Realization of closed manifolds as A_5-fixed point sets

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§ Our Problem

G finite group

M manifold

$\mathfrak{F}_G(M)$ the family of all manifolds F obtainable as $F = M^G$

$\mathfrak{M}$ family of smooth manifolds

$\mathfrak{F}_G(\mathfrak{M}) = \bigcup_{M \in \mathfrak{M}} \mathfrak{F}_G(M)$

$\mathfrak{F}_G(\{D^n\})$ for $G \notin \mathcal{P}$ $\cdots\cdots$ B. Oliver

$\mathfrak{F}_G(\{S^n\})$ for various G with $n_G = 1$ $\cdots\cdots$ M. M. & K. Pawałowski

$\mathfrak{F}_G(\{P^n_{\mathbb{C}}\})$ various G with $n_G = 1$

($G = A_n$ with $n \geq 7$, A_5) $\cdots\cdots$ M. Kaluba

In the cases above, $\pi_1(X) = 1$

Prob. How about $\mathfrak{F}_G(\{P^n_{\mathbb{R}}\})$, $\mathfrak{F}_G(\{L^{2k-1}\})$?

For these cases, $|\pi_1(X)| < \infty$

Basic Idea due to Kaluba

G nontrivial perfect group

Z compact G-manifold with $z_0 \in Z^G$

$F \in \mathfrak{F}(\{D^n\})$ $(F \neq \emptyset)$

Find G-action S^n with $n = \dim Z$,

$$S^G = F \amalg F', \; a_0 \in F' \cong_{\text{diff}} F, \; T_{a_0}S \cong_G T_{z_0}Z$$

Take conn. sum $Y = S^n \# Z$ at (a_0, z_0)

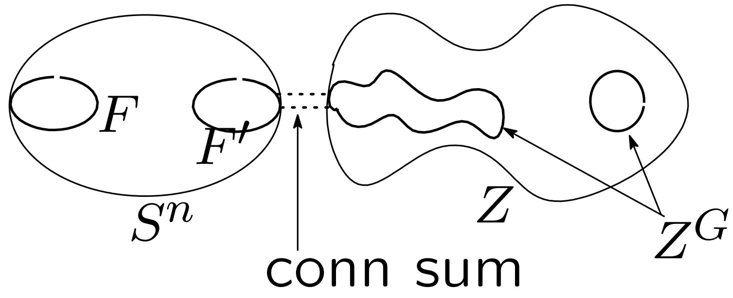


Fig. $Y = S^n \# Z$

Remove $F' \# Z^G$ from Y by G-surgery

Question. Does there exist a G-map $f : X \to Y$ s.t.

f is homotopic to diffeo. & $X^G = F$?

If Answer is Yes then $F \in \mathfrak{F}(Z)$

§ Known Results

Families of finite groups.

$G \in \mathcal{G}_{\mathbb{R}} \overset{\text{def}}{\Longleftrightarrow} G$ possesses a subquotient K/H isomorphic to a dihedral group of order $2pq$ for some distinct primes p and q, where $H \triangleleft K \leq G$.

$G \in \mathcal{G}_{\mathbb{C}}^{\sigma} \overset{\text{def}}{\Longleftrightarrow} G$ contains an element g being conjugate to its inverse of order pq for some distinct primes p and q.

$G \in \mathcal{G}_{\mathbb{C}} \overset{\text{def}}{\Longleftrightarrow} G$ contains an element g of order pq for some distinct primes p and q.

$G \in \mathcal{E} \overset{\text{def}}{\Longleftrightarrow}$ A Sylow 2-subgroup of G is not normal in G, and any element of G is of prime power order.

Thm. (B. Oliver) Let $G \in \mathcal{G}_{\mathbb{C}} \cup \mathcal{E}$. Then

$F \in \mathfrak{F}_G(\{D^n\}) \iff \chi(F) \equiv 1 \mod n_G$ and

Case $G \in \mathcal{G}_{\mathbb{R}}$: no restrictions on TF.

Case $G \in \mathcal{G}_{\mathbb{C}}^{\sigma} \smallsetminus \mathcal{G}_{\mathbb{R}}$: $c_{\mathbb{R}}([TF]) \in c_{\mathbb{H}}\left(\widetilde{KSp}(F)\right) + \mathrm{Tor}\left(\widetilde{KU}(F)\right)$.

Case $G \in \mathcal{G}_{\mathbb{C}} \smallsetminus \mathcal{G}_{\mathbb{C}}^{\sigma}$: $[TF] \in r_{\mathbb{C}}\left(\widetilde{KU}(F)\right) + \mathrm{Tor}\left(\widetilde{KO}(F)\right)$.

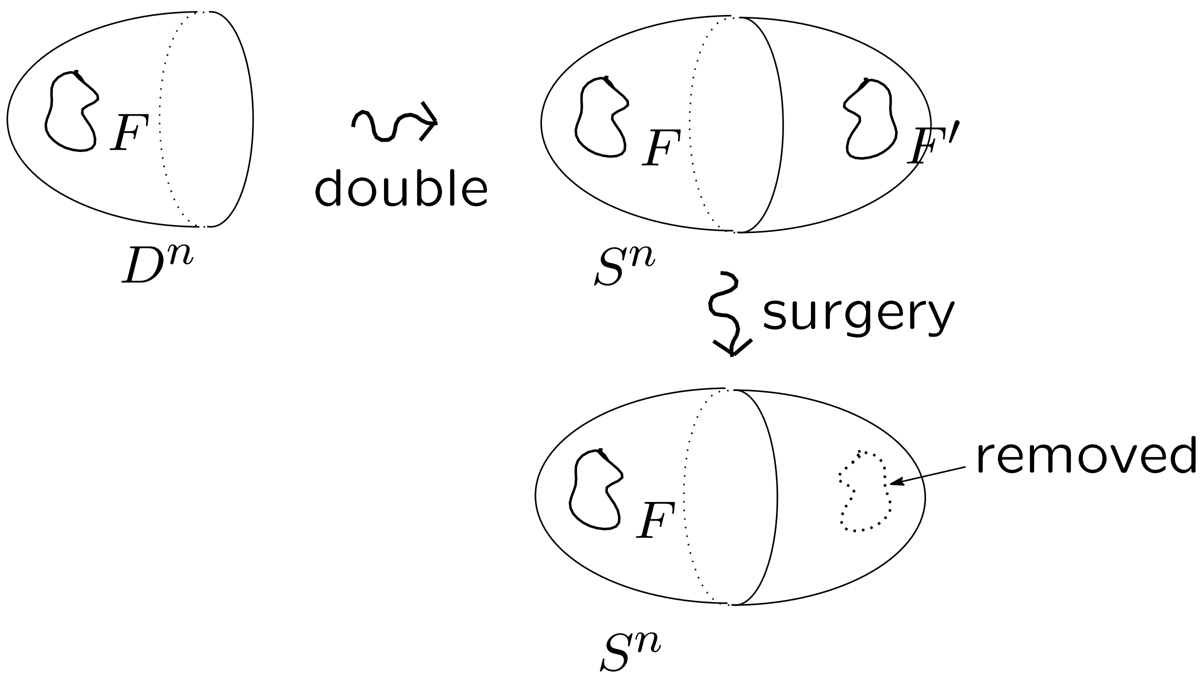
Case $G \in \mathcal{E}$: $[TF] \in \mathrm{Tor}\left(\widetilde{KO}(F)\right)$.

G nontrivial nonsolvable $\Longrightarrow n_G = 1$

$G \in \mathcal{E}$ and $F \in \mathfrak{F}_G(\{D^n\}) \Longrightarrow \dim F_i = \dim F_j$ (conn. copmp.'s)

Thm. (M. M.–K. Pawałowski) G nontrivial perfect group.

$$F \in \mathfrak{F}_G(\{D^n\}) \Longrightarrow F \in \mathfrak{F}_G(\{S^n\})$$



Thm. (M. Kaluba) G nontrivial perfect group $\in \mathcal{G}_{\mathbb{C}}$

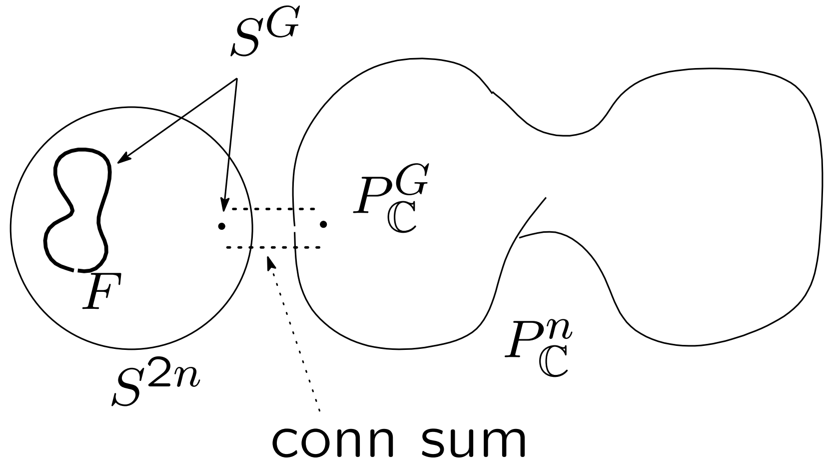
$F \in \mathfrak{F}_G(\{D^n\})$ & dim F_i is even for some $F_i \subset F$ if $G \in \mathcal{G}_{\mathbb{C}} \setminus \mathcal{G}_{\mathbb{R}}$

$$\Longrightarrow F \in \mathfrak{F}_G(\{P^n_{\mathbb{C}}\})$$

Idea. Assumption $\overset{\text{Oliver}}{\Longrightarrow} F \amalg \{a_0\} \in \mathfrak{F}_G(\{D^n\})$

$$\overset{\text{M.–P.}}{\Longrightarrow} F \amalg \{a_0\} \in \mathfrak{F}_G(\{S^n\})$$

$$\Longrightarrow F \in \mathfrak{F}_G(\{P^n_{\mathbb{C}}\})$$

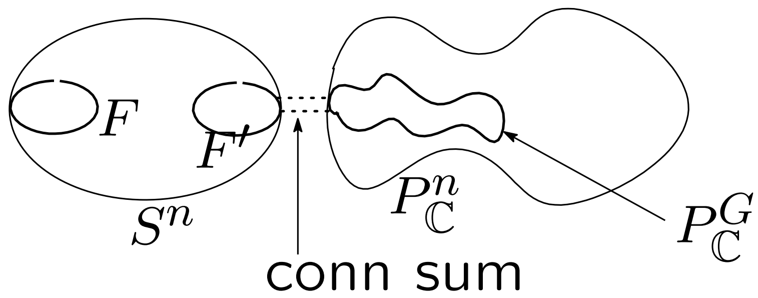


$G = A_5$, $F \in \mathfrak{F}(\{D^n\})$, $F_i, F_j \subset F \implies \dim F_i = \dim F_j$

Thm. (M. Kaluba) $G = A_5$, $F \in \mathfrak{F}_G(\{D^n\})$, $\dim F \equiv 0 \mod 2$

$$\implies F \in \mathfrak{F}_G(\{P_{\mathbb{C}}^n\})$$

Idea. Assumption $\implies F \amalg F' \in \mathfrak{F}(\{S^n\})$



$P_{\mathbb{C}}^G = P_{\mathbb{C}}^k$, $k = \dim F'$

Remove $F' \# P_{\mathbb{C}}^G$ from $S^n \# P_{\mathbb{C}}^n$

§ Our Result

Thm. $K = \mathbb{R}$ or $\mathbb{C}$, W $K[G]$-module

$\{X_n\}$ sequence of compact conn. manifolds s.t.

$\exists$ G-actions on X_n satisfying

$X_n^G = \{x_n\}$ (one fixed pt action)

$T_{x_n}(X_n) \cong_{K[G]} W \oplus (K[G] - K)^{m_n}$ with $\lim_{n\to\infty} m_n = \infty$

G nontrivial perfect group $\in \mathcal{G}_K$, $F \in \mathfrak{F}(\{D^n\})$

$\Longrightarrow F \in \mathfrak{F}(\{X_n\})$

§ Flow of Modifications of G-manifolds

$F \in \mathfrak{F}(\{D^n\})$, Y G-mfd s.t. $Y^G \supset F$

Goal. Obtain X G-mfd s.t. $X^G = F$ and $X \cong_{\text{diff}} Y$

(1) Costruct $f : X \to Y$ G-framed map s.t. $X^G = F$

(2) Modify f so that $f^H : X^H \to Y^H$ is diffeo ($E \neq \forall\, H < G$)

(3) Replace f by $f' : X' \to Y$ (by G-surgery on free part) s.t.

$$X' \cong_{\text{diff}} Y$$

§ Subgroup Category and Inverse Limit

G finite group

$\mathcal{S} = \mathcal{S}(G) = \{H \mid H \leq G\}$

$\mathbb{S}$ the category:

$$\text{Obj}(\mathbb{S}) = \mathcal{S}$$
$$\text{Mor}(\mathbb{S}) = \bigcup_{H,K \leq G} \text{Mor}(H,K)$$
$$\text{Mor}(H,K) = \{(H,g,K) \mid g \in G, gHg^{-1} \leq K\}$$
$$(K,g',L) \circ (H,g,K) = (H,g'g,L)$$

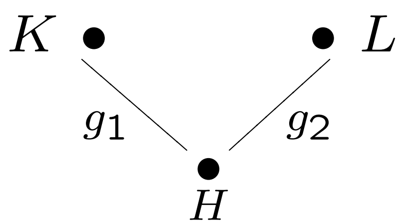
$\mathbb{A}$ the category of abelian groups

$M : \mathbb{S} \to \mathbb{A}$ contravariant functor

For $\mathbb{F} = (A, B)$ s.t. $A \subset \mathcal{S}$, $B \subset \mathrm{Mor}(\mathbb{S})$

Inverse limit $M_{\mathbb{F}} = \varprojlim_{\mathbb{F}} M(-)$ is

$$M_{\mathbb{F}} = \Big\{ (x_H)_H \in \prod_{H \in A} M(H) \mid f_1^* x_K = f_2^* x_L$$
$$\text{for } f_1,\ f_2 \in B,\ K,\ L \in A,$$
$$f_1 = (H, g_1, K),\ f_2 = (H, g_2, L) \in B \Big\}$$



For lower-closed, conj.-invariant $\mathcal{F} \subset \mathcal{S}$

$\mathbb{F} = \mathbb{F}(G, \mathcal{F})$ full subcategory of $\mathbb{S}$ s.t. $\mathrm{Obj}(\mathbb{F}) = \mathcal{F}$

§ G-Framed Maps and G-Surgery

G-framed map $\boldsymbol{f} = (f, b)$ consists of

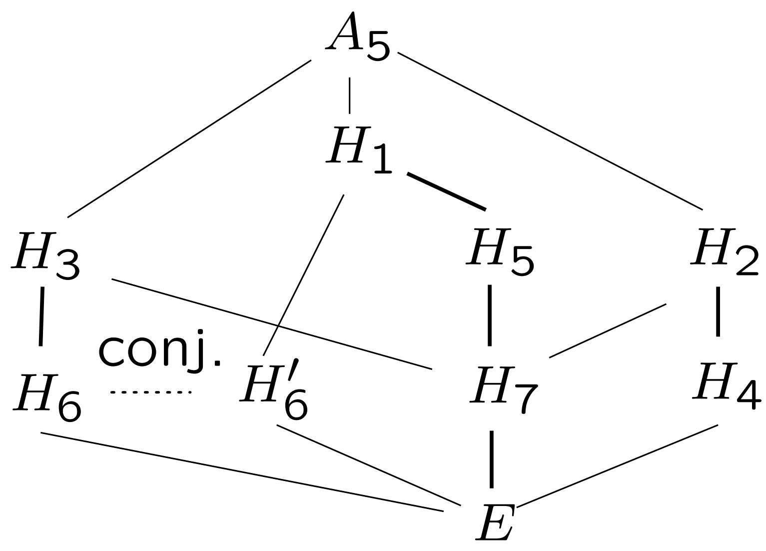
$$f : (X, \partial X) \to (Y, \partial Y)$$

$$b : TX \oplus \varepsilon_X(\mathbb{R}^m) \to f^*TY \oplus \varepsilon_X(\mathbb{R}^m)$$

§ Subgroup System of A_5

$$H_1 = A_4,$$
$$H_2 = \langle (1,3,4,2,5),\ (1,2)(3,4) \rangle \cong D_{10},$$
$$H_3 = \langle (1,2,5),\ (1,2)(3,4) \rangle \cong D_6,$$
$$H_4 = \langle (1,3,4,2,5) \rangle \cong C_5,$$
$$H_5 = \langle (1,2)(3,4),\ (1,3)(2,4) \rangle \cong D_4,$$
$$H_6 = \langle (1,2,5) \rangle \cong C_3,$$
$$H_6' = \langle (1,2,4) \rangle \cong C_3,$$
$$H_7 = \langle (1,2)(3,4) \rangle \cong C_2,$$
$$H_8 = E = \{e\}$$

$$a = (1,2)(4,5)$$



$$\mathfrak{q}_1 = (H_2, H_7, e, H_1),\ \mathfrak{q}_2 = (H_3, H_7, e, H_1),\ \mathfrak{q}_3 = (H_3, H_6, a, H_1)$$

$$\mathcal{M} = \{H_1,\ H_2,\ H_3\}$$

$G = A_5$

$M : \mathbb{S} \to \mathbb{A}$ contravariant functor s.t.

$$(1) \quad (H, g, H)^* = id \quad (H \leq G,\ g \in H)$$
$$(2) \quad (H, g, H)^* = id \quad (H \leq G,\ g \in C_G(H))$$

$$\mathbb{M} = \Big(\mathcal{M},\ \bigcup_{i=1}^{3} \{(H, e, L),\ (H, g, K) \mid (L, H, g, K) = \mathfrak{q}_i\}\Big)$$

$$M_{\mathbb{M}} = \{(x_T)_{T \in \mathcal{M}} \mid x_T \in M(T),\ (H.e.L)^* x_L = (H, g, K)^* x_K,$$
$$\text{for } (L, H, g, K) = \mathfrak{q}_i,\ i = 1, 2, 3\}$$

Lem. For $\mathcal{F} = \mathcal{S} \setminus \{G\}$, $\mathbb{F} = \mathbb{F}(G, \mathcal{F})$,

$$M_{\mathbb{F}} \xrightarrow{\cong} M_{\mathbb{M}}$$

§ Equiv Cohomotopy and Equiv Surgery

Let $\omega_G^n(Y)$ denote equiv stable cohomotopy group

Sketch.

$$\begin{array}{ccccccc}
\omega_G^n(Y) & & \xrightarrow{\quad\mathrm{res}_{\mathcal{F}}\quad} & & \omega^n(Y)_{\mathbb{F}} & & \\
\downarrow{\scriptstyle q_{\mathbb{F}}} & & & & & & \\
Q_{G,\mathcal{F}}^n(Y) & \xrightarrow[\overline{\mathrm{res}_{\mathcal{F}}}]{} & \mathrm{Im}(\mathrm{res}_{\mathcal{F}}) & \hookrightarrow & \omega^n(Y)_{\mathbb{F}} & \hookrightarrow & \prod_{H\in\mathcal{F}} \omega_H^n(Y)
\end{array}$$

$$\omega_G^n(Y) \xrightarrow{\sqcap_{H\in\mathcal{F}}\mathrm{res}_H^G} \prod_{H\in\mathcal{F}} \omega_H^n(Y)$$

$$\mathcal{F} = \mathcal{S} \smallsetminus \{G\}, \ \mathbb{F} = \mathbb{F}(G,\mathcal{F})$$

$\omega_G^0(pt) \cong A(G)$ the Burnside ring

$Q_{G,\mathcal{F}}^n(Y)$ is important to obtain $\boldsymbol{G}$-framed maps

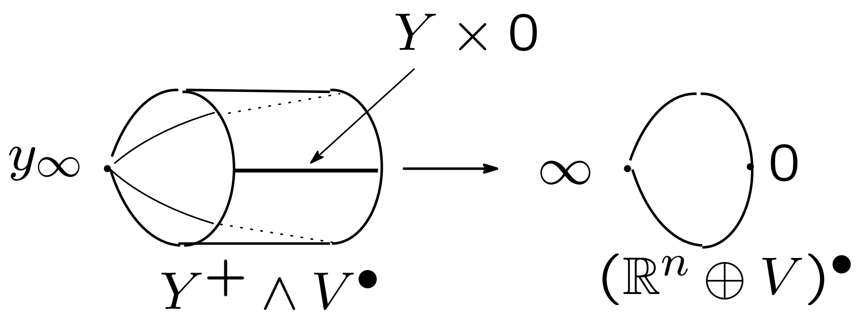
Cohomotopy groups $\omega_G^n(Y)$

$$M(X,Y)_0^G = \{f : X \to Y \text{ b.p.pres. } G\text{-map}\}$$

$$[X,Y]_0^G = M(X,Y)_0^G / \sim_{G\text{-ht}} \qquad Y^+ = Y \amalg \{y_\infty\} \qquad V^\bullet = V \cup \{\infty\}$$

$$\omega_G^n(Y) \stackrel{\text{def}}{=} \{Y^+, \mathbb{R}^{n\bullet}\}_0^G = \lim_n \, [Y^+ \wedge V^\bullet, (\mathbb{R}^n \oplus V)^\bullet]_0^G \;\; (V = \mathbb{C}[G]^n)$$

(equivariant stable) cohomotopy group



Def. $\omega_G^0 \stackrel{\text{def}}{=} \omega_G^0(pt)$

Def. $A(G) = \{[X] \mid X \text{ finite } G\text{-CW complex}\}$

$$[X_1] = [X_2] \overset{\text{def}}{\iff} \chi(X_1^H) = \chi(X_2^H) \ (\forall\, H \leq G)$$

$$\mathfrak{N}(G, Y) = \{\boldsymbol{f} = (f, b) \mid f : X \to Y\}$$

where $\boldsymbol{f}$ is G-framed map

Thm. $\exists\, \Phi \ : \ \omega_G^0(Y) \xrightarrow{\cong} \mathfrak{N}(G, Y)/\sim_{G\text{-cob}}$

Equivalence Relation $\sim_{\mathbb{F}}$ on $\omega^n_G(Y)$

$x,\ y \in \omega^n_G(Y),\ x = [\alpha],\ y = [\beta],\quad \alpha,\ \beta : Y^+ \wedge V^\bullet \to (\mathbb{R}^n \oplus V)^\bullet$

Def. $x \sim_{\mathbb{F}} y \overset{\text{def}}{\Longleftrightarrow} \exists\ \{h_H : \alpha \sim_H \beta \mid H \in \mathcal{F}\}$ s.t.

$$h_H \sim_H (H, g, K)^\# h_K \text{ rel. } \alpha,\ \beta$$

$$\text{where } (H, g, K)\text{: } H,\ K \in \mathcal{F},\ gHg^{-1} \subset K$$

Def. $Q^n_{G,\mathcal{F}}(Y) = \omega^n_G(Y)/\sim_{\mathbb{F}}, \qquad Q^n_{G,\mathcal{F}} = \omega^n_G/\sim_{\mathbb{F}}$

$$\begin{array}{ccccccc}
\omega^n_G(Y) & & \xrightarrow{\quad\quad\quad\quad\text{res}_{\mathcal{F}}\quad\quad\quad\quad} & & \omega^n(Y)_{\mathbb{F}} & & \\
\downarrow {\scriptstyle q_{\mathbb{F}}} & & & & & & \\
Q^n_{G,\mathcal{F}}(Y) & \xrightarrow[\overline{\text{res}}_{\mathbb{F}}]{} & \text{Im}(\text{res}_{\mathcal{F}}) & \hookrightarrow & \omega^n(Y)_{\mathbb{F}} & = & \varprojlim_{\mathbb{F}} \omega^n_{-}(Y)
\end{array}$$

For $H \leq G$, $\chi_H : A(G) \to \mathbb{Z}$;

$$\chi_H(x) = |X_1^H| - |X_2^H| \quad (x = [X_1] - [X_2] \in A(G))$$

$$\overline{A}(G) \overset{\text{def}}{=} \text{Im}\left(\prod_H \chi_H : A(G) \to \prod_{H \leq G} \mathbb{Z}\right)$$

Lem. (Burnside Congrulence)

$(x_H) \in \left(\prod_{H \leq G} \mathbb{Z}\right)^G$ lies in $\overline{A}(G) \iff$

$$\sum_{s \in WH} x_K \equiv 0 \mod |WH| \quad (\forall\ H \leq G)$$

where $WH = N_G(H)/H$, $\langle s \rangle = K/H$ with $H \leq K \leq N_G(H)$

(*) G nontrivial perfect group, $\mathcal{F} = \mathcal{S}(G) \smallsetminus \{G\}$, $\mathbb{F} = \mathbb{F}(G, \mathcal{F})$

Lem. (*) $\Longrightarrow \exists\ \beta_G \in A(G) = \omega_G^0$ s.t.

$$\chi_G(\beta_G) = 1 \text{ and } \chi_K(\beta_G) = 0 \text{ for all } K < G.$$

Observ. $\exists$ finite G-CW complex D s.t.

$$D^G = \emptyset \ \ \& \ \ \chi(D^H) = 1 \ (H < G)$$

Then $\quad \beta_G = [G/G] - [D]$ □

(*) G nontrivial perfect group, $\mathcal{F} = \mathcal{S}(G) \smallsetminus \{G\}$, $\mathbb{F} = \mathbb{F}(G, \mathcal{F})$

Thm. (Y. Hara–M. M.) (*) $\Longrightarrow$ $q_{\mathbb{F}}(\beta_G) = 0$ in $Q^0_{G,\mathcal{F}}$

Cor. (*) $\Longrightarrow$ $q_{\mathbb{F}}(\beta_G\, \omega^n_G(Y)) = \{0\}$ $(\subset Q^n_{G,\mathcal{F}}(Y))$

Thm. (Y. Hara–M. M.) (*) $\Longrightarrow$ $Q^0_{G,\mathcal{F}} \to \omega^0_{\mathbb{F}}$ is injective