Combination of Lorentzian transformation groups

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Topics

Introduction

Question!

Consider properly discontinuous actions onto $(E^2_1, \mathbb{R}^2_1)$, which is not cocompact.

Question

When is an action on $(E^2_1, \mathbb{R}^2_1)$ properly discontinuous ?

Only the case

$$\mathbb{F}_n \cong \pi_1(S_{g,b})$$

Definition

A group Γ acts a topology space X *properly discontinuously* if for any compact set K in X, $\sharp\{\gamma \in \Gamma \mid \gamma(K) \cap K \neq \emptyset\} < \infty$.

Question!

Question'

When is an action of $\pi_1(S_{g,b})$ on E_1^2 properly discontinuously ?

- Linear action : $\pi_1(S_{g,b}) \hookrightarrow \mathrm{PSL}(2,\mathbb{R}) \cong SO^0(2,1)$: **not** properly discontinuous
- Affine action : $\pi_1(S_{g,b}) \hookrightarrow SO^0(2,1) \ltimes \mathbb{R}_1^2$: **can** be properly discontinuous

Goal

Study the deformation space of these actions

Lorentzian spacetime and Margulis spacetime

Lorentzian spacetime

Definition

$(\mathbb{R}^2_1, B)$: $(2+1)$-dim. Lorentzian spacetime

- $\mathbb{R}^2_1$: 3-dim. vector space
- B : Lorentzian inner product

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2_1$, we can represent as:

$$B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \mathbf{y}.$$

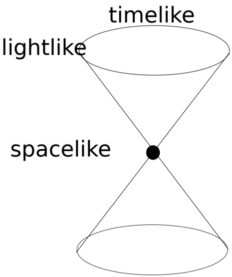
Lorentzian spacetime

For a vector $\mathbf{x} \in \mathbb{R}^2_1$, we say

- *spacelike* $\Leftrightarrow B(\mathbf{x}, \mathbf{x}) > 0$,
- *lightlike* or *null* $\Leftrightarrow B(\mathbf{x}, \mathbf{x}) = 0$,
- *timelike* $\Leftrightarrow B(\mathbf{x}, \mathbf{x}) < 0$.

Definition

A *light cone* (or *null cone*) is the set of lightlike vectors.



Action

Lorentzian transformation group

$$O(2,1) > SO(2,1) > SO^0(2,1).$$

1. $O(2,1)$: preserve B,
2. $SO(2,1)$: also preserve the orientation of $\mathbb{R}^2_1$,
3. $SO^0(2,1)$: also preserve the orientation of timelike vectors.

$\mathrm{PSL}(2,\mathbb{R}) \cong SO^0(2,1)$

Consider a Klein model disk ; put a disk in the light cone: namely,

$$\mathbb{H}^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^2_1 \mid x_1^2 + x_2^2 < 1, x_3 = 1/\sqrt{2}\}.$$

Then

$$\mathrm{PSL}(2,\mathbb{R}) \cong SO^0(2,1)$$

$$\begin{array}{ccc} \mathbb{R}^2_1 & \xrightarrow{SO^0(2,1)} & \mathbb{R}^2_1 \\ {\scriptstyle proj.}\downarrow & & \downarrow{\scriptstyle proj.} \\ \mathbb{H}^2 & \xrightarrow{\mathrm{PSL}(2,\mathbb{R})} & \mathbb{H}^2 \end{array}$$

Definition

Through this correspondence, we call an element of $SO^0(2,1)$ *hyperbolic*, *parabolic*, or *elliptic*.

Basis

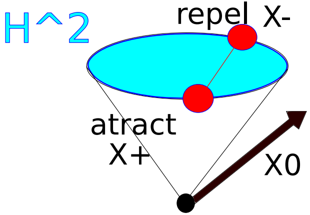
Definition

An ordered triple $\{\mathbf{X}^0, \mathbf{X}^-, \mathbf{X}^+\}$ is a *null frame* if

1. $B(\mathbf{X}^0, \mathbf{X}^0) = 1,\ B(\mathbf{X}^0, \mathbf{X}^\pm) = 0$,
2. $\mathbf{X}^\pm$ have each endpoint on $\partial\mathbb{H}^2$,
3. $det[\mathbf{X}^0, \mathbf{X}^-, \mathbf{X}^+] > 0$.

In fact, $\mathbf{X}^0$ determines $\mathbf{X}^\pm$ uniquely.

A null frame is a base of $\mathbb{R}^2_1$.



Base and null frame

Take a hyperbolic element g

- as $\mathrm{PSL}(2,\mathbb{R})$:
  g has two fixed points in $\partial\mathbb{H}^2$.
- as $SO^0(2,1)$:
  g has three eigenvectors in $\mathbb{R}^2_1$.

eigenvectors

- $\mathbf{X}_g^0$: corresponds to the invariant axis of g in $\mathbb{H}^2$,
- $\mathbf{X}_g^-$: corresponds to the repelling point in $\partial\mathbb{H}^2$,
- $\mathbf{X}_g^+$: corresponds to the attracting point in $\partial\mathbb{H}^2$.

The triple $\{\mathbf{X}_g^0, \mathbf{X}_g^-, \mathbf{X}_g^+\}$ is a null frame.

Minkowski spacetime

Definition

$(E_1^2, \mathbb{R}_1^2, B)$: $(2+1)$-dim. Minkowski spacetime

- $(E_1^2, \mathbb{R}_1^2)$: affine space
- $(\mathbb{R}_1^2, B)$: Lorenzian spacetime

(affine) transformation group acting E_1^2

- $O(2,1) \ltimes \mathbb{R}_1^2$: the transformation group preserving B,
- $SO^0(2,1) \ltimes \mathbb{R}_1^2 < O(2,1) \ltimes \mathbb{R}_1^2$.

Margulis spacetime

$\Gamma < O(2,1) \ltimes \mathbb{R}^2_1$: discrete subgroup

$\Gamma \curvearrowright E^2_1$: pro. dis. and free $\Rightarrow E^2_1/\mathbb{R}^2_1$ is a 3-dim. manifold.

In particular,

Definition

Let Γ be isomorphic to a free group. Then we say

- the transf. grp. Γ is a *Margulis group*,
- the quotient manifold E^2_1/Γ is a *Magulis manifold*.

Affine deformation

deformation

$$\begin{array}{ccc} SO(2,1)^0 & \xrightarrow{\rho_{\mathbf{u}}} & SO(2,1)^0 \ltimes \mathbb{R}^2_1 \\ \rho_0 \uparrow & & \downarrow proj. \\ \Gamma & \xrightarrow{\mathbf{u}} & \mathbb{R}^2_1 \end{array}$$

- ρ_0 : holonomy (injective and discrete homomorphism), fixed
- $\rho_{\mathbf{u}}$: affine deformation
- $\mathbf{u}$: cocycle

Consider that $\Gamma = \pi_1(S_{g,b})$.

Let $S_{g,b}$ denote a hyperbolic surface with boundaries.

cocycles

Definition

A map $\mathbf{u} : \Gamma \to \mathbb{R}^2_1$ is a *cocycle* if, for $\forall g, h \in \Gamma$,

$$\mathbf{u}(gh) = g\mathbf{u}(h) + \mathbf{u}(g).$$

Definition

($\mathbf{v} \in \mathbb{R}^2_1$) Call a cocycle $\delta_{\mathbf{v}}$ a *coboundary* [a], which is represented by,

$$\delta_{\mathbf{v}}(g) = \mathbf{v} - g\mathbf{v}.$$

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[a] corresponding to a conjugation with translation $\mathbf{v}$

cohomology : ‘deformation space’

- $\mathrm{Z}^1(\Gamma, \mathbb{R}^2_1)$: a set of cocycles,
- $\mathrm{B}^1(\Gamma, \mathbb{R}^2_1)$: a set of coboundaries,

Definition (cohomology)

$$\begin{aligned} \mathrm{H}^1(\Gamma, \mathbb{R}^2_1) &:= \mathrm{Z}^1(\Gamma, \mathbb{R}^2_1)/\mathrm{B}^1(\Gamma, \mathbb{R}^2_1) \\ &= \{[\mathbf{u}] \mid \mathbf{u} \in \mathrm{Z}^1(\Gamma, \mathbb{R}^2_1)\} \end{aligned}$$

affine deformation

Definition(affine deformation)

$$\rho_{\mathbf{u}} : \Gamma = \pi_1(S_{g,b}) \ni g \mapsto (g, \mathbf{u}(g)) \in SO^0(2,1) \ltimes \mathbb{R}^2_1.$$

$$\begin{array}{ccc} SO(2,1)^0 & \xrightarrow{\rho_{\mathbf{u}}} & SO(2,1)^0 \ltimes \mathbb{R}^2_1 \\ {\scriptstyle \rho_0}\uparrow & & \downarrow{\scriptstyle proj.} \\ \Gamma & \xrightarrow{\mathbf{u}} & \mathbb{R}^2_1 \end{array}$$

- A representation in $SO^0(2,1) \ltimes \mathbb{R}^2_1 \Leftrightarrow$ an element of $\mathrm{H}^1(\Gamma, \mathbb{R}^2_1)$,
- P.D. actions $\Leftrightarrow$ a subset of $\mathrm{H}^1(\Gamma, \mathbb{R}^2_1)$ ($:=$ **Proper**)

Margulis invariant

Let $\rho_{\mathbf{u}}$ be an affine deformation.

Definition (Margulis invariant)

A map $\alpha_{\mathbf{u}} : \Gamma \to \mathbb{R}$,

$$\alpha_{\mathbf{u}}(g) := B(\mathbf{X}_g^0, \rho_{\mathbf{u}}(g)(x)) = B(\mathbf{X}_g^0, \mathbf{u}(g) + gx) \quad , \ x \in E_1^2.$$

(1) independent on $x \in E_1^2$,

(2) decide affine deformation up tp a conj. (Drumm_Goldman),

(3) can represent as $\mathbf{u}(g) = \alpha_{\mathbf{u}}(g)\mathbf{X}_g^0 + c^-\mathbf{X}_g^- + c^+\mathbf{X}_g^+$,

Lemma (Margulis)

$\exists h_1, h_2 \in \pi_1(S_{g,b})$ *s.t.* $\alpha_{\mathbf{u}}(h_1) \leq 0 \leq \alpha_{\mathbf{u}}(h_2)$
$\Rightarrow \rho_{\mathbf{u}}(\Gamma) \curvearrowright E_1^2$: **not** *properly discontinuous*

Proper cocycles

Theorem (Goldman, Labourie, Margulis, 2009)

Proper $\subset \mathrm{H}^1(\Gamma, \mathbb{R}^2_1)$ *consists of two* [a] *symmetric open convex cones.*

[a] Denote them by **Proper**$_+$ and **Proper**$_-$.

Examples of Margulis spacetimes

Free groups of rank 2

$\Gamma = \langle \gamma_1, \gamma_2 \rangle$. Consider all affine deformations of Γ :

Lemma (Drumm, Goldman)

The following corresponding is canonically linearly isomorphic:

$$\mathrm{H}^1(\Gamma, \mathbb{R}^2_1) \ni [\mathbf{u}] \longleftrightarrow (\alpha_{\mathbf{u}}(\gamma_1), \alpha_{\mathbf{u}}(\gamma_2), \alpha_{\mathbf{u}}(\gamma_2^{-1}\gamma_1^{-1})) \in \mathbb{R}^3.$$

The quotient manifold is an open handle body of genus two.

Example

Introduce two examples :

- $\Gamma = \pi_1(S_{0,3})$,
- $\Gamma = \pi_1(S_{1,1})$,

affine deformations of thrice holed sphere

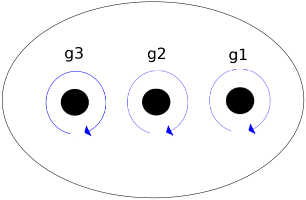
$$\pi_1(S_{0,3}) = \langle g_1, g_2, g_3 \mid g_1 \cdot g_2 \cdot g_3 = id. \rangle$$

Theorem (Charette, Drumm, Goldman, 2010)

$$[\mathbf{u}] \in \mathbf{Proper}_+ \Leftrightarrow \alpha_{\mathbf{u}}(g_1), \alpha_{\mathbf{u}}(g_2), \alpha_{\mathbf{u}}(g_3) > 0.$$

Proof.

Assign four crooked planes disjointly. □



affine deformations of once holed torus

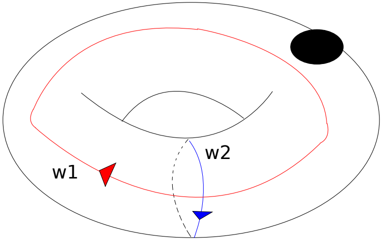
$$\pi_1(S_{1,1}) = \langle w_1, w_2, w, K \mid w = w_2^{-1} w_1^{-1}, K = [w_1, w_2] \rangle$$

Theorem (Charette, Drumm, Goldman, 2015)

$[\mathbf{u}] \in \mathbf{Proper}_+$ *if and only if* $[\mathbf{u}]$ *is in 'Tile's.*

Proof.

Assign four crooked planes disjointly. □



Main theorem
Consider free groups of rank > 2

hyperbolic surfaces with boundaries

Let $S_{g,b}$ be a hyperbolic surface of genus g, punctures b.
Assume that $b > 0$ and the hyperbolic length of each boundary is positive.

Fenchel-Nielsen

The dimensions of Teichmuller space of (top.) surface of (g, b)-type are $6g - 6 + b$.

Fix two hyperbolic surfaces S^1, S^2,
which $\exists$ boud. comp. $g^i \subset \partial S^i$ $s.t.$ $\ell_{\mathbb{H}_2}(g^1) = \ell_{\mathbb{H}_2}(g^2)$.

- S denote a surface glued by S_1 and S_2 along g^i.
- $g \in \pi_1(S)$, $g := g^1 = g^2$.

Combination of cocycles

Take $\mathbf{u}^i \in \mathrm{Z}^1(\pi_1(S^i), \mathbb{R}^2_1)$: cocycles
Assume that $\alpha_{\mathbf{u}^1}(g^1) = \alpha_{\mathbf{u}^2}(g^2) =: \alpha_g$.

Definition (Combination of cocycles)

Define the map $\mathbf{u}^1 \#_g \mathbf{u}^2 : \pi_1(S) \to \mathbb{R}^2_1$ as follows:

$$\mathbf{u}^1 \#_g \mathbf{u}^2(h) = \begin{cases} \mathbf{u}^1(h), & (h \in \pi_1(S^1)), \\ \mathbf{u}^2(h) + \delta_{\textbf{trans.}}(h), & (h \in \pi_1(S^2)), \\ \text{by cocycle condition} & (\text{the others}), \end{cases}$$

where $\mathbf{u}^2(g) + \delta_{\textbf{trans.}}(g) = \alpha_g \mathbf{X}^0_1$ holds.

Affine twist cocycle

Definition (Affine twist cocycle)

Define the map $\mathrm{AT}_g : \pi_1(S) \to \mathbb{R}^2_1$ as follows:

$$\mathbf{AT}_g(h) = \begin{cases} \mathbf{0}, & (h \in \pi_1(S^1)) \\ \mathbf{X}^0_g - h\mathbf{X}^0_g, & (h \in \pi_1(S^2)) \\ \text{by cocycle condition} & \text{(the others)}. \end{cases}$$

Theorem (M.)

$\tau \in \mathbb{R}$, *cocycles* $\mathbf{u}^1 \#_g \mathbf{u}^2 + \tau \mathbf{AT}_g$ *generate the cohomology* $\mathrm{H}^1(\pi_1(S), \mathbb{R}^2_1)$.

Proof.

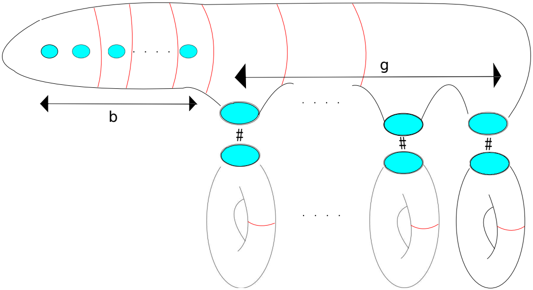
Show that their dimensions are equal. □

decomposition of a surface

Let $S_{g,b}$ be a hyperbolic surface with boudaries.

- $\{p_i\}_i$: original boundary components ($i = 1, \ldots, b$),
- $\{q_j\}_j$: dividing curves ($i = 1, \ldots, 3g + b - 3$).

- $t_{\mathbf{u}}(q_i)$: coefficient of the affine twist along q_i



decomposition of a surface

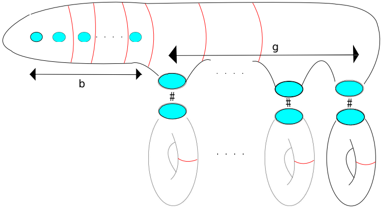
Corollary

The following correspondence is canonically linearly isomorphic.

$$\mathrm{H}_1(\pi_1(S_{g,b}), \mathbb{R}^2_1) \ni [\mathbf{u}] \leftrightarrow (\alpha, \beta, \tau) \in \mathbb{R}^{6g-6+3b},$$

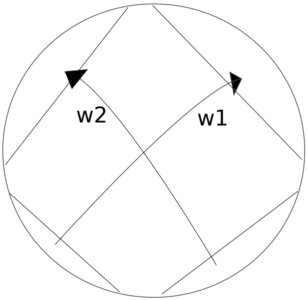
where 'the angle of each torus' is not $\pi/2$, and

- $\alpha := (\alpha_{\mathbf{u}}(p_1), \ldots, \alpha_{\mathbf{u}}(p_b))$, $\beta := (\alpha_{\mathbf{u}}(q_1), \ldots, \alpha_{\mathbf{u}}(q_{3g+b-3}))$,
- $\tau := (t_{\mathbf{u}}(q_1), \ldots, t_{\mathbf{u}}(q_{3g+b-3}))$.



the angle of torus

Set $\pi_1(S_{1,1}) = \langle w_1, w_2 \rangle$.



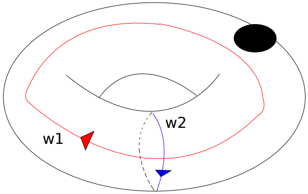


Figure: Action by w_1, w_2 on $\mathbb{H}^2$, and 'the angle of torus' $\theta_{w_1}^{w_2}$.

Proof of corollary

the pair of pants $\langle g_1, g_2, g_3 \mid g_1 \cdot g_2 \cdot g_3 = id \rangle$

- $g_1 := w_1 \cdot w_2 \cdot w_1^{-1} \cdot w_2^{-1}$,
- $g_2 = w_2$,
- $g_3 := w_1 \cdot w_2^{-1} \cdot w_1^{-1}$.

null flame

- $w_i \leftrightarrow \{\mathbf{Y}_i^0, \mathbf{Y}_i^-, \mathbf{Y}_i^+\}$, $(i = 1, 2)$.
- $g_j \leftrightarrow \{\mathbf{X}_j^0, \mathbf{X}_j^-, \mathbf{X}_j^+\}$, $(j = 1, 2, 3)$.
- $g_2 = w_2$.

Remark :

We would like to obtain $\mathrm{H}^1(\pi_1(S_{1,1}), \mathbb{R}_1^2) \cong \{(\alpha_1, \alpha_2, t) \in \mathbb{R}^3\}$.

representation of cocycles

Example (thrice-holed sphere)

- $\mathbf{u}_P(g_1) = \alpha_1 \mathbf{X}_1^0$
- $\mathbf{u}_P(g_2) = \alpha_2 \mathbf{X}_2^0 + c_2^+ \mathbf{X}_2^+$,
- $\mathbf{u}_P(g_3) = \alpha_3 \mathbf{X}_3^0 + c_3^- \mathbf{X}_3^- + c_3^+ \mathbf{X}_3^+$.

Lemma (once-holed torus)

When $\theta_{w_1}^{w^2} \neq \frac{\pi}{2}$; $\forall \alpha_1, \alpha_2, t \in \mathbb{R}, \exists d_1^\pm, d_2^+$,

- $\mathbf{u}_T(w_1) = t\mathbf{Y}_1^0 + d_1^- \mathbf{Y}_1^- + d_1^+ \mathbf{Y}_1^+$,
- $\mathbf{u}_T(w_2) = \alpha_2 \mathbf{Y}_2^0 + d_2^+ \mathbf{Y}_2^+$,
- $\mathbf{u}_T(g_1) = \alpha_1 \mathbf{X}_1^0$.

Proof of Lemma

Proof.

For any cocycle $\mathbf{u}$ on $S_{1,1}$, we can have (by direct calculation)

$$\mathbf{u}(g_1) = (Id - g_3^{-1})\mathbf{u}(w_1) + (w_1 - g_1)\mathbf{u}(w_2)$$

Next, set

$$\mathbf{u}(w_1) = t\mathbf{Y}_1^0 + a\mathbf{Y}_1^- + b\mathbf{Y}_1^+, \mathbf{u}(w_2) = \alpha_2\mathbf{Y}_2^0 + c\mathbf{Y}_2^- + d\mathbf{Y}_2^+.$$

Then

1. $\theta_{w_1}^{w_2} \neq \frac{\pi}{2} \Rightarrow \alpha_1$ dependes linearly on t, α_2, a, b, c, d.
2. $\theta_{w_1}^{w_2} \neq \frac{\pi}{2} \Rightarrow \alpha_1$ does **not** depend on a, b, c, d at all.

Finally an appropriate translation by coboundary makes the claimed representation. □