A normal generating set for the Torelli group of a compact non-orientable surface

Ryoma Kobayashi

Ishikawa National College of Technology

November 27, 2015

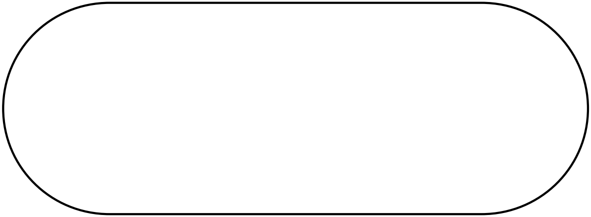
Contents

Notations

N_g^b : a genus g compact non-orientable surface with b boundary components.

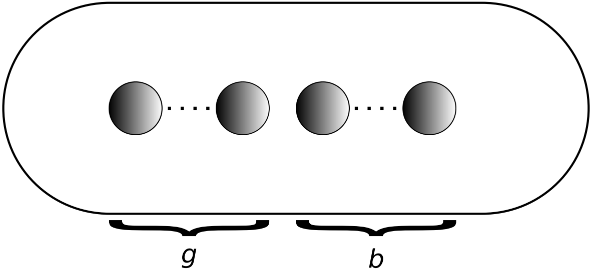
Notations

N_g^b : a genus g compact non-orientable surface with b boundary components.



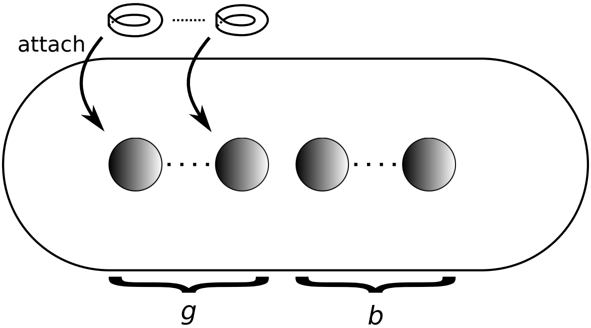
Notations

N_g^b : a genus g compact non-orientable surface with b boundary components.



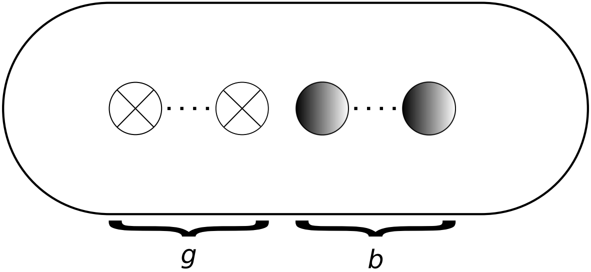
Notations

N_g^b : a genus g compact non-orientable surface with b boundary components.



Notations

N_g^b : a genus g compact non-orientable surface with b boundary components.



Notations

N_g^b : a genus g compact non-orientable surface with b boundary components.
The mapping class group of N_g^b is defined as

$$\mathcal{M}(N_g^b) = \mathrm{Diff}(N_g^b, \partial N_g^b)/\mathrm{isotopy}.$$

Notations

N_g^b : a genus g compact non-orientable surface with b boundary components.
The mapping class group of N_g^b is defined as

$$\mathcal{M}(N_g^b) = \mathrm{Diff}(N_g^b, \partial N_g^b)/\mathrm{isotopy}.$$

The Torelli group of N_g^b is defined as

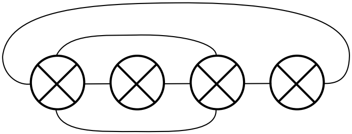
$$\mathcal{I}(N_g^b) = \ker(\mathcal{M}(N_g^b) \to \mathrm{Aut}(H_1(N_g^b; \mathbb{Z}))).$$

Notations

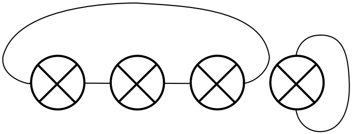
c : a simple closed curve on N_g^b.

(a) c is an A-circle $\underset{\text{def}}{\Longleftrightarrow}$ the regular neighborhood is an annulus,

(b) c is an M-circle $\underset{\text{def}}{\Longleftrightarrow}$ the regular neighborhood is an Möbius band.



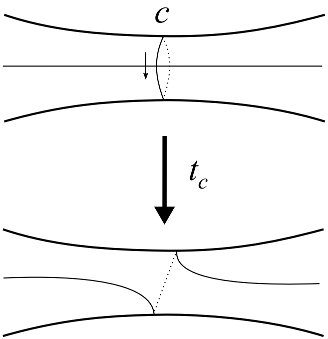
(a) A-circles



(b) M-circles

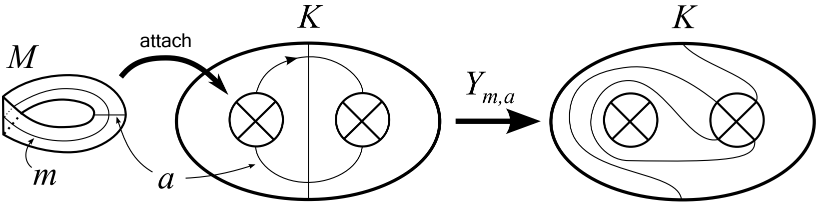
Notations

For an A-circle c, the Dehn twist t_c is defined as



Notations

For an oriented A-circle a and an M-circle m (w/ $\sharp\{a \cap m\} = 1$), the Y-homeomorphism $Y_{m,a}$ is defined as

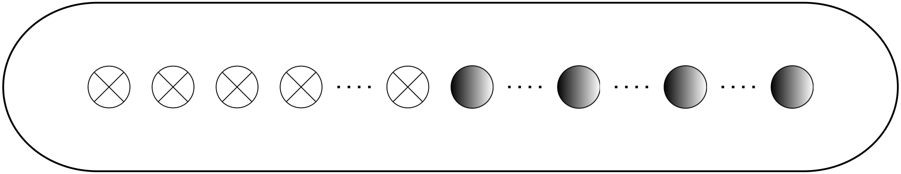


Main result

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if* $g = 4$*).*

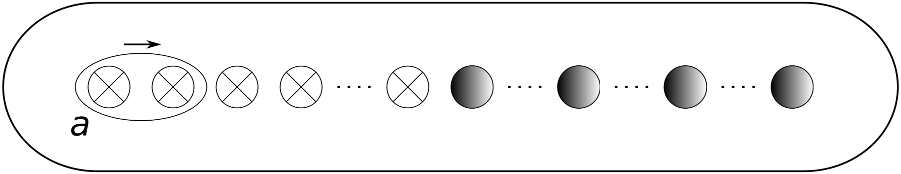


Main result

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if $g = 4$).*

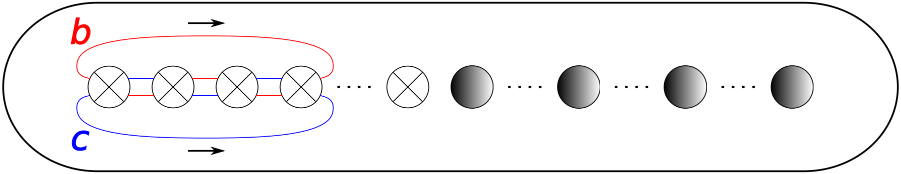


Main result

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if $g = 4$).*

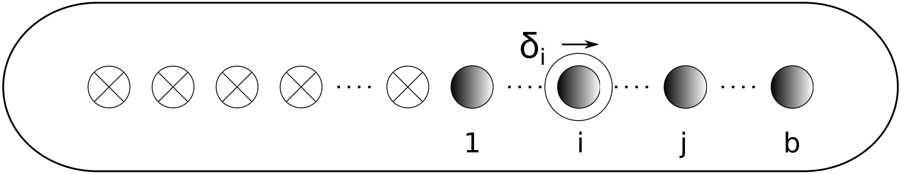


Main result

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if* $g = 4$*).*

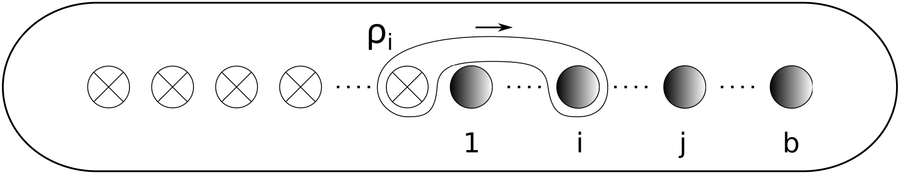


Main result

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if $g = 4$).*

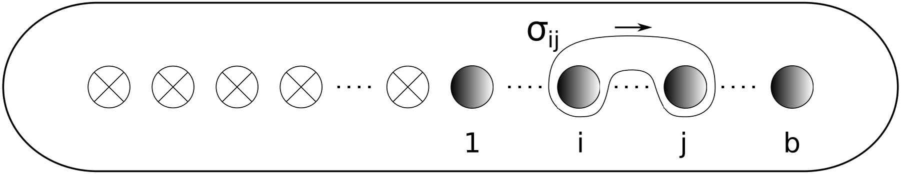


Main result

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if $g = 4$).*

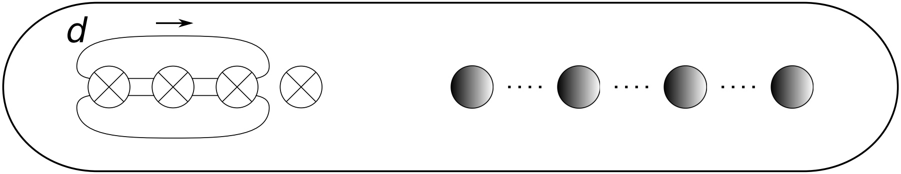


Main result

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if $g = 4$).*



Main result

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if $g = 4$).*

Theorem (Hirose-K. (arXiv:1412.2222))

For $g \geq 4$, $\mathcal{I}(N_g^0)$ is normally generated by

- t_a, $t_b t_c^{-1}$ *and*
- t_d *(only if $g = 4$).*

On mapping class groups for non-orientable surfaces

- $\mathcal{M}(N_g^b)$ is generated by Dehn twists and Y-homeomorphism.
  (The mapping class group of an orientable surface is generated by only Dehn twists.)
- $\mathcal{M}(N_g^b)$ is not generated by only Dehn twists.
  Y-homeomorphisms are needed!
- Finite presentations for $\mathcal{M}(N_g^0)$ and $\mathcal{M}(N_g^1)$ ware given by Paris-Szepietowski (arXiv:1308.5856).
  A finite presentation for $\mathcal{M}(N_g^b)$ are not known.

On mapping class groups for non-orientable surfaces

- $\mathcal{M}(N_g^b)$ is generated by Dehn twists and Y-homeomorphism.
  (The mapping class group of an orientable surface is generated by only Dehn twists.)
- $\mathcal{M}(N_g^b)$ is not generated by only Dehn twists.
  Y-homeomorphisms are needed!
- Finite presentations for $\mathcal{M}(N_g^0)$ and $\mathcal{M}(N_g^1)$ ware given by Paris-Szepietowski (arXiv:1308.5856).
  A finite presentation for $\mathcal{M}(N_g^b)$ are not known.

On mapping class groups for non-orientable surfaces

- $\mathcal{M}(N_g^b)$ is generated by Dehn twists and Y-homeomorphism. (The mapping class group of an orientable surface is generated by only Dehn twists.)
- $\mathcal{M}(N_g^b)$ is not generated by only Dehn twists. Y-homeomorphisms are needed!
- Finite presentations for $\mathcal{M}(N_g^0)$ and $\mathcal{M}(N_g^1)$ ware given by Paris-Szepietowski (arXiv:1308.5856). A finite presentation for $\mathcal{M}(N_g^b)$ are not known.

On mapping class groups for non-orientable surfaces

- $\mathcal{M}(N_g^b)$ is generated by Dehn twists and Y-homeomorphism. (The mapping class group of an orientable surface is generated by only Dehn twists.)
- $\mathcal{M}(N_g^b)$ is not generated by only Dehn twists. Y-homeomorphisms are needed!
- Finite presentations for $\mathcal{M}(N_g^0)$ and $\mathcal{M}(N_g^1)$ ware given by Paris-Szepietowski (arXiv:1308.5856). A finite presentation for $\mathcal{M}(N_g^b)$ are not known.

On mapping class groups for non-orientable surfaces

- $\mathcal{M}(N_g^b)$ is generated by Dehn twists and Y-homeomorphism. (The mapping class group of an orientable surface is generated by only Dehn twists.)
- $\mathcal{M}(N_g^b)$ is not generated by only Dehn twists. Y-homeomorphisms are needed!
- Finite presentations for $\mathcal{M}(N_g^0)$ and $\mathcal{M}(N_g^1)$ ware given by Paris-Szepietowski (arXiv:1308.5856). A finite presentation for $\mathcal{M}(N_g^b)$ are not known.

On mapping class groups for non-orientable surfaces

- $\mathcal{M}(N_g^b)$ is generated by Dehn twists and Y-homeomorphism. (The mapping class group of an orientable surface is generated by only Dehn twists.)
- $\mathcal{M}(N_g^b)$ is not generated by only Dehn twists. Y-homeomorphisms are needed!
- Finite presentations for $\mathcal{M}(N_g^0)$ and $\mathcal{M}(N_g^1)$ ware given by Paris-Szepietowski (arXiv:1308.5856).
  A finite presentation for $\mathcal{M}(N_g^b)$ are not known.

On Torelli groups for non-orientable surfaces

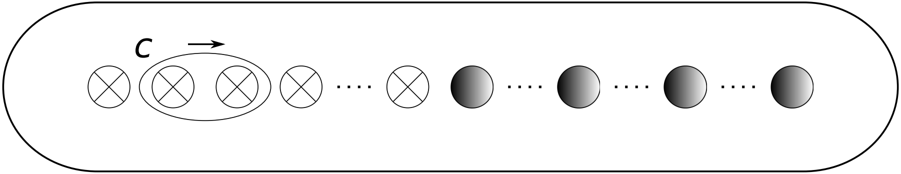
Definition

1. t_c *is a BSCC map* $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c$ *is not connected.*
2. $t_{c_1}t_{c_2}^{-1}$ *is a BP map* $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c_i$ *is connected,* $N_g^b \setminus (c_1 \cup c_2)$ *is not connected and one of its connected components is an orientable surface with two boundary components.*

On Torelli groups for non-orientable surfaces

Definition

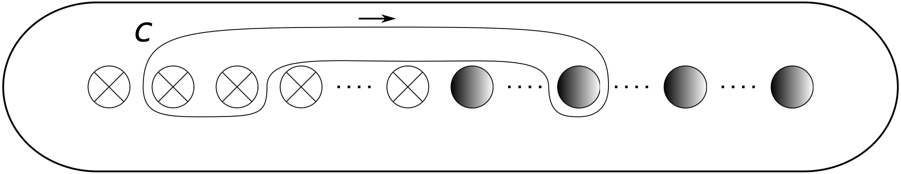
1. t_c *is a BSCC map* $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c$ *is not connected.*
2. $t_{c_1} t_{c_2}^{-1}$ *is a BP map* $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c_i$ *is connected,* $N_g^b \setminus (c_1 \cup c_2)$ *is not connected and one of its connected components is an orientable surface with two boundary components.*



On Torelli groups for non-orientable surfaces

Definition

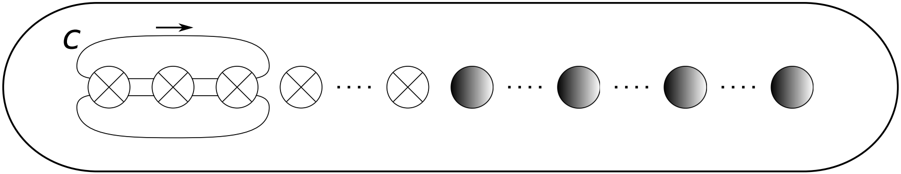
1. t_c *is a BSCC map* $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c$ *is not connected.*
2. $t_{c_1}t_{c_2}^{-1}$ *is a BP map* $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c_i$ *is connected,* $N_g^b \setminus (c_1 \cup c_2)$ *is not connected and one of its connected components is an orientable surface with two boundary components.*



On Torelli groups for non-orientable surfaces

Definition

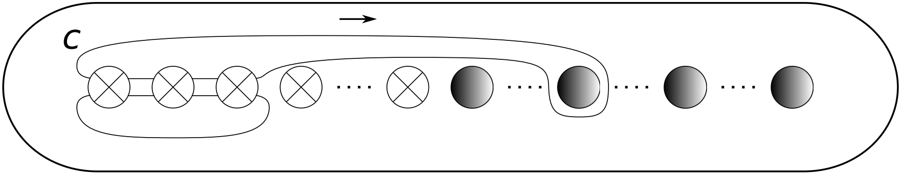
1. *t_c is a BSCC map $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c$ is not connected.*
2. *$t_{c_1} t_{c_2}^{-1}$ is a BP map $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c_i$ is connected, $N_g^b \setminus (c_1 \cup c_2)$ is not connected and one of its connected components is an orientable surface with two boundary components.*



On Torelli groups for non-orientable surfaces

Definition

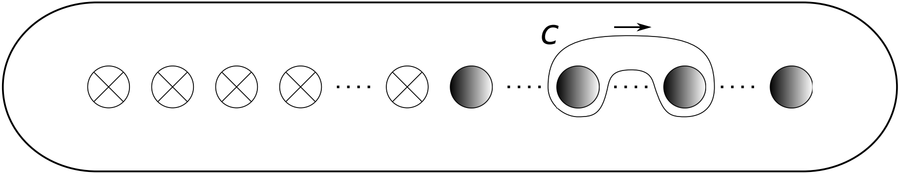
1. *t_c is a BSCC map $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c$ is not connected.*
2. *$t_{c_1}t_{c_2}^{-1}$ is a BP map $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c_i$ is connected, $N_g^b \setminus (c_1 \cup c_2)$ is not connected and one of its connected components is an orientable surface with two boundary components.*



On Torelli groups for non-orientable surfaces

Definition

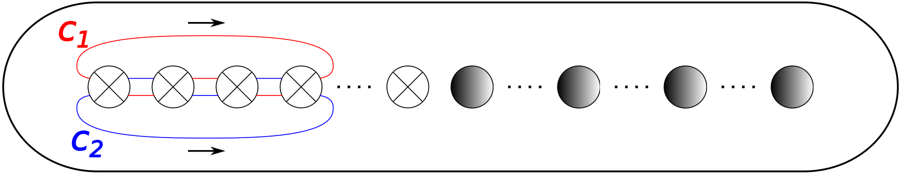
1. t_c *is a BSCC map* $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c$ *is not connected.*
2. $t_{c_1}t_{c_2}^{-1}$ *is a BP map* $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c_i$ *is connected,* $N_g^b \setminus (c_1 \cup c_2)$ *is not connected and one of its connected components is an orientable surface with two boundary components.*



On Torelli groups for non-orientable surfaces

Definition

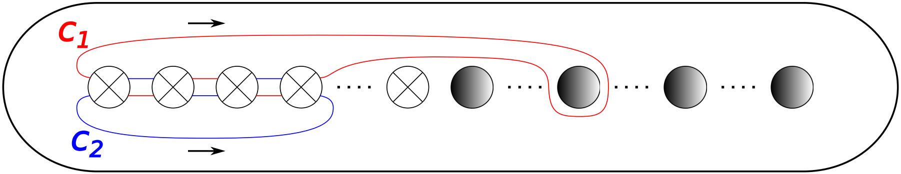
1. *t_c is a BSCC map* $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c$ *is not connected.*
2. *$t_{c_1}t_{c_2}^{-1}$ is a BP map* $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c_i$ *is connected,* $N_g^b \setminus (c_1 \cup c_2)$ *is not connected and one of its connected components is an orientable surface with two boundary components.*



On Torelli groups for non-orientable surfaces

Definition

1. t_c *is a BSCC map* $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c$ *is not connected.*
2. $t_{c_1} t_{c_2}^{-1}$ *is a BP map* $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c_i$ *is connected,* $N_g^b \setminus (c_1 \cup c_2)$ *is not connected and one of its connected components is an orientable surface with two boundary components.*



On Torelli groups for non-orientable surfaces

Definition

1. t_c *is a BSCC map* $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c$ *is not connected.*
2. $t_{c_1}t_{c_2}^{-1}$ *is a BP map* $\underset{\text{def}}{\Longleftrightarrow}$ $N_g^b \setminus c_i$ *is connected,* $N_g^b \setminus (c_1 \cup c_2)$ *is not connected and one of its connected components is an orientable surface with two boundary components.*

Remark

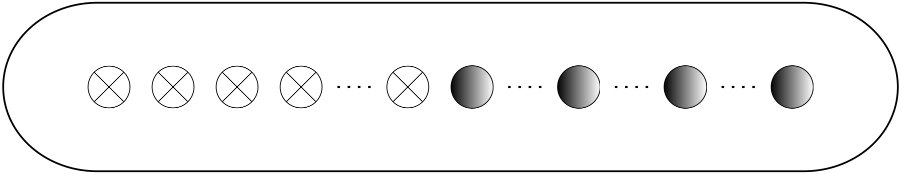
All BSCC maps and BP maps are in $\mathcal{I}(N_g^b)$.

On Torelli groups for non-orientable surfaces

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if* $g = 4$*).*

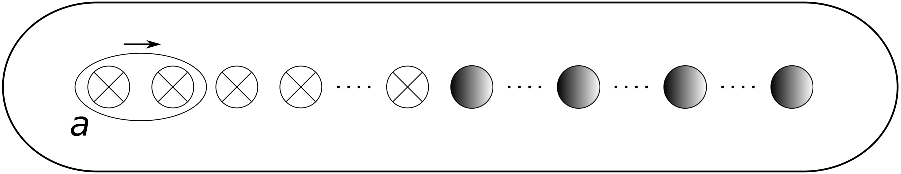


On Torelli groups for non-orientable surfaces

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if $g = 4$).*

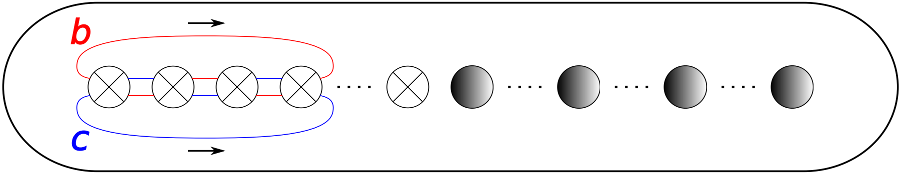


On Torelli groups for non-orientable surfaces

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if $g = 4$).*

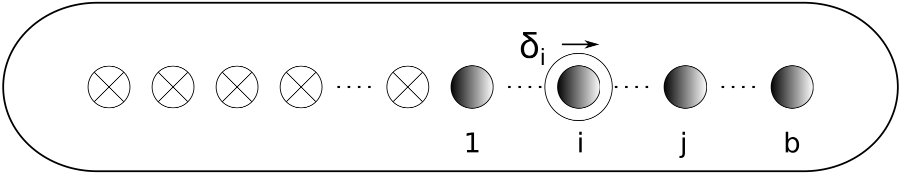


On Torelli groups for non-orientable surfaces

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if $g = 4$).*

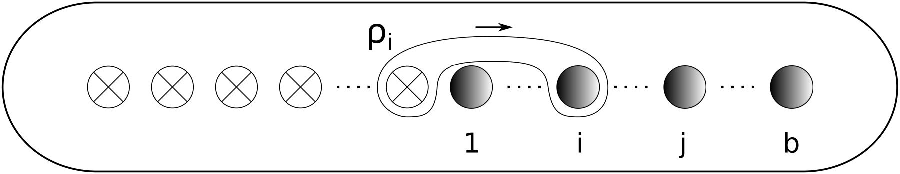


On Torelli groups for non-orientable surfaces

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if $g = 4$).*

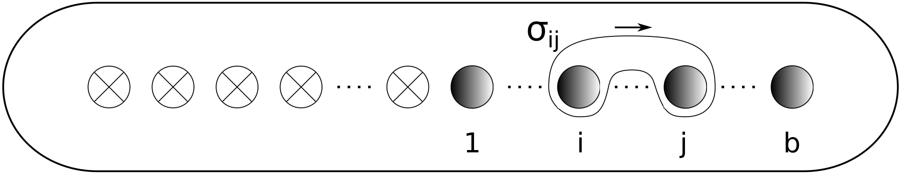


On Torelli groups for non-orientable surfaces

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if $g = 4$).*

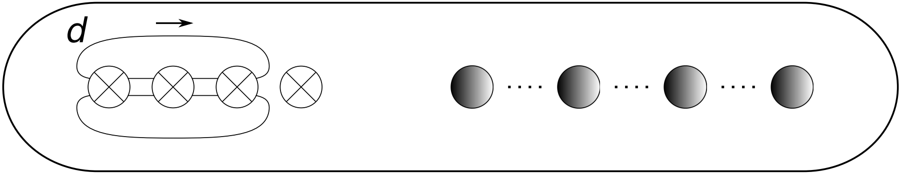


On Torelli groups for non-orientable surfaces

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if $g = 4$).*



On Torelli groups for non-orientable surfaces

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if* $g = 4$*).*

Corollary

$\mathcal{I}(N_g^b)$ is generated by BSCC maps and BP maps.

Capping homomorphisms

$* \in N_g^{b-1}$: a point in the interior of N_g^{b-1}.

$$\mathcal{M}(N_g^{b-1}, *) = \mathrm{Diff}(N_g^{b-1}, \partial N_g^{b-1} \cup \{*\})/\mathrm{isotopy}$$

Capping homomorphisms

$* \in N_g^{b-1}$: a point in the interior of N_g^{b-1}.

$$\mathcal{M}(N_g^{b-1}, *) = \mathrm{Diff}(N_g^{b-1}, \partial N_g^{b-1} \cup \{*\})/\mathrm{isotopy}$$

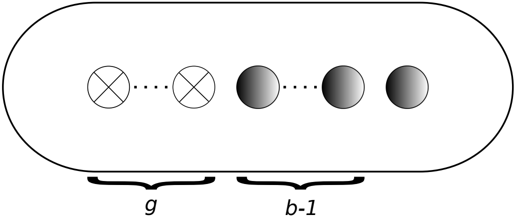
We can regard N_g^b as a subsurface of N_g^{b-1} not containing $*$, by the natural embedding $N_g^b \hookrightarrow N_g^{b-1}$.

Capping homomorphisms

$* \in N_g^{b-1}$: a point in the interior of N_g^{b-1}.

$$\mathcal{M}(N_g^{b-1}, *) = \mathrm{Diff}(N_g^{b-1}, \partial N_g^{b-1} \cup \{*\})/\mathrm{isotopy}$$

We can regard N_g^b as a subsurface of N_g^{b-1} not containing $*$, by the natural embedding $N_g^b \hookrightarrow N_g^{b-1}$.

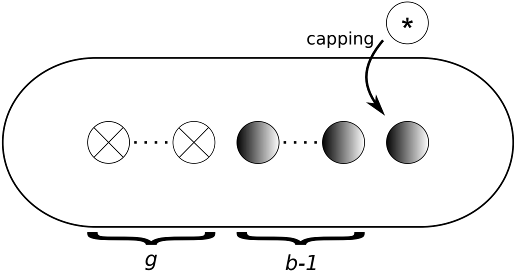


Capping homomorphisms

$* \in N_g^{b-1}$: a point in the interior of N_g^{b-1}.

$$\mathcal{M}(N_g^{b-1}, *) = \mathrm{Diff}(N_g^{b-1}, \partial N_g^{b-1} \cup \{*\})/\text{isotopy}$$

We can regard N_g^b as a subsurface of N_g^{b-1} not containing $*$, by the natural embedding $N_g^b \hookrightarrow N_g^{b-1}$.

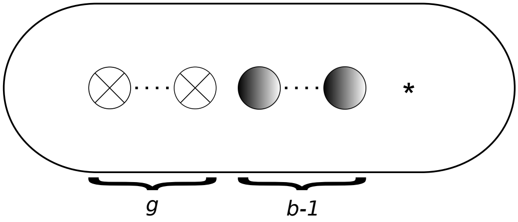


Capping homomorphisms

$* \in N_g^{b-1}$: a point in the interior of N_g^{b-1}.

$$\mathcal{M}(N_g^{b-1}, *) = \mathrm{Diff}(N_g^{b-1}, \partial N_g^{b-1} \cup \{*\})/\mathrm{isotopy}$$

We can regard N_g^b as a subsurface of N_g^{b-1} not containing $*$, by the natural embedding $N_g^b \hookrightarrow N_g^{b-1}$.



Capping homomorphisms

$* \in N_g^{b-1}$: a point in the interior of N_g^{b-1}.

$$\mathcal{M}(N_g^{b-1}, *) = \mathrm{Diff}(N_g^{b-1}, \partial N_g^{b-1} \cup \{*\})/\mathrm{isotopy}$$

We can regard N_g^b as a subsurface of N_g^{b-1} not containing $*$, by the natural embedding $N_g^b \hookrightarrow N_g^{b-1}$.

The capping homomorphism is

$$\mathcal{C}_g^b : \mathcal{M}(N_g^b) \to \mathcal{M}(N_g^{b-1}, *)$$

Capping homomorphisms

$* \in N_g^{b-1}$: a point in the interior of N_g^{b-1}.

$$\mathcal{M}(N_g^{b-1}, *) = \mathrm{Diff}(N_g^{b-1}, \partial N_g^{b-1} \cup \{*\})/\text{isotopy}$$

We can regard N_g^b as a subsurface of N_g^{b-1} not containing $*$, by the natural embedding $N_g^b \hookrightarrow N_g^{b-1}$.

The capping homomorphism is

$$\mathcal{C}_g^b : \mathcal{M}(N_g^b) \to \mathcal{M}(N_g^{b-1}, *)$$

Remark

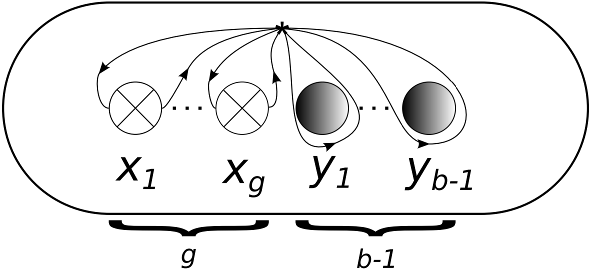
- $\ker \mathcal{C}_g^b$ is generated by t_{δ_b}.
- $\ker \mathcal{C}_g^b|_{\mathcal{I}(N_g^b)}$ is generated by t_{δ_b}.

Pushing and Forgetful homomorphisms

- $\mathcal{P}_g^{b-1} : \pi_1(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1}, *)$: the pushing homomorphism.
- $\mathcal{F}_g^{b-1} : \mathcal{M}(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1})$: the forgetful homomorphism.

Pushing and Forgetful homomorphisms

- $\mathcal{P}_g^{b-1} : \pi_1(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1}, *)$: the pushing homomorphism.
- $\mathcal{F}_g^{b-1} : \mathcal{M}(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1})$: the forgetful homomorphism.



Pushing and Forgetful homomorphisms

- $\mathcal{P}_g^{b-1} : \pi_1(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1}, *)$: the pushing homomorphism.
- $\mathcal{F}_g^{b-1} : \mathcal{M}(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1})$: the forgetful homomorphism.

$$\mathrm{Diff}(N_g^{b-1}, \partial N_g^{b-1} \cup \{*\}) \to \mathrm{Diff}(N_g^{b-1}, \partial N_g^{b-1})$$

Pushing and Forgetful homomorphisms

- $\mathcal{P}_g^{b-1} : \pi_1(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1}, *)$: the pushing homomorphism.
- $\mathcal{F}_g^{b-1} : \mathcal{M}(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1})$: the forgetful homomorphism.

Remark

We have short exact sequences

$$\begin{array}{ccccccc} \pi_1(N_g^{b-1}, *) & \xrightarrow{\mathcal{P}_g^{b-1}} & \mathcal{M}(N_g^{b-1}, *) & \xrightarrow{\mathcal{F}_g^{b-1}} & \mathcal{M}(N_g^{b-1}) & \longrightarrow & 1, \\ \pi_1(N_g^{b-1}, *) & \longrightarrow & \mathcal{I}(N_g^{b-1}, *) & \longrightarrow & \mathcal{I}(N_g^{b-1}) & \longrightarrow & 1. \end{array}$$

Outline of Proof

$$
\begin{array}{ccccccc}
 & & \mathcal{I}(N_g^b) & & & & \\
 & & \downarrow & & & & \\
\pi_1(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1}) & \to & 1
\end{array}
$$

Outline of Proof

$$\begin{array}{ccccccc} & & \mathcal{I}(N_g^b) & & & & \\ & & \downarrow & & & & \\ \pi_1(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1}) & \to & 1 \end{array}$$

$\mathcal{I}(N_g^{b-1},*)$ is normally generated by

- lifts of normal generators of $\mathcal{I}(N_g^{b-1})$ and
- the images of the generators of $\pi_1(N_g^{b-1},*)$.

Outline of Proof

$$\begin{array}{ccccccc} & & \mathcal{I}(N_g^b) & & & & \\ & & \downarrow & & & & \\ \pi_1(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1}) & \to & 1 \end{array}$$

$\mathcal{I}(N_g^{b-1},*)$ is normally generated by

- lifts of normal generators of $\mathcal{I}(N_g^{b-1})$ and
- the images of the generators of $\pi_1(N_g^{b-1},*)$.

$\mathcal{I}(N_g^b)$ is normally generated by

- lifts of normal generators of $\mathcal{C}_g^b(\mathcal{I}(N_g^b))$ and
- $\ker(\mathcal{C}_g^b|_{\mathcal{I}(N_g^b)} : \mathcal{I}(N_g^b) \to \mathcal{I}(N_g^{b-1},*))$.

Outline of Proof

$$\begin{array}{ccccccc} & & \mathcal{I}(N_g^b) & & & & \\ & & \downarrow & & & & \\ \pi_1(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1}) & \to & 1 \end{array}$$

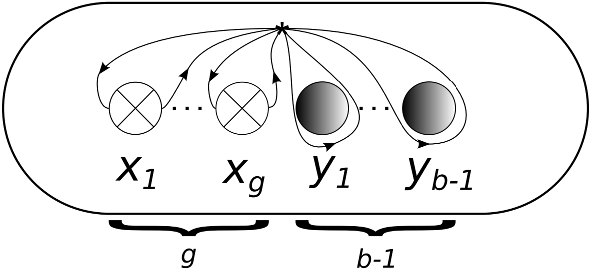
$\mathcal{I}(N_g^{b-1},*)$ is normally generated by

- lifts of normal generators of $\mathcal{I}(N_g^{b-1})$ and
- the images of the generators of $\pi_1(N_g^{b-1},*)$.

$\mathcal{I}(N_g^b)$ is normally generated by

- lifts of normal generators of $\mathcal{C}_g^b(\mathcal{I}(N_g^b))$ and
- t_{δ_b}.

x_i, y_j : the generators of $\pi_1(N_g^{b-1}, *)$.



x_i, y_j : the generators of $\pi_1(N_g^{b-1},*)$.
$p:\pi_1(N_g^{b-1},*)\to\pi_1(N_g^0,*)$: the projection (w/ $p(x_i)=x_i$, $p(y_j)=1$).

x_i, y_j : the generators of $\pi_1(N_g^{b-1}, *)$.
$p : \pi_1(N_g^{b-1}, *) \to \pi_1(N_g^0, *)$: the projection (w/ $p(x_i) = x_i$, $p(y_j) = 1$).
For $x \in \pi_1(N_g^{b-1}, *)$ we can denote $p(x) = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_t}^{\varepsilon_t}$
($\varepsilon_k = \pm 1$, t is the word length of $p(x)$).

x_i, y_j : the generators of $\pi_1(N_g^{b-1}, *)$.
$p : \pi_1(N_g^{b-1}, *) \to \pi_1(N_g^0, *)$: the projection (w/ $p(x_i) = x_i$, $p(y_j) = 1$).
For $x \in \pi_1(N_g^{b-1}, *)$ we can denote $p(x) = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_t}^{\varepsilon_t}$
($\varepsilon_k = \pm 1$, t is the word length of $p(x)$).

$$\begin{aligned} O_i(x) &= \sharp\{i_{2k-1} \mid i_{2k-1} = i\}, \\ E_i(x) &= \sharp\{i_{2k} \mid i_{2k} = i\}, \\ \Gamma_g^{b-1} &= \{x \in \pi_1(N_g^{b-1}, *) \mid O_i(x) = E_i(x), 1 \leq i \leq g\} \end{aligned}$$

x_i, y_j : the generators of $\pi_1(N_g^{b-1}, *)$.
$p : \pi_1(N_g^{b-1}, *) \to \pi_1(N_g^0, *)$: the projection (w/ $p(x_i) = x_i$, $p(y_j) = 1$).
For $x \in \pi_1(N_g^{b-1}, *)$ we can denote $p(x) = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_t}^{\varepsilon_t}$
($\varepsilon_k = \pm 1$, t is the word length of $p(x)$).

$$\begin{aligned} O_i(x) &= \sharp\{i_{2k-1} \mid i_{2k-1} = i\}, \\ E_i(x) &= \sharp\{i_{2k} \mid i_{2k} = i\}, \\ \Gamma_g^{b-1} &= \{x \in \pi_1(N_g^{b-1}, *) \mid O_i(x) = E_i(x), 1 \leq i \leq g\} \end{aligned}$$

Example

$x_1 y_2 x_2 x_3^{-1} y_5 y_1^{-2} x_1 x_2^{-1} y_4^3 x_3^{-1} \Longrightarrow p(x) = x_1 x_2 x_3^{-1} x_1 x_2^{-1} x_3^{-1}$.
$O_1(x) = E_1(x) = 1$, $O_2(x) = E_2(x) = 1$, $O_3(x) = E_3(x) = 1$,
$O_i(x) = E_i(x) = 0$ $(i \geq 4)$.
$\rightsquigarrow x \in \Gamma_g^{b-1}$.

x_i, y_j : the generators of $\pi_1(N_g^{b-1}, *)$.
$p : \pi_1(N_g^{b-1}, *) \to \pi_1(N_g^0, *)$: the projection (w/ $p(x_i) = x_i$, $p(y_j) = 1$).
For $x \in \pi_1(N_g^{b-1}, *)$ we can denote $p(x) = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_t}^{\varepsilon_t}$
($\varepsilon_k = \pm 1$, t is the word length of $p(x)$).

$$\begin{aligned} O_i(x) &= \sharp\{i_{2k-1} \mid i_{2k-1} = i\}, \\ E_i(x) &= \sharp\{i_{2k} \mid i_{2k} = i\}, \\ \Gamma_g^{b-1} &= \{x \in \pi_1(N_g^{b-1}, *) \mid O_i(x) = E_i(x), 1 \leq i \leq g\} \end{aligned}$$

Example

$x_1 y_2 x_2 x_3^{-1} y_5 y_1^{-2} x_1 x_2^{-1} y_4^3 x_3^{-1} \Longrightarrow p(x) = x_1 x_2 x_3^{-1} x_1 x_2^{-1} x_3^{-1}$.
$O_1(x) = E_1(x) = 1$, $O_2(x) = E_2(x) = 1$, $O_3(x) = E_3(x) = 1$,
$O_i(x) = E_i(x) = 0$ $(i \geq 4)$.
$\rightsquigarrow x \in \Gamma_g^{b-1}$.

x_i, y_j : the generators of $\pi_1(N_g^{b-1},*)$.
$p:\pi_1(N_g^{b-1},*)\to\pi_1(N_g^0,*)$: the projection (w/ $p(x_i)=x_i$, $p(y_j)=1$).
For $x\in\pi_1(N_g^{b-1},*)$ we can denote $p(x)=x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}\cdots x_{i_t}^{\varepsilon_t}$
($\varepsilon_k=\pm1$, t is the word length of $p(x)$).

$$
\begin{aligned}
O_i(x) &= \sharp\{i_{2k-1}\mid i_{2k-1}=i\},\\
E_i(x) &= \sharp\{i_{2k}\mid i_{2k}=i\},\\
\Gamma_g^{b-1} &= \{x\in\pi_1(N_g^{b-1},*)\mid O_i(x)=E_i(x), 1\le i\le g\}
\end{aligned}
$$

Example

$x_1y_2x_2x_3^{-1}y_5y_1^{-2}x_1x_2^{-1}y_4^3x_3^{-1}\Longrightarrow p(x)=x_1x_2x_3^{-1}x_1x_2^{-1}x_3^{-1}$.
$O_1(x)=E_1(x)=1$, $O_2(x)=E_2(x)=1$, $O_3(x)=E_3(x)=1$,
$O_i(x)=E_i(x)=0$ $(i\ge4)$.
$\rightsquigarrow x\in\Gamma_g^{b-1}$.

x_i, y_j : the generators of $\pi_1(N_g^{b-1}, *)$.
$p : \pi_1(N_g^{b-1}, *) \to \pi_1(N_g^0, *)$: the projection (w/ $p(x_i) = x_i$, $p(y_j) = 1$).
For $x \in \pi_1(N_g^{b-1}, *)$ we can denote $p(x) = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_t}^{\varepsilon_t}$
($\varepsilon_k = \pm 1$, t is the word length of $p(x)$).

$$\begin{aligned} O_i(x) &= \sharp\{i_{2k-1} \mid i_{2k-1} = i\}, \\ E_i(x) &= \sharp\{i_{2k} \mid i_{2k} = i\}, \\ \Gamma_g^{b-1} &= \{x \in \pi_1(N_g^{b-1}, *) \mid O_i(x) = E_i(x), 1 \leq i \leq g\} \end{aligned}$$

Example

$x_1 y_2 x_2 x_3^{-1} y_5 y_1^{-2} x_1 x_2^{-1} y_4^3 x_3^{-1} \Longrightarrow p(x) = x_1 x_2 x_3^{-1} x_1 x_2^{-1} x_3^{-1}$.
$O_1(x) = E_1(x) = 1$, $O_2(x) = E_2(x) = 1$, $O_3(x) = E_3(x) = 1$,
$O_i(x) = E_i(x) = 0$ $(i \geq 4)$.
$\rightsquigarrow x \in \Gamma_g^{b-1}$.

x_i, y_j : the generators of $\pi_1(N_g^{b-1}, *)$.
$p : \pi_1(N_g^{b-1}, *) \to \pi_1(N_g^0, *)$: the projection (w/ $p(x_i) = x_i$, $p(y_j) = 1$).
For $x \in \pi_1(N_g^{b-1}, *)$ we can denote $p(x) = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_t}^{\varepsilon_t}$
($\varepsilon_k = \pm 1$, t is the word length of $p(x)$).

$$\begin{aligned} O_i(x) &= \sharp\{i_{2k-1} \mid i_{2k-1} = i\}, \\ E_i(x) &= \sharp\{i_{2k} \mid i_{2k} = i\}, \\ \Gamma_g^{b-1} &= \{x \in \pi_1(N_g^{b-1}, *) \mid O_i(x) = E_i(x), 1 \leq i \leq g\} \end{aligned}$$

Example

$x_1 y_2 x_2 x_3^{-1} y_5 y_1^{-2} x_1 x_2^{-1} y_4^3 x_3^{-1} \Longrightarrow p(x) = x_1 x_2 x_3^{-1} x_1 x_2^{-1} x_3^{-1}$.
$O_1(x) = E_1(x) = 1$, $O_2(x) = E_2(x) = 1$, $O_3(x) = E_3(x) = 1$,
$O_i(x) = E_i(x) = 0$ $(i \geq 4)$.
$\rightsquigarrow x \in \Gamma_g^{b-1}$.

x_i, y_j : the generators of $\pi_1(N_g^{b-1}, *)$.
$p : \pi_1(N_g^{b-1}, *) \to \pi_1(N_g^0, *)$: the projection (w/ $p(x_i) = x_i$, $p(y_j) = 1$).
For $x \in \pi_1(N_g^{b-1}, *)$ we can denote $p(x) = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_t}^{\varepsilon_t}$
($\varepsilon_k = \pm 1$, t is the word length of $p(x)$).

$$\begin{aligned} O_i(x) &= \sharp\{i_{2k-1} \mid i_{2k-1} = i\}, \\ E_i(x) &= \sharp\{i_{2k} \mid i_{2k} = i\}, \\ \Gamma_g^{b-1} &= \{x \in \pi_1(N_g^{b-1}, *) \mid O_i(x) = E_i(x), 1 \leq i \leq g\} \end{aligned}$$

Example

$x_1 y_2 x_2 x_3^{-1} y_5 y_1^{-2} x_1 x_2^{-1} y_4^3 x_3^{-1} \Longrightarrow p(x) = x_1 x_2 x_3^{-1} x_1 x_2^{-1} x_3^{-1}$.
$O_1(x) = E_1(x) = 1$, $O_2(x) = E_2(x) = 1$, $O_3(x) = E_3(x) = 1$,
$O_i(x) = E_i(x) = 0$ $(i \geq 4)$.
$\rightsquigarrow x \in \Gamma_g^{b-1}$.

x_i, y_j : the generators of $\pi_1(N_g^{b-1},*)$.
$p : \pi_1(N_g^{b-1},*) \to \pi_1(N_g^0,*)$: the projection (w/ $p(x_i)=x_i$, $p(y_j)=1$).
For $x \in \pi_1(N_g^{b-1},*)$ we can denote $p(x)=x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}\cdots x_{i_t}^{\varepsilon_t}$
($\varepsilon_k = \pm 1$, t is the word length of $p(x)$).

$$\begin{aligned} O_i(x) &= \sharp\{i_{2k-1} \mid i_{2k-1}=i\}, \\ E_i(x) &= \sharp\{i_{2k} \mid i_{2k}=i\}, \\ \Gamma_g^{b-1} &= \{x \in \pi_1(N_g^{b-1},*) \mid O_i(x)=E_i(x), 1 \le i \le g\} \end{aligned}$$

Example

$x_1y_2x_2x_3^{-1}y_5y_1^{-2}x_1x_2^{-1}y_4^3x_3^{-1} \Longrightarrow p(x)=x_1x_2x_3^{-1}x_1x_2^{-1}x_3^{-1}$.
$O_1(x)=E_1(x)=1$, $O_2(x)=E_2(x)=1$, $O_3(x)=E_3(x)=1$,
$O_i(x)=E_i(x)=0$ $(i \ge 4)$.
$\rightsquigarrow x \in \Gamma_g^{b-1}$.

$\mathcal{C}_g^b : \mathcal{M}(N_g^b) \to \mathcal{M}(N_g^{b-1}, *)$
$\mathcal{P}_g^{b-1} : \pi_1(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1}, *)$
$\mathcal{F}_g^{b-1} : \mathcal{M}(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1})$

$\mathcal{C}_g^b : \mathcal{M}(N_g^b) \to \mathcal{M}(N_g^{b-1}, *)$
$\mathcal{P}_g^{b-1} : \pi_1(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1}, *)$
$\mathcal{F}_g^{b-1} : \mathcal{M}(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1})$

Proposition

$\mathcal{C}_g^b(\mathcal{I}(N_g^b))$ is the normal closure of $\mathcal{P}_g^{b-1}(\Gamma_g^{b-1})$ and lifts by $\mathcal{F}_g^{b-1}$ of normal generators of $\mathcal{I}(N_g^{b-1})$.

$\mathcal{C}_g^b : \mathcal{M}(N_g^b) \to \mathcal{M}(N_g^{b-1}, *)$
$\mathcal{P}_g^{b-1} : \pi_1(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1}, *)$
$\mathcal{F}_g^{b-1} : \mathcal{M}(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1})$

Proposition

$\mathcal{C}_g^b(\mathcal{I}(N_g^b))$ is the normal closure of $\mathcal{P}_g^{b-1}(\Gamma_g^{b-1})$ and lifts by $\mathcal{F}_g^{b-1}$ of normal generators of $\mathcal{I}(N_g^{b-1})$.

Proposition

$\mathcal{P}_g^{b-1}(\Gamma_g^{b-1})$ is the normal closure of $\mathcal{P}_g^{b-1}(x_g^2)$ and $\mathcal{P}_g^{b-1}(y_j)$ $(1 \leq j \leq b-1)$.

$\mathcal{C}_g^b : \mathcal{M}(N_g^b) \to \mathcal{M}(N_g^{b-1}, *)$
$\mathcal{P}_g^{b-1} : \pi_1(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1}, *)$
$\mathcal{F}_g^{b-1} : \mathcal{M}(N_g^{b-1}, *) \to \mathcal{M}(N_g^{b-1})$

Proposition

$\mathcal{C}_g^b(\mathcal{I}(N_g^b))$ is the normal closure of $\mathcal{P}_g^{b-1}(\Gamma_g^{b-1})$ and lifts by $\mathcal{F}_g^{b-1}$ of normal generators of $\mathcal{I}(N_g^{b-1})$.

Proposition

$\mathcal{P}_g^{b-1}(\Gamma_g^{b-1})$ is the normal closure of $\mathcal{P}_g^{b-1}(x_g^2)$ and $\mathcal{P}_g^{b-1}(y_j)$ $(1 \leq j \leq b-1)$.

Corollary

$\mathcal{C}_g^b(\mathcal{I}(N_g^b))$ is normally generated by $\mathcal{P}_g^{b-1}(x_g^2)$, $\mathcal{P}_g^{b-1}(y_j)$ $(1 \leq j \leq b-1)$ and lifts by $\mathcal{F}_g^{b-1}$ of normal generators of $\mathcal{I}(N_g^{b-1})$.

$$\begin{array}{ccccccc} & & \mathcal{I}(N_g^b) & & & & \\ & & \downarrow & & & & \\ \pi_1(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1}) & \to & 1 \end{array}$$

$\mathcal{I}(N_g^b)$ is normally generated by

- t_{δ_b} and
- lifts by $\mathcal{C}_g^b$ of normal generators of $\mathcal{C}_g^b(\mathcal{I}(N_g^b))$.

$$\begin{array}{ccccccc} & & \mathcal{I}(N_g^b) & & & & \\ & & \downarrow & & & & \\ \pi_1(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1}) & \to & 1 \end{array}$$

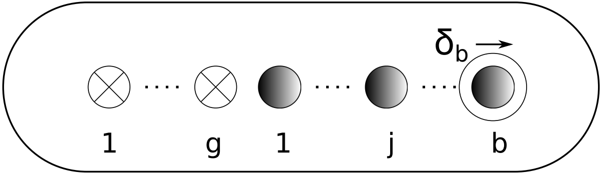
$\mathcal{I}(N_g^b)$ is normally generated by

- t_{δ_b} and
- lifts by $\mathcal{C}_g^b$ of
  - $\mathcal{P}_g^{b-1}(x_g^2)$, $\mathcal{P}_g^{b-1}(y_j)$ $(1 \leq j \leq b-1)$ and
  - lifts by $\mathcal{F}_g^{b-1}$ of normal generators of $\mathcal{I}(N_g^{b-1})$.

$$\begin{array}{ccccccc} & & \mathcal{I}(N_g^b) & & & & \\ & & \downarrow & & & & \\ \pi_1(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1}) & \to & 1 \end{array}$$

$\mathcal{I}(N_g^b)$ is normally generated by

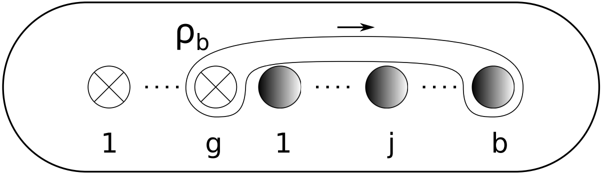
- t_{δ_b} and
- lifts by $\mathcal{C}_g^b$ of
  - $\mathcal{P}_g^{b-1}(x_g^2)$, $\mathcal{P}_g^{b-1}(y_j)$ $(1 \leq j \leq b-1)$ and
  - lifts by $\mathcal{F}_g^{b-1}$ of normal generators of $\mathcal{I}(N_g^{b-1})$.



$$\begin{array}{ccccccc} & & \mathcal{I}(N_g^b) & & & & \\ & & \downarrow & & & & \\ \pi_1(N_g^{b-1}, *) & \to & \mathcal{I}(N_g^{b-1}, *) & \to & \mathcal{I}(N_g^{b-1}) & \to & 1 \end{array}$$

$\mathcal{I}(N_g^b)$ is normally generated by

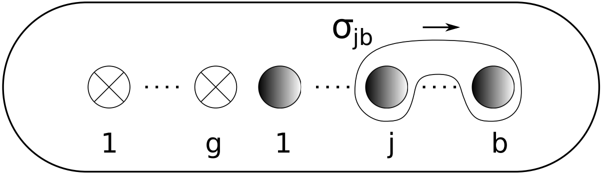
- t_{δ_b} and
- lifts by $\mathcal{C}_g^b$ of
  - $\mathcal{P}_g^{b-1}(x_g^2)$, $\mathcal{P}_g^{b-1}(y_j)$ $(1 \leq j \leq b-1)$ and
  - lifts by $\mathcal{F}_g^{b-1}$ of normal generators of $\mathcal{I}(N_g^{b-1})$.



$$\begin{array}{ccccccc} & & \mathcal{I}(N_g^b) & & & & \\ & & \downarrow & & & & \\ \pi_1(N_g^{b-1}, *) & \to & \mathcal{I}(N_g^{b-1}, *) & \to & \mathcal{I}(N_g^{b-1}) & \to & 1 \end{array}$$

$\mathcal{I}(N_g^b)$ is normally generated by

- t_{δ_b} and
- lifts by $\mathcal{C}_g^b$ of
  - $\mathcal{P}_g^{b-1}(x_g^2)$, $\mathcal{P}_g^{b-1}(y_j)$ $(1 \leq j \leq b-1)$ and
  - lifts by $\mathcal{F}_g^{b-1}$ of normal generators of $\mathcal{I}(N_g^{b-1})$.



$$\begin{array}{ccccccc} & & \mathcal{I}(N_g^b) & & & & \\ & & \downarrow & & & & \\ \pi_1(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1},*) & \to & \mathcal{I}(N_g^{b-1}) & \to & 1 \end{array}$$

$\mathcal{I}(N_g^b)$ is normally generated by

- t_{δ_b},
- t_{ρ_b}, $t_{\sigma_{jb}}$,
- lifts by $\mathcal{F}_g^{b-1} \circ \mathcal{C}_g^b$ of normal generators of $\mathcal{I}(N_g^{b-1})$.

Theorem (Hirose-K. (arXiv:1412.2222))

For $g \geq 4$, $\mathcal{I}(N_g^0)$ is normally generated by t_a, $t_b t_c^{-1}$ (and t_d).

Theorem (Hirose-K. (arXiv:1412.2222))

For $g \geq 4$, $\mathcal{I}(N_g^0)$ is normally generated by t_a, $t_b t_c^{-1}$ (and t_d).

$\mathcal{I}(N_g^1)$ is normally generated by t_a, $t_b t_c^{-1}$, t_{δ_1}, t_{ρ_1} (and t_d).

Theorem (Hirose-K. (arXiv:1412.2222))

For $g \geq 4$, $\mathcal{I}(N_g^0)$ is normally generated by t_a, $t_b t_c^{-1}$ (and t_d).

$\mathcal{I}(N_g^1)$ is normally generated by t_a, $t_b t_c^{-1}$, t_{δ_1}, t_{ρ_1} (and t_d).
$\mathcal{I}(N_g^2)$ is normally generated by t_a, $t_b t_c^{-1}$, t_{δ_1}, t_{ρ_1}, t_{δ_2}, t_{ρ_2}, $t_{\sigma_{12}}$ (and t_d).

Theorem (Hirose-K. (arXiv:1412.2222))

For $g \geq 4$, $\mathcal{I}(N_g^0)$ is normally generated by t_a, $t_b t_c^{-1}$ (and t_d).

$\mathcal{I}(N_g^1)$ is normally generated by t_a, $t_b t_c^{-1}$, t_{δ_1}, t_{ρ_1} (and t_d).
$\mathcal{I}(N_g^2)$ is normally generated by t_a, $t_b t_c^{-1}$, t_{δ_1}, t_{ρ_1}, t_{δ_2}, t_{ρ_2}, $t_{\sigma_{12}}$ (and t_d).

$\vdots$

Theorem (Hirose-K. (arXiv:1412.2222))

For $g \geq 4$, $\mathcal{I}(N_g^0)$ is normally generated by t_a, $t_b t_c^{-1}$ (and t_d).

$\mathcal{I}(N_g^1)$ is normally generated by t_a, $t_b t_c^{-1}$, t_{δ_1}, t_{ρ_1} (and t_d).
$\mathcal{I}(N_g^2)$ is normally generated by t_a, $t_b t_c^{-1}$, t_{δ_1}, t_{ρ_1}, t_{δ_2}, t_{ρ_2}, $t_{\sigma_{12}}$ (and t_d).

$\vdots$

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b)$ *and*
- t_d *(only if $g = 4$).*

Theorem (Hirose-K. (arXiv:1412.2222))

For $g \geq 4$, $\mathcal{I}(N_g^0)$ is normally generated by t_a, $t_b t_c^{-1}$ (and t_d).

$\mathcal{I}(N_g^1)$ is normally generated by t_a, $t_b t_c^{-1}$, t_{δ_1}, t_{ρ_1} (and t_d).
$\mathcal{I}(N_g^2)$ is normally generated by t_a, $t_b t_c^{-1}$, t_{δ_1}, t_{ρ_1}, t_{δ_2}, t_{ρ_2}, $t_{\sigma_{12}}$ (and t_d).

$\vdots$

Theorem (K.)

For $g \geq 4$ and $b \geq 0$, $\mathcal{I}(N_g^b)$ is normally generated by

- t_a, $t_b t_c^{-1}$,
- t_{δ_i}, t_{ρ_i} $(1 \leq i \leq b-1)$,
- $t_{\sigma_{ij}}$ $(1 \leq i < j \leq b-1)$ *and*
- t_d *(only if $g = 4$).*

Thank you for your attention!