let $\psi \in \mathbb{C}^2 \otimes \mathbb{C}^2$ be a unit vector such that $\sigma = \mathrm{Tr}_1 |\psi\rangle\langle\psi|$ is represented as (22), and let $\rho_\theta := \Gamma_\theta(\sigma)$. Then by a direct computation, we obtain

$$\min_{\sqrt{-1}X \in u(2)} f_\theta(\sigma, X) = \frac{3 - (a^2 + b^2 + c^2)}{4} - \det \rho_\theta - \frac{a^2}{16 \det \rho_\theta} \tag{27}$$

at $\theta \notin \Theta_{\mathrm{sing}}(|\psi\rangle\langle\psi|) = \{\theta\,;\, \det \rho_\theta = 0\}$, where

$$\det \rho_\theta = \frac{1}{8}\left[2 - (b^2 + c^2) - (b^2 - c^2)\cos 2\theta - 2bc \sin 2\theta\right].$$

Note that $\Theta_{\mathrm{sing}}(|\psi\rangle\langle\psi|) \neq \emptyset$ if and only if $b^2 + c^2 = 1$.

We first assume that $b^2 + c^2 < 1$: in this case $\Theta_{\mathrm{sing}}(|\psi\rangle\langle\psi|) = \emptyset$. With (b, c) fixed, (27) is monotone decreasing in a^2, so that

$$\begin{aligned} J((\mathrm{id} \otimes \Gamma_\theta)(|\psi\rangle\langle\psi|)) &= 4 \min_{\sqrt{-1}X \in u(2)} f_\theta(\sigma, X) \\ &\leq 4 \min_{\sqrt{-1}X \in u(2)} f_\theta(\sigma, X)\Big|_{a=0} = 2 - r^2 \sin^2(\theta - \phi) \leq 2. \end{aligned} \tag{28}$$

Here we have set $(a, b, c) = (0, r\cos\phi, r\sin\phi)$ in the second equality.

We next assume that $b^2 + c^2 = 1$: in this case, we need to pay special attention to the fact that $\Theta_{\mathrm{sing}}(|\psi\rangle\langle\psi|) \neq \emptyset$. Let us fix a point $\theta_0 \in \Theta$ arbitrarily, and let us take a unit vector $\psi_0 \in \mathbb{C}^2 \otimes \mathbb{C}^2$ such that $\sigma_0 := \mathrm{Tr}_1 |\psi_0\rangle\langle\psi_0|$ is represented by $(a, b, c) = (0, \cos\theta_0, \sin\theta_0)$. Then $\det \Gamma_\theta(\sigma_0) = \sin^2(\theta - \theta_0)/4$, so that $\theta_0 \in \Theta_{\mathrm{sing}}(|\psi_0\rangle\langle\psi_0|)$. On the other hand, we see from (27) that

$$J((\mathrm{id} \otimes \Gamma_\theta)(|\psi_0\rangle\langle\psi_0|)) = 4 \min_{\sqrt{-1}X \in u(2)} f_\theta(\sigma_0, X) = 2 - \sin^2(\theta - \theta_0) \tag{29}$$

for all $\theta \notin \Theta_{\mathrm{sing}}(|\psi_0\rangle\langle\psi_0|)$. Taking the limit $\theta \to \theta_0$, and using the convention (3) at a singular point, we have $J((\mathrm{id} \otimes \Gamma_{\theta_0})(|\psi_0\rangle\langle\psi_0|)) = 2$. This implies that the formula (29) holds for all $\theta \in \Theta$.

In summary, we see from (28) and (29) that

$$J((\mathrm{id} \otimes \Gamma_\theta)(|\psi\rangle\langle\psi|)) \leq 2$$

for all $\psi \in \mathbb{C}^2 \otimes \mathbb{C}^2$ and $\theta \in \Theta$. Moreover, this upper bound is achieved, for instance, by a tensor product state $\psi = \phi_1 \otimes \phi_2$, where ϕ_1 and ϕ_2 are unit vectors such that $|\phi_2\rangle\langle\phi_2|$ is specified by the Stokes parameter $(a, b, c) = (0, \cos\theta, \sin\theta)$. This completes the proof of (24) and (25).

Finally, we prove that $\beta_1 \neq 0$ for any generator $\mathcal{A}(\theta)$ of Γ_θ. To this end, we show that the equation (19) does not have a Hermitian solution $X = (x_{k\ell})$. By a direct computation using the parametrization (26) of X, (19) is reduced to

$$\left(\frac{x + w}{2}\right) I + (y \cos\theta)\, \sigma_2 + (y \sin\theta)\, \sigma_3 = \frac{1}{2}\sigma_1.$$

Since $\{I,\, \sigma_1,\, \sigma_2,\, \sigma_3\}$ is linearly independent, this equation does not have a solution.

5.3 Full-rank quasi-unitary channel

While Theorem 8 asserts that $\tilde{J}((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n}) = O(n)$ for any family of full-rank channels Γ_θ, it does not always imply the additivity $\tilde{J}((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n}) = n\,\tilde{J}(\mathrm{id}\otimes\Gamma_\theta)$. In this section we demonstrate the superadditivity by an example.

Given $\varepsilon\in[0,1/3)$, let $\Gamma_\theta^\varepsilon:\mathcal{S}(\mathbb{C}^2)\to\mathcal{S}(\mathbb{C}^2)$ be defined by

$$\Gamma_\theta^\varepsilon(\tau) := (1-3\varepsilon)\,U_\theta\tau U_\theta^* + \varepsilon\sum_{i=1}^{3}\sigma_i\tau\sigma_i^*$$

where

$$U_\theta := \exp(\sqrt{-1}\,\theta\,\sigma_1) = I\cos\theta + \sqrt{-1}\,\sigma_1\sin\theta$$

with $\theta\in(-\pi/2,\pi/2)$. The channel $\Gamma_\theta^\varepsilon$ is full-rank if and only if $\varepsilon\neq 0$. For sufficiently small $\varepsilon>0$, the channel is regarded as an "almost" unitary channel, perturbed by a fixed depolarizing noise. When $\varepsilon=0$, on the other hand, the channel is reduced to a genuine unitary channel, and it enjoys $\tilde{J}((\mathrm{id}\otimes\Gamma_\theta^0)^{\otimes n}) = O(n^2)$ [13]. It is therefore probable that $\tilde{J}((\mathrm{id}\otimes\Gamma_\theta^\varepsilon)^{\otimes n}) > n\,\tilde{J}(\mathrm{id}\otimes\Gamma_\theta^\varepsilon)$ for sufficiently small ε.

We first show that

$$\tilde{J}(\mathrm{id}\otimes\Gamma_\theta^\varepsilon) = \frac{2(1-3\varepsilon)(2-(5+\cos\theta)\varepsilon)}{1-2\varepsilon}. \tag{30}$$

Let us take

$$B_0(\theta) := \sqrt{1-3\varepsilon}\,U_\theta, \qquad B_i(\theta) := \sqrt{\varepsilon}\,\sigma_i \quad (1\le i\le 3)$$

as a reference generator to define $f_\theta(\tau,X)$. It then follows from (9) that $\tilde{J}(\mathrm{id}\otimes\Gamma_\theta^\varepsilon)$ is the quadruple of the saddle value of $f_\theta(\tau,X)$. It can be shown that the function $(\tau,X)\mapsto f_\theta(\tau,X)$ has a unique saddle point (τ_0,X_0), where

$$\tau_0 := \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

and

$$X_0 := \frac{\sqrt{\varepsilon(1-3\varepsilon)}\,\cos\theta}{1-2\varepsilon}\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}.$$

The formula (30) is then obtained by computing $4f_\theta(\tau_0,X_0)$. Note that $\max_{\tilde\tau} J((\mathrm{id}\otimes\Gamma_\theta^\varepsilon)(\tilde\tau))$ is attained by a maximally entangled $\tilde\tau\in\partial_e\mathcal{S}(\mathbb{C}^2\otimes\mathbb{C}^2)$ because τ_0 is the barycentre of $\mathcal{S}(\mathbb{C}^2)$.

In order to prove the superadditivity of $\tilde{J}((\mathrm{id}\otimes\Gamma_\theta^\varepsilon)^{\otimes n})$, it suffices to show that there is a $\psi\in(\mathbb{C}^2\otimes\mathbb{C}^2)^{\otimes 2}$ that satisfies

$$J\left((\mathrm{id}\otimes\Gamma_\theta^\varepsilon)^{\otimes 2}(|\psi\rangle\langle\psi|)\right) > 2\tilde{J}(\mathrm{id}\otimes\Gamma_\theta^\varepsilon).$$

As in Section 4.1, we identify $(\mathrm{id}\otimes\Gamma_\theta^\varepsilon)^{\otimes 2}$ with $(\mathrm{id}^{\otimes 2}\otimes\Gamma_\theta^{\varepsilon\otimes 2})$. Let $\{e_i\}_{i=1,2}$ be the standard basis of $\mathbb{C}^2$, and let

$$\tilde{f}_1 := e_1\otimes e_1, \quad \tilde{f}_2 := \frac{1}{\sqrt{2}}(e_1\otimes e_2 + e_2\otimes e_1), \quad \tilde{f}_3 := e_2\otimes e_2.$$

Then $\{\tilde{f}_i\}_{1\le i\le 3}$ forms an orthonormal basis of an irreducible subspace of $\mathbb{C}^2\otimes\mathbb{C}^2$ under the $SU(2)$ action. It is known [13] that when $\varepsilon=0$, the maximally entangled vector

$$\psi_{ME} := \frac{1}{\sqrt{3}}\sum_{i=1}^{3}\tilde{f}_i\otimes\tilde{f}_i$$

gives an optimal (more precisely, an admissible) input state to the extended channel $(\mathrm{id}^{\otimes 2}\otimes\Gamma_\theta^{0\otimes 2})$. Therefore, we can expect that for sufficiently small ε, ψ_{ME} would give a nearly optimal input to the extended channel $(\mathrm{id}^{\otimes 2}\otimes\Gamma_\theta^{\varepsilon\otimes 2})$. By computing explicitly the SLD of the output state $(\mathrm{id}^{\otimes 2}\otimes\Gamma_\theta^{\varepsilon\otimes 2})(|\psi_{ME}\rangle\langle\psi_{ME}|)$ at $\theta=0$, we obtain

$$J\left((\mathrm{id}^{\otimes 2}\otimes\Gamma_\theta^{\varepsilon\otimes 2})(|\psi_{ME}\rangle\langle\psi_{ME}|)\right)\Big|_{\theta=0}=\frac{32(1-3\varepsilon)^2(3-15\varepsilon+20\varepsilon^2)}{9-42\varepsilon+48\varepsilon^2}. \tag{31}$$

Comparing (31) with (30) at $\theta=0$, we see that

$$J\left((\mathrm{id}^{\otimes 2}\otimes\Gamma_\theta^{\varepsilon\otimes 2})(|\psi_{ME}\rangle\langle\psi_{ME}|)\right)\Big|_{\theta=0}>2\tilde J(\mathrm{id}\otimes\Gamma_\theta^{\varepsilon})\Big|_{\theta=0}$$

for $0\le\varepsilon<(9-\sqrt{21})/40\,(=0.110\cdots)$. This completes the proof of the superadditivity.

Incidentally, (30) and (31) suggest that $\tilde J((\mathrm{id}\otimes\Gamma_\theta^{\varepsilon})^{\otimes n})$ would be of the form

$$\tilde J((\mathrm{id}\otimes\Gamma_\theta^{\varepsilon})^{\otimes n})=\frac{O(n^2)}{1+\varepsilon\, O(n)}.$$

Deriving the explicit formula of $\tilde J((\mathrm{id}\otimes\Gamma_\theta^{\varepsilon})^{\otimes n})$ is a challenging open problem.

6 Concluding remarks

We introduced a fibre bundle structure over manifolds of quantum channels. Under mild regularity conditions, it was shown that the SLD Fisher information $\tilde J(\mathrm{id}\otimes\Gamma_\theta)$ of a one-paremeter family of quantum channels Γ_θ is expressed by means of the operator norm of the horizontal lift of the tangent vector on the base manifold. Using this formula, we proved that $\tilde J((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n})=O(n)$ for any family of full-rank channels Γ_θ. This result asserts that for almost all quantum channels, the maximum SLD Fisher information increases in the order of $O(n)$. We presented several illustrative examples for the sake of demonstration.

There are many open problems left. Among others, investigating the order of $\tilde J((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n})$ for channels Γ_θ of ranks in-between would be of primary importance. We observe that the solution (16) also leads to the following evaluation:

$$n(n-1)\left\|\beta_1\right\|^2-n\left\|\alpha_1\right\|\le\left\|\alpha_n\right\|\le n(n-1)\left\|\beta_1\right\|^2+n\left\|\alpha_1\right\| \tag{32}$$

for any generator $\mathcal{A}(\theta)$ of Γ_θ. This suggests the following dichotomy: the order of $\tilde J((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n})$ is either $O(n)$ or $O(n^2)$, and is $O(n)$ if and only if Γ_θ has a generator $\mathcal{A}(\theta)$ that satisfies $\beta_1=0$. If this were true, the rank-two quasi-unitary channel investigated in Section 5.2 would give an example of non-unitary channel that exhibits $\tilde J((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n})=O(n^2)$. Unfortunately, because

$$\tilde J((\mathrm{id}\otimes\Gamma_\theta)^{\otimes n})\le 4\min_{\mathcal{A}(\theta)}\|\alpha_n\|,$$

(32) does not conclude anything about this conjecture at present.

Another important subject is to establish a perturbation theory of quantum channels. In an experiment, noise from the environment is inevitable. According to Theorem 8, a slight perturbation applied to a unitary channel induces a transition of the increasing order from $O(n^2)$ to $O(n)$. Does this mean that the $O(n^2)$ increase of the quantum Fisher information cannot be detected by an experiment? Actually, given a family $\Gamma_\theta^{\varepsilon}$ of perturbed unitary channels, the SLD Fisher information $\tilde J((\mathrm{id}\otimes\Gamma_\theta^{\varepsilon})^{\otimes n})$ would be continuous in the magnitude ε of perturbation for each n, as demonstrated in Section 5.3, and there is a hope for detecting the $O(n^2)$ increase approximately. A detailed analysis of the transition from $O(n^2)$ to $O(n)$ is, therefore, important not only from a theoretical point of view but also from an experimental point of view.

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