One-to-one parametrization of quantum channels
(An extended version)

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Abstract

In contrast to the many-to-one and quadratic nature of correspondence in the conventional representation of quantum channels, such as the operator-sum or the unitary representation, we present a one-to-one and linear (affine) parametrization of quantum channels. This parametrization enables us to visualize the convex structure of the space of quantum channels. To demonstrate its usefulness, we apply it to the analysis of quantum binary channels.

1 Introduction

Let $\mathcal{H}, \mathcal{H}'$ be Hilbert spaces which represent the physical systems of interest and let $\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}')$ be the algebras of all bounded operators on $\mathcal{H}, \mathcal{H}'$. A linear map $\kappa : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ is called positive if it sends positive operators to positive operators[1]. It is called *completely positive* [1] if for every positive integer n, the induced map

$$\kappa^{(n)} = \kappa \otimes I^{(n)} \ : \ \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}^{(n)} \ \longrightarrow \ \mathcal{B}(\mathcal{H}') \otimes \mathcal{M}^{(n)}$$

(where $I^{(n)}$ is the identity map acting on the algebra $\mathcal{M}^{(n)}$ of $n \times n$ complex matrices) is positive. We denote by $\mathcal{CP}(\mathcal{H}, \mathcal{H}')$ the totality of completely positive (CP) maps. Note that $\mathcal{CP}(\mathcal{H}, \mathcal{H}')$ is a convex cone.

As is well-known, CP maps play an important role in quantum theory [2]. A dynamical change of the system, called a *quantum channel*, is described by the dual κ^* of a CP map κ which satisfies $\kappa(I) = I'$. The condition that κ is identity-preserving (or equivalently, κ^* is trace-preserving) is necessary to ensure that κ^* maps the set $\mathcal{S}(\mathcal{H}')$ of all density operators on $\mathcal{H}'$ into $\mathcal{S}(\mathcal{H})$. To see why κ must be CP, let the elements of $\mathcal{M}^{(n)}$ be viewed as operators on an ancillary quantum system $\mathcal{H}_a$ with dimension $\dim \mathcal{H}_a = n$. Then for density operators $\rho \in \mathcal{S}(\mathcal{H}')$ and $\rho_a \in \mathcal{S}(\mathcal{H}_a)$, the induced map $\kappa^{(n)*}$ maps the composite state $\rho \otimes \rho_a$ on $\mathcal{H}' \otimes \mathcal{H}_a$ to $\kappa^*(\rho) \otimes \rho_a$. This means that κ does not influence $\mathcal{H}_a$ directly. The condition that $\kappa^{(n)}$ is positive is needed to ensure that every state on the total system $\mathcal{H}' \otimes \mathcal{H}_a$ (which may be entangled) is mapped to a state [3]. We denote by $\mathcal{CP}_1(\mathcal{H}, \mathcal{H}')$ the totality of identity-preserving CP maps. Note that $\mathcal{CP}_1(\mathcal{H}, \mathcal{H}')$ is a convex subset of $\mathcal{CP}(\mathcal{H}, \mathcal{H}')$.

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[1] A linear operator on a Hilbert space is called positive if it is positive semidefinite.

Some representation theorems for CP maps are known [2]. For example, the operator-sum representation asserts that a linear map $\kappa : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ is CP iff it can be represented in the form

$$\kappa(X) = \sum_j A_j^* X A_j, \tag{1}$$

where $\mathcal{A} = \{A_j\}_j$ is a collection of bounded operators in $\mathcal{B}(\mathcal{H}', \mathcal{H})$, see Appendix. When a CP map κ is represented in this way, we call $\mathcal{A}$ a generator of κ and denote κ by $\kappa_{\mathcal{A}}$.

While there is no room for doubt as to the usefulness of the conventional representation, applications are occasionally inconvenient because of their many-to-one and non-affine nature. (In the operator-sum representation, the correspondence $\mathcal{A} \mapsto \kappa_{\mathcal{A}}$ is many-to-one and quadratic.) It will be useful if we have a parametrization of $\mathcal{CP}_1(\mathcal{H}, \mathcal{H}')$ that is both one-to-one and affine, whereby we can visualize the convex structure of the space of quantum channels. The purpose of this paper is to provide such a parametrization explicitly for the case where $\mathcal{H}$ and $\mathcal{H}'$ are both finite dimensional.

In Section 2, we extend Choi's argument [4] to establish a one-to-one affine correspondence between $\mathcal{CP}_1(\mathbf{C}^m, \mathbf{C}^n)$ and a subset of the convex cone of hermitian positive semidefinite matrices in $\mathbf{C}^{mn\times mn}$. This correspondence leads to the desired one-to-one affine parametrization of $\mathcal{CP}_1(\mathbf{C}^m, \mathbf{C}^n)$. In Section 3, we demonstrate its usefulness through an analysis of quantum binary channels: We show that not every ellipsoid inside the unit ball in Stokes' parameter space can be the image of a channel. The final Section 4 gives conclusions.

2 Affine parametrization of $\mathcal{CP}_1(\mathbf{C}^m, \mathbf{C}^n)$

Let $\dim \mathcal{H} = m$, $\dim \mathcal{H}' = n$ $(m, n < \infty)$. Given an orthonormal basis for $\mathcal{H}$, one may identify vectors in $\mathcal{H}$ with column vectors in $\mathbf{C}^m$ and operators in $\mathcal{B}(\mathcal{H})$ with $m \times m$ matrices in $\mathbf{C}^{m\times m}$. In the same way, vectors in $\mathcal{H}'$ and operators in $\mathcal{B}(\mathcal{H}')$ are represented by column vectors in $\mathbf{C}^n$ and $n \times n$ matrices in $\mathbf{C}^{n\times n}$. A CP map $\kappa : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ is a superoperator that maps operators to operators. If $X = (X_{\sigma\tau})$ is an operator with matrix entries $X_{\sigma\tau}$, then $Y = \kappa(X)$ is an operator with matrix entries

$$Y_{\mu\nu} = \sum_{\sigma\tau} K_{\mu\nu,\sigma\tau} X_{\sigma\tau}.$$

(The subscripts σ, τ run from 1 to m, and μ, ν from 1 to n.) The coefficients $K_{\mu\nu,\sigma\tau}$ are the matrix entries of $\kappa(E^{\sigma\tau})$, where $E^{\sigma\tau}$ is the matrix with entries $E^{\sigma\tau}_{\sigma'\tau'} = \delta_{\sigma\sigma'}\delta_{\tau\tau'}$.

For every operator or matrix $X = (X_{\sigma\tau})$ in $\mathbf{C}^{m\times m}$ let $\mathbf{X}$ denote the column vector with height m^2 that is obtained by stacking the successive columns of X on top of each other. The element $X_{\sigma\tau}$ of the matrix X ends up in line $\langle\!\langle\sigma\tau\rangle\!\rangle$ of the column vector $\mathbf{X}$, where

$$\langle\!\langle\sigma\tau\rangle\!\rangle = (\tau - 1)m + \sigma.$$

In the same way, a matrix $Y = (Y_{\mu\nu})$ in $\mathbf{C}^{n\times n}$ is represented by the column vector $\mathbf{Y}$ in $\mathbf{C}^{n^2}$, in which the element $Y_{\mu\nu}$ appears in line

$$\langle\mu\nu\rangle = (\nu - 1)n + \mu.$$

The functional relationship $Y = \kappa(X)$ is then represented by the matrix equation $\mathbf{Y} = \mathbf{K}\,\mathbf{X}$, where $\mathbf{K}$ is the $n^2 \times m^2$ matrix with entries

$$\mathbf{K}_{\langle\mu\nu\rangle,\langle\!\langle\sigma\tau\rangle\!\rangle} = K_{\mu\nu,\sigma\tau}.$$

Let $\mathcal{A} = \{A_j\}_{1\leq j\leq J}$ be a generator for κ, where $A_j \in \mathcal{B}(\mathbf{C}^n, \mathbf{C}^m)$ are represented by $m \times n$ matrices in $\mathbf{C}^{m\times n}$. Then for any matrix $X = (X_{\sigma\tau})$, $\kappa(X)$ is a matrix with entries

$$\kappa(X)_{\mu\nu} = \sum_j \sum_{\sigma\tau} (A_j^*)_{\mu\sigma} X_{\sigma\tau} (A_j)_{\tau\nu} = \sum_{\sigma\tau} K_{\mu\nu,\sigma\tau} X_{\sigma\tau},$$

where

$$K_{\mu\nu,\sigma\tau} = \sum_j (A_j)_{\tau\nu} (A_j^*)_{\mu\sigma}.$$

All the information in the generator $\mathcal{A} = \{A_j\}_j$ that pertains to the definition of the map $\kappa = \kappa_{\mathcal{A}}$ is compressed in the m^2n^2 numbers $K_{\mu\nu,\sigma\tau}$.

Let $\mathbf{A}_j$ be the column vector in $\mathbf{C}^{mn}$ that represents the matrix A_j: the element $(A_j)_{\sigma\mu}$ appears in line

$$\langle\!\langle \sigma\mu \rangle = (\mu - 1)m + \sigma$$

of $\mathbf{A}_j$. Let $\mathbf{A}$ denote the $mn \times J$ matrix whose columns are the vectors $\mathbf{A}_j$. Then the matrix $\mathbf{AA}^*$ can be written in the form

$$\mathbf{AA}^* = \sum_j \mathbf{A}_j \mathbf{A}_j^*.$$

The $mn \times mn$ matrix $\mathbf{AA}^*$ contains the m^2n^2 elements of the matrix $\mathbf{K} = \mathbf{K}_{\mathcal{A}}$ in some order. In fact,

$$\mathbf{K}_{\langle\mu\nu\rangle,\langle\!\langle\sigma\tau\rangle\!\rangle} = K_{\mu\nu,\sigma\tau} = \sum_j (A_j)_{\tau\nu}(A_j^*)_{\mu\sigma} = (\mathbf{AA}^*)_{\langle\!\langle\tau\nu\rangle,\langle\!\langle\sigma\mu\rangle}. \tag{2}$$

As a consequence we have

Lemma 1. *Let $\mathcal{A} = \{A_j\}_j$ and $\mathcal{B} = \{B_k\}_k$ be two collections of linear operators in $\mathcal{B}(\mathbf{C}^n, \mathbf{C}^m)$. Then $\kappa_{\mathcal{A}} = \kappa_{\mathcal{B}}$ iff $\mathbf{AA}^* = \mathbf{BB}^*$.*

Lemma 1 implies that the map $\kappa_{\mathcal{A}} \mapsto \mathbf{AA}^*$ gives a one-to-one affine correspondence between $\mathcal{CP}(\mathbf{C}^m, \mathbf{C}^n)$ and the cone

$$\mathcal{P}(\mathbf{C}^{mn}) := \left\{ \mathbf{M} \in \mathbf{C}^{mn\times mn} \ ; \ \mathbf{M} \geq 0 \right\}$$

of hermitian positive semidefinite matrices in $\mathbf{C}^{mn\times mn}$. In particular, dim $\mathcal{CP}(\mathbf{C}^m, \mathbf{C}^n) = m^2n^2$, and a generator $\mathcal{A}$ can be found by spectral factorization of $\mathbf{AA}^*$.

Thus far there is nothing new: we have just recast Choi's argument [4] (see Appendix) in a slightly different manner. We now proceed to a characterization of the condition that $\kappa(I_m) = I_n$. Let us identify $\mathbf{C}^{mn}$ with $\mathcal{H}_1 \otimes \mathcal{H}_2$ where $\mathcal{H}_1 = \mathbf{C}^n$, $\mathcal{H}_2 = \mathbf{C}^m$. Given an array $M_{\sigma\mu,\tau\nu}$ of m^2n^2 complex numbers ($\sigma, \tau = 1, ..., m$; $\mu, \nu = 1, ..., n$), we consider the $mn \times mn$ matrix $\mathbf{M}$ with entries $\mathbf{M}_{\langle\!\langle\sigma\mu\rangle,\langle\!\langle\tau\nu\rangle} = M_{\sigma\mu,\tau\nu}$. The partial trace[2] $\mathrm{Tr}_{\mathcal{H}_2}\mathbf{M}$ with respect to $\mathcal{H}_2$ is intrinsically defined as the

[2] The other partial trance $\mathrm{Tr}_{\mathcal{H}_1}\mathbf{M}$ is defined as the $m \times m$ matrix which satisfies

$$\mathrm{Tr}_{\mathbf{C}^{n^2}}(\mathbf{M}(I_n \otimes Y)) = \mathrm{Tr}_{\mathcal{H}_2}((\mathrm{Tr}_{\mathcal{H}_1}\mathbf{M})Y),$$

for all $m \times m$ matrices Y. It is explicitly written as

$$(\mathrm{Tr}_{\mathcal{H}_1}\mathbf{M})_{\tau\sigma} = \sum_{\mu\nu} M_{\tau\nu,\sigma\mu}\delta_{\mu\nu} = \sum_{\mu} \mathbf{M}_{\langle\!\langle\tau\mu\rangle,\langle\!\langle\sigma\mu\rangle}.$$

$n \times n$ matrix which satisfies

$$\mathrm{Tr}_{\mathbf{C}^{n^2}}(\,(X \otimes I_m)\,\mathbf{M}\,) = \mathrm{Tr}_{\mathcal{H}_1}(\,X\,(\mathrm{Tr}_{\mathcal{H}_2}\,\mathbf{M})\,),$$

for all $n \times n$ matrices X. Here the Kronecker product $X \otimes Y$ of $X \in \mathbf{C}^{n\times n}$ and $Y \in \mathbf{C}^{m\times m}$ is defined as the $mn \times mn$ matrix with entries $(X \otimes Y)_{\langle\!\langle \sigma\mu\rangle\!\rangle, \langle\!\langle \tau\nu\rangle\!\rangle} = X_{\mu\nu}Y_{\sigma\tau}$. Thus we get

$$\mathrm{Tr}_{\mathbf{C}^{n^2}}(\,(X \otimes I_m)\,\mathbf{M}\,) = \sum_{\mu\nu,\sigma\tau} (X_{\mu\nu}\delta_{\sigma\tau})M_{\tau\nu,\sigma\mu} = \sum_{\mu\nu} X_{\mu\nu}\left(\sum_{\sigma\tau}\delta_{\sigma\tau}M_{\tau\nu,\sigma\mu}\right),$$

and consequently

$$(\mathrm{Tr}_{\mathcal{H}_2}\mathbf{M})_{\nu\mu} = \sum_{\sigma\tau}\delta_{\sigma\tau}M_{\tau\nu,\sigma\mu} = \sum_{\sigma}\mathbf{M}_{\langle\!\langle \sigma\nu\rangle\!\rangle, \langle\!\langle \sigma\mu\rangle\!\rangle}.$$

Since

$$\kappa_{\mathcal{A}}(I_m)_{\mu\nu} = \sum_{\sigma\tau} K_{\mu\nu,\sigma\tau}\delta_{\sigma\tau} = \sum_{\sigma\tau}(\mathbf{A}\mathbf{A}^*)_{\langle\!\langle \tau\nu\rangle\!\rangle, \langle\!\langle \sigma\mu\rangle\!\rangle}\delta_{\sigma\tau} = \sum_{\sigma}(\mathbf{A}\mathbf{A}^*)_{\langle\!\langle \sigma\nu\rangle\!\rangle, \langle\!\langle \sigma\mu\rangle\!\rangle},$$

we obtain

Lemma 2. *$\kappa_{\mathcal{A}}(I_m) = X$ is equivalent to $\mathrm{Tr}_{\mathcal{H}_2}\mathbf{A}\mathbf{A}^* = {}^tX$.*

Now we reach the main claim. Let us define the convex subset

$$\mathcal{P}_1(\mathbf{C}^{mn}) := \{\mathbf{M} \in \mathcal{P}(\mathbf{C}^{mn})\ ;\ \mathrm{Tr}_{\mathcal{H}_2}\mathbf{M} = I_n\}$$

of the cone $\mathcal{P}(\mathbf{C}^{mn})$. If $\kappa_{\mathcal{A}}(I_m) = I_n$, then $\mathrm{Tr}_{\mathcal{H}_2}\mathbf{A}\mathbf{A}^* = I_n$ by Lemma 2, hence $\mathbf{A}\mathbf{A}^* \in \mathcal{P}_1(\mathbf{C}^{mn})$. We consider the map

$$\kappa_{\mathcal{A}} \in \mathcal{CP}_1(\mathbf{C}^m, \mathbf{C}^n) \ \longmapsto\ \mathbf{A}\mathbf{A}^* \in \mathcal{P}_1(\mathbf{C}^{mn}). \tag{3}$$

Then (3) is injective by Lemma 1, and is surjective by Lemma 2. (The spectral decomposition of an element in $\mathcal{P}_1(\mathbf{C}^{mn})$ gives a generator.) Thus (3) gives a one-to-one affine correspondence between $\mathcal{CP}_1(\mathbf{C}^m, \mathbf{C}^n)$ and $\mathcal{P}_1(\mathbf{C}^{mn})$. In particular, dim $\mathcal{CP}_1(\mathbf{C}^n) = m^2n^2 - n^2$, and the correspondence (2) of matrix elements leads to the following

Theorem 3. *Given an array $K_{\mu\nu,\sigma\tau}$ of m^2n^2 complex numbers, there is a map $\kappa \in \mathcal{CP}_1(\mathbf{C}^m, \mathbf{C}^n)$ such that $K_{\mu\nu,\sigma\tau} = (\kappa(E^{\sigma\tau}))_{\mu\nu}$ iff the matrix $\mathbf{M}$ with entries $\mathbf{M}_{\langle\!\langle \tau\nu\rangle\!\rangle, \langle\!\langle \sigma\mu\rangle\!\rangle} = K_{\mu\nu,\sigma\tau}$ belongs to $\mathcal{P}_1(\mathbf{C}^{mn})$.*

It is straightforward to obtain an analogous correspondence for a general constraint of the form $\kappa_{\mathcal{A}}(I_m) = X$ where X is an arbitrary $n \times n$ positive matrix. Physical implication of such a general constraint is presented in [2].

Theorem 3 establishes a one-to-one affine parametrization for $\mathcal{CP}_1(\mathbf{C}^m, \mathbf{C}^n)$, since $\mathcal{P}_1(\mathbf{C}^{mn})$ has a standard affine parametrization. In fact, recall that there is a simple way to visualize calculation of partial trace. Given an $mn \times mn$ matrix $\mathbf{M}$, let us partition $\mathbf{M}$ into n^2 blocks:

$$\mathbf{M} = \begin{bmatrix} \tilde{M}_{11} & \cdots & \tilde{M}_{1n} \\ \vdots & & \vdots \\ \tilde{M}_{n1} & \cdots & \tilde{M}_{nn} \end{bmatrix},$$

where $\tilde{M}_{\mu\nu}$ are $m \times m$ matrices. Then

$$(\mathrm{Tr}_{\mathcal{H}_2}\mathbf{M})_{\mu\nu} = \sum_\sigma \mathbf{M}_{\langle\!\langle\sigma\mu\rangle,\langle\!\langle\sigma\nu\rangle} = \sum_\sigma \mathbf{M}_{(\mu-1)m+\sigma,(\nu-1)m+\sigma} = \mathrm{Tr}\,\tilde{M}_{\mu\nu}.$$

Thus the condition $\mathrm{Tr}_{\mathcal{H}_2}\mathbf{M} = I_n$ in the definition of $\mathcal{P}_1(\mathbf{C}^{mn})$ is equivalent to $\mathrm{Tr}\,\tilde{M}_{\mu\nu} = \delta_{\mu\nu}$ for each (μ, ν) block.

In order to parametrize quantum channels directly, we only need a translation formula between a CP map $\kappa_{\mathcal{A}}$ and its dual $\kappa_{\mathcal{A}}^*$. The dual $\kappa_{\mathcal{A}}^*$ is defined by the requirement that $\mathrm{Tr}\,\kappa^*(S)X = \mathrm{Tr}\,S\kappa(X)$ for all $S \in \mathcal{B}(\mathbf{C}^n)$ and $X \in \mathcal{B}(\mathbf{C}^m)$, and is explicitly given by

$$\kappa_{\mathcal{A}}^*(S) = \sum_j A_j S A_j^*.$$

Hence

$$\kappa_{\mathcal{A}}^*(S)_{\sigma\tau} = \sum_j \sum_{\mu\nu} (A_j)_{\sigma\mu}(S)_{\mu\nu}(A_j^*)_{\nu\tau} = \sum_{\mu\nu} \hat{K}_{\sigma\tau,\mu\nu} S_{\mu\nu},$$

where

$$\hat{K}_{\sigma\tau,\mu\nu} = \sum_j (A_j)_{\sigma\mu}(A_j^*)_{\nu\tau} = (\mathbf{A}\mathbf{A}^*)_{\langle\!\langle\sigma\mu\rangle,\langle\!\langle\tau\nu\rangle} = K_{\nu\mu,\tau\sigma}.$$

The column vector that represents the matrix $T = \kappa_{\mathcal{A}}^*(S)$ is given by

$$\mathbf{T} = \hat{\mathbf{K}}_{\mathcal{A}}\,\mathbf{S},$$

where $\hat{\mathbf{K}}_{\mathcal{A}} = \hat{\mathbf{K}}$ is the $m^2 \times n^2$ matrix with entries

$$\hat{\mathbf{K}}_{\langle\!\langle\sigma\tau\rangle\!\rangle,\langle\mu\nu\rangle} = \hat{K}_{\sigma\tau,\mu\nu} = K_{\nu\mu,\tau\sigma} = \mathbf{K}_{\langle\nu\mu\rangle,\langle\!\langle\tau\sigma\rangle\!\rangle} = (\mathbf{A}\mathbf{A}^*)_{\langle\!\langle\sigma\mu\rangle,\langle\!\langle\tau\nu\rangle}. \tag{4}$$

This gives the desired translation formula.

3 Application to quantum binary channels

When $\dim\mathcal{H} = \dim\mathcal{H}' = 2$, a quantum channel $\kappa^* : \mathcal{S}(\mathcal{H}') \to \mathcal{S}(\mathcal{H})$ is called a *quantum binary channel* [5]. In this section we apply the parametrization to the analysis of quantum binary channels.

A state of a two-level quantum system $\mathcal{H} \cong \mathbf{C}^2$ can be represented by a 2×2 Hermitian matrix of the form

$$\rho_\theta = \frac{1}{2}\begin{bmatrix} 1+\theta_z & \theta_x - i\theta_y \\ \theta_x + i\theta_y & 1-\theta_z \end{bmatrix},$$

where $\theta = {}^t(\theta_x, \theta_y, \theta_z)$ is a vector in the unit ball

$$\Theta = \{\theta \in \mathbf{R}^3 \; ; \; \|\theta\|^2 = \theta_x^2 + \theta_y^2 + \theta_z^2 \leq 1\}.$$

The correspondence $\theta \mapsto \rho_\theta$ is often called the Stokes parametrization of two-level quantum states.

By Theorem 3, $\mathcal{CP}_1(\mathbf{C}^2, \mathbf{C}^2)$ can be identified with $\mathcal{P}_1(\mathbf{C}^4)$, each element of which has the form

$$\mathbf{A}\mathbf{A}^* = \begin{bmatrix} \frac{1}{2}+p & x & r & w \\ \bar{x} & \frac{1}{2}-p & y & -r \\ \bar{r} & \bar{y} & \frac{1}{2}+q & z \\ \bar{w} & -\bar{r} & \bar{z} & \frac{1}{2}-q \end{bmatrix}. \tag{5}$$

Here $p, q \in \mathbf{R}$ and $r, x, y, z, w \in \mathbf{C}$ are taken so that the matrix becomes positive semidefinite. Using (4), we see that the dual $\kappa_{\mathcal{A}}^*$ of the CP map $\kappa_{\mathcal{A}}$ defined by (5) corresponds to

$$\hat{\mathbf{K}}_{\mathcal{A}} = \begin{bmatrix} \frac{1}{2}+p & \bar{r} & r & \frac{1}{2}+q \\ \bar{x} & \bar{w} & y & \bar{z} \\ x & \bar{y} & w & z \\ \frac{1}{2}-p & -\bar{r} & -r & \frac{1}{2}-q \end{bmatrix}.$$

Since the column vectors representing ρ_θ and $\rho_{\theta'} = \kappa_{\mathcal{A}}^*(\rho_\theta)$ are related by $\hat{\mathbf{K}}_{\mathcal{A}}$, we can easily verify that $\kappa_{\mathcal{A}}^*$ induces the following affine map on the Stokes parameter space:

$$\begin{bmatrix} \theta'_x \\ \theta'_y \\ \theta'_z \end{bmatrix} = \begin{bmatrix} y_R + w_R & y_I + w_I & x_R - z_R \\ y_I - w_I & -y_R + w_R & -x_I + z_I \\ 2r_R & 2r_I & p-q \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} + \begin{bmatrix} x_R + z_R \\ -x_I - z_I \\ p+q \end{bmatrix}. \tag{6}$$

(The subscripts R and I denote the real and imaginary parts.) Thus there exist a matrix $A \in \mathbf{R}^{3\times 3}$ and a vector $b \in \mathbf{R}^3$ such that $\theta' = A\theta + b$. The transformation $\theta \mapsto A\theta + b$ has $n^4 - n^2 = 12$ degrees of freedom, but is not completely arbitrary since (5) must be positive semidefinite.

In the paper [5], the notion of pseudoclassicality of a quantum channel was introduced and studied intensively. In particular, a necessary and sufficient condition for a quantum binary channel to be pseudoclassical was expressed only in terms of the shape of the image of the channel [5, Theorem 23]. In that paper, it was assumed for the sake of simplicity that the channel was just an affine map (i.e. not necessarily the dual of a CP map), so that every ellipsoid in the unit ball could be the image of a channel. To emphasize that we are dealing now with a more restricted class of channels, we shall call the dual of an identity preserving CP map a *CP channel.* Now a question arises naturally: Does the additional condition that the channel shall be CP impose restrictions on the shape of the image? In other words, can every ellipsoid inside the unit ball be realized as the image of a CP channel? The answer is given in the next theorem.

Theorem 4. *There exists an ellipsoid inside the unit ball that cannot be realized as the image of a binary CP channel.*

Proof It is sufficient to consider affine maps of the form $\theta' = A\theta + b$ where $A \in \mathbf{R}^{3\times 3}$ is regular. We first observe that the matrix A can be expressed as

$$A = \sigma \tilde{A} Q, \tag{7}$$

where σ is the sign of $\det A$, $\tilde{A} = \sqrt{A\,{}^t\!A}$ is a symmetric positive definite matrix, and Q is an orthogonal matrix with $\det Q = 1$. Next, there exists an orthogonal matrix P with $\det P = 1$ and a diagonal matrix Λ with nonzero diagional elements all having sign σ such that

$$\sigma \tilde{A} = {}^t\!P \Lambda P. \tag{8}$$

According to (7) and (8), the map $\theta' = A\theta + b$ can be decomposed into three stages:

$$\begin{aligned} \theta'' &= PQ\,\theta, \\ \theta''' &= \Lambda\theta'' + \beta, \quad \text{where } \beta = Pb, \\ \theta' &= {}^t\!P\,\theta'''. \end{aligned}$$

Thus we have the following commutative diagram:

$$\begin{array}{ccc} \theta & \longrightarrow & \theta'' \\ \Big\downarrow & & \Big\downarrow \\ \theta' & \longleftarrow & \theta''' \end{array}$$

The first stage $\theta \mapsto \theta'' = (PQ)\theta$ maps the unit ball Θ to the unit ball and does not affect the image $A\Theta + b = \sigma\tilde{A}\Theta + b$. The first and third stages represent invertible CP channels since they correspond to rotations of $\mathbf{R}^3$ or unitary evolutions on the quantum state space. It follows that the process $\theta \mapsto \theta'$ is a CP channel iff the process $\theta'' \mapsto \theta'''$ is a CP channel. Therefore it suffices to consider special maps of the form $\theta' = \Lambda\theta + \beta$, where Λ is a diagonal matrix with nonzero diagonal elements all having the same sign.

Now we show that there is an ellipsoid $\Lambda\Theta + \beta$ inside the unit ball Θ which is not the image of a CP channel. We take the vector β to be parallel to one of the principal axes of the ellipsoid. By a suitable choice of the reference frame, such a map $\theta \mapsto \Lambda\theta + \beta$ can be represented in the form

$$\begin{bmatrix} \theta'_x \\ \theta'_y \\ \theta'_z \end{bmatrix} = \begin{bmatrix} y_R + w_R & & \\ & -y_R + w_R & \\ & & p - q \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ p + q \end{bmatrix}, \tag{9}$$

where $p+q > 0$. According to [5, Eqn.(28)], the ellipsoid $\Lambda\Theta+\beta$ is inside the unit ball iff $|p|, |q| \le \frac{1}{2}$ and

$$\begin{aligned} (\pm y_R + w_R)^2 \quad &\le \quad \frac{1 - 4pq + \sqrt{(1-4p^2)(1-4q^2)}}{2} \\ &= \quad \left(\sqrt{\left(\frac{1}{2} - p\right)\left(\frac{1}{2} + q\right)} + \sqrt{\left(\frac{1}{2} + p\right)\left(\frac{1}{2} - q\right)} \right)^2 . \end{aligned} \tag{10}$$

Using (5) and (6), it is easy to see that (9) is a CP channel iff the matrix

$$\begin{bmatrix} \frac{1}{2} + p & 0 & 0 & w_R \\ 0 & \frac{1}{2} - p & y_R & 0 \\ 0 & y_R & \frac{1}{2} + q & 0 \\ w_R & 0 & 0 & \frac{1}{2} - q \end{bmatrix} \tag{11}$$

is positive semidefinite. By a suitable orthogonal decomposition of $\mathbf{C}^4$, (11) is represented as

$$\begin{bmatrix} \frac{1}{2} - p & y_R \\ y_R & \frac{1}{2} + q \end{bmatrix} \oplus \begin{bmatrix} \frac{1}{2} + p & w_R \\ w_R & \frac{1}{2} - q \end{bmatrix}.$$

Therefore (11) becomes positive semidefinite iff $|p|, |q| \le \frac{1}{2}$ and

$$y_R^2 \le \left(\frac{1}{2} - p\right)\left(\frac{1}{2} + q\right), \qquad w_R^2 \le \left(\frac{1}{2} + p\right)\left(\frac{1}{2} - q\right). \tag{12}$$

We first consider the case where the diagonal elements $\pm y_R + w_R$ and $p - q$ of Λ are all positive. Suppose we are given p, q such that $|p|, |q| \le \frac{1}{2}$ and $p \pm q > 0$. In this case, the conditions (10) together with $\pm y_R + w_R > 0$ define a square region in the upper-half (y_R, w_R)-plane, whereas the conditions (12) together with $\pm y_R + w_R > 0$ define a “home base”-shaped pentagonal region which is a proper subset of the former region.

Next we consider the case where the diagonal elements $\pm y_R + w_R$ and $p - q$ of Λ are all negative. Suppose we are given p, q such that $|p|, |q| \leq \frac{1}{2}$, $p - q < 0$, and $p + q > 0$. In this case, the conditions (10) together with $\pm y_R + w_R < 0$ define a square region in the lower-half (y_R, w_R)-plane, whereas the conditions (12) together with $\pm y_R + w_R < 0$ define a triangular region which is again a proper subset of the former region. □

Theorem 4 is in no sense contrary to [5, Theorem 23]. It only implies that physical law imposes restrictions on shapes of images of quantum binary channels to which [5, Theorem 23] is applied.

4 Conclusions

In this paper, we derived a one-to-one affine correspondence between $\mathcal{CP}_1(\mathbf{C}^m, \mathbf{C}^n)$ and $\mathcal{P}_1(\mathbf{C}^{mn})$. This establishes a one-to-one affine parametrization of all finite dimensional quantum channels. The parametrization was applied to the analysis of quantum binary channels.

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Appendix: Operator-sum representation

Theorem 5. *Given a linear map $\kappa : \mathcal{B}(\mathbf{C}^m) \to \mathcal{B}(\mathbf{C}^n)$, the following are equivalent.*

(a) *κ is CP.*

(b) *κ is m-positive.*

(c) $\displaystyle\sum_{\sigma,\tau=1}^{m} \kappa(E^{\sigma\tau}) \otimes E^{\sigma\tau}$ *is positive.*

(d) *κ is represented in the form* $\displaystyle\kappa(X) = \sum_{j=1}^{J} A_j^* X A_j$.

Proof (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are obvious. (Recall that $\sum_{\sigma\tau} E^{\sigma\tau} \otimes E^{\sigma\tau}$ is a positive operator.) Now assume (c), and let $K_{\mu\nu,\sigma\tau} = (\kappa(E^{\sigma\tau}))_{\mu\nu}$. Then

$$\sum_{\sigma\tau} \kappa(E^{\sigma\tau}) \otimes E^{\sigma\tau}. \tag{13}$$

is an $mn \times mn$ matrix whose $(\langle\!\langle \sigma\mu \rangle\!\rangle, \langle\!\langle \tau\nu \rangle\!\rangle)$ entry is $K_{\mu\nu,\sigma\tau}$. Letting $\mathbf{A}\mathbf{A}^*$ be a decomposition of the transpose of (13), and recalling the relation (2), we have an operator-sum representation. Finally (d) $\Rightarrow$ (a) is obvious by Stinespring's theorem [1]. □

Corollary 6. *For every CP map $\kappa = \kappa_{\mathcal{A}}$, the cardinality J of a generator $\mathcal{A}$ can be taken at most mn.*

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