Remarks on the Burnside ring of A_5

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Topics

1. **Burnside ring of G**
2. **Inverse Limit of $A(G)$**
3. **Motivation**
4. **The case of $G = A_5$**

Burnside ring

- G **: finite group**
- $[X]$ **: equivalent class of finite** G**-set** X

$$[X_1] - [X_2] = [Y_1] - [Y_2] \overset{def}{\Leftrightarrow} |X_1^H \amalg Y_2^H| = |Y_1^H \amalg X_2^H| \quad (\forall H \leq G)$$

$$A(G) := \{[X_1] - [X_2] \mid X_1, X_2 : \textbf{finite } G\textbf{-sets}\}$$

<Addition>
$([X_1] - [X_2]) + ([Y_1] - [Y_2]) \overset{def}{=} [X_1 \amalg Y_1] - [X_2 \amalg Y_2]$
<Multiplication>
$([X_1] - [X_2])([Y_1] - [Y_2])$

$$\overset{def}{=} [(X_1 \times Y_1) \amalg (X_2 \times Y_2)] - [(X_1 \times Y_2) \amalg (X_2 \times Y_1)]$$

Burnside ring

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<Addition>
$([X_1]-[X_2])+([Y_1]-[Y_2]) \overset{def}{=} [X_1 \amalg Y_1]-[X_2 \amalg Y_2]$
<Multiplication>
$([X_1]-[X_2])([Y_1]-[Y_2])$

$$\overset{def}{=} [(X_1 \times Y_1) \amalg (X_2 \times Y_2)] - [(X_1 \times Y_2) \amalg (X_2 \times Y_1)]$$

A(G) : *Burnside Ring*

- $\chi_H : A(G) \to \mathbb{Z}$: **map**

$$\chi_H([X_1]-[X_2]) \overset{def}{=} |X_1{}^H| - |X_2{}^H| \quad ([X_1]-[X_2] \in A(G), H \leq G)$$

Then, χ_H is ring homomorphism.

$$\begin{aligned}[X]=[Y] &\Leftrightarrow |X^H| = |Y^H| \quad (\forall H \leq G)\\ &\Leftrightarrow \chi_H([X]) = \chi_H([Y]) \quad (\forall H \leq G)\end{aligned}$$

$$\begin{array}{cccc} \bigoplus_{H\leq G}\chi_H : & A(G) & \longrightarrow & \bigoplus_{H\leq G}\mathbb{Z} \\ & \Uplus & & \Uplus \\ & \alpha & \longmapsto & (\chi_H(\alpha))_{H\leq G} \end{array}$$

is injective.

Setting

- G : **finite nontrivial group**
- $\mathcal{S}(G)$: **set of all subgroups of** G
- $\mathcal{F}$:G**-conjugation invariant,lower closed set of subgroups of** G

i.e. $\mathcal{F}$ **satisfies**

1. $H \in \mathcal{F}, g \in G \Longrightarrow gHg^{-1} \in \mathcal{F}$
2. $H \in \mathcal{F} \Longrightarrow S(H) \subset \mathcal{F}$

- $\mathfrak{F}$: **category s.t.**

1. **Obj**$(\mathfrak{F}) = \mathcal{F}$
2. **Mor**$(\mathfrak{F}) = \{(H, g, K) : H \to K | H, K \in \mathcal{F}, g \in G, gHg^{-1} \leq K\}$
3. $(K, h, L) \circ (H, g, K) = (H, hg, L)$

For $H \leq K \leq G$**,**
the restriction homomorphism res^K_H **:** $A(K) \to A(H)$

For $H \in S(G)$ **and** $g \in G$**,**
the conjugation map c_g **:** $H \to gHg^{-1}$ **;** $c_g(h) = ghg^{-1}$ $(h \in H)$

$\rightsquigarrow$ **homomorphism** c_g^* **:** $A(gHg^{-1}) \to A(H)$

For $(H, g, K) \in$ **Mor**$(\mathfrak{F})$**,**
the homomorphism $(H, g, K)^*$ **:** $A(K) \to A(H)$

$$A(K) \xrightarrow{res^K_{gHg^{-1}}} A(gHg^{-1}) \xrightarrow{c_g^*} A(H)$$

i.e. $(H, g, K)^* = c_g^* \circ res^K_{gHg^{-1}}$

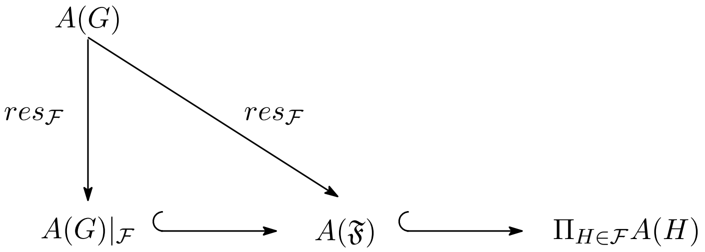
Inverse Limit of $A(G)$

$A(\mathfrak{F})$ **: subring of** $\prod_{H\in\mathcal{F}} A(H)$

$$A(\mathfrak{F}) := \lim_{\longleftarrow \mathcal{F}} A(H) \quad (H \in \mathcal{F})$$

$$\lim_{\leftarrow \mathcal{F}} A(H) = \{(x_H) \in \prod_{H\in\mathcal{F}} A(H) \mid (P,g,Q)^* x_Q = x_P \textbf{ for } \forall (P,g,Q) \in \mathbf{Mor}(\mathfrak{F})\}$$

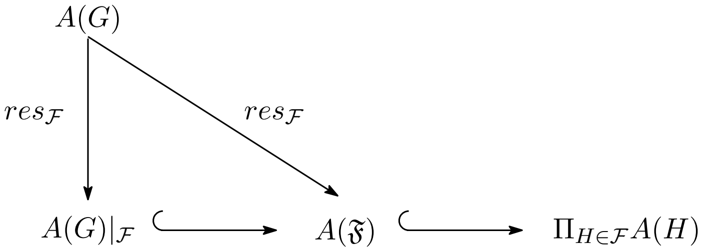
$A(G)|_{\mathcal{F}}$ **: image of** $res_{\mathcal{F}} : A(G) \to \prod_{H\in\mathcal{F}} A(H)$



Question

$$\begin{array}{ccccc} A(G) & & & & \\ \downarrow res_{\mathcal{F}} & \searrow^{res_{\mathcal{F}}} & & & \\ A(G)|_{\mathcal{F}} & \hookrightarrow & A(\mathfrak{F}) & \hookrightarrow & \Pi_{H \in \mathcal{F}} A(H) \end{array}$$

Question



Question

Does $A(G)|_{\mathcal{F}}$ coincide with $A(\mathfrak{F})$?

Setting

$\mathcal{F}_G = \mathcal{S}(G) \setminus \{G\}$
$\mathcal{M}$ **: the set of representations of conjugacy classes of maximal subgroups of** G**.**

$$A(G)$$

$$\downarrow res_{\mathcal{F}}$$

$$\prod_{H \in \mathcal{F}} A(H) \xrightarrow{\text{projection } \pi} \prod_{H \in \mathcal{M}} A(H) \supset A(\mathfrak{F}) \supset A(G)|_{\mathcal{F}}$$

$$\cup$$

$$A(\mathfrak{F}) = \mathbf{invlim}_{\mathcal{F}} A(H) \xrightarrow{\pi|_{A(\mathfrak{F})}\text{:injective}}$$

Setting

$\mathcal{F}_G = \mathcal{S}(G) \setminus \{G\}$
$\mathcal{M}$ **: the set of representations of conjugacy classes of maximal subgroups of** G**.**

$$
\begin{array}{ccc}
A(G) & & \\
\downarrow \, res_{\mathcal{F}} & & \\
\prod_{H \in \mathcal{F}} A(H) & \xrightarrow{\textbf{projection } \pi} & \\
\cup & & \prod_{H \in \mathcal{M}} A(H) \supset A(\mathfrak{F}) \supset A(G)|_{\mathcal{F}} \\
A(\mathfrak{F}) = \textbf{invlim}_{\mathcal{F}} A(H) & \xrightarrow[\pi|_{A(\mathfrak{F})} \textbf{:injective}]{} &
\end{array}
$$

Therefore, in order to show that $A(G)|_{\mathcal{F}} = A(\mathfrak{F})$**, it suffices to show that** $A(\mathfrak{F}) \subset A(G)|_{\mathcal{F}} \cong Im(res_{\mathcal{M}})$**.**

Motivation

Theorem 1 (Y.Hara,M.Morimoto)

Let G a finite nontrivial nilpotent group, $\mathcal{F} = \mathcal{F}_G$, and $\mathfrak{F} = \mathfrak{F}_G$. Then $A(G)|_{\mathcal{F}} = A(\mathfrak{F})$ if and only if G is a cyclic group of which the order is a prime or a product of distinct primes.

Theorem 2 (Y.Hara,M.Morimoto)

Set $\mathcal{F} = \mathcal{F}_G$. For $G = A_4$, the alternating group on four letters, $A(G)|_{\mathcal{F}} = A(\mathfrak{F})$.

Proof of Theorem 2

$A_4 \cong D_4 \rtimes C_3$
$\Rightarrow \mathcal{F} = (D_4) \amalg (C_3) \amalg (C_2) \amalg (E)$, $\mathcal{M} = \{D_4, C_3\}$

$\omega_{A_4} \in A(A_4)$ **is written in the following form.**

$$\omega_{A_4} = x_1[A_4/A_4] + x_2[A_4/D_4] + x_3[A_4/C_3] + x_4[A_4/C_2] + x_5[A_4/E] \ (x_i \in \mathbb{Z})$$

($[A_4/H]$ **: basis of** $A(A_4)$ **(**$H \leq A_4$**))**

$$\omega_{D_4} = a_1[D_4/D_4] + a_2[D_4/C_2] + a_3[D_4/C_2'] + a_4[D_4/C_2''] + a_5[D_4/E] \in A(D_4) \quad (a_i \in \mathbb{Z})$$

$$\omega_{C_3} = b_1[C_3/C_3] + b_2[C_3/E] \in A(C_3) \quad (c_i \in \mathbb{Z})$$

Proof of Theorem 2

	$res^{A_4}_{D_4}$					$res^{A_4}_{C_3}$	
	$[D_4/D_4]$	$[D_4/C_2]$	$[D_4/C_2']$	$[D_4/C_2'']$	$[D_4/E]$	$[C_3/C_3]$	$[C_3/E]$
$[A_4/A_4]$	1	0	0	0	0	1	0
$[A_4/D_4]$	3	0	0	0	0	0	1
$[A_4/C_3]$	0	0	0	0	1	1	1
$[A_4/C_2]$	0	1	1	1	0	0	2
$[A_4/E]$	0	0	0	0	3	0	4

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 & 0 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof of Theorem 2

Set

$$\begin{cases} \boldsymbol{v_1} & = & [1,0,0,0,0,1,0], \\ \boldsymbol{v_2} & = & [0,1,1,1,0,0,2], \\ \boldsymbol{v_3} & = & [0,0,0,0,1,1,1], \\ \boldsymbol{v_4} & = & [0,0,0,0,0,3,-1], \end{cases}$$

then $\omega \in A(\mathfrak{F})$ **is written in the following form,**

$$\omega = \alpha_1 \boldsymbol{v_1} + \alpha_2 \boldsymbol{v_2} + \alpha_3 \boldsymbol{v_3} + \alpha_4 \boldsymbol{v_4} \qquad (\alpha_i \in \mathbb{Q}).$$

$$A(\mathfrak{F}) = (< \boldsymbol{v_1}, \cdots, \boldsymbol{v_4} >_{\mathbb{Q}}) \cap (A(D_4) \times A(C_3)),$$

if $\alpha_1, \cdots, \alpha_4 \in \mathbb{Z}$**, we can see that** $A(\mathfrak{F}) \subset Im(res_{\mathcal{M}})$**.**

Proof of Theorem 2

$$A(\mathfrak{F}) \longrightarrow A(D_4) = \sum_{(K_j)_{D_4}} \mathbb{Z} \cdot [D_4/K_j] \cong \mathbb{Z}^3$$

$$A(\mathfrak{F}) \longrightarrow A(C_3) = \sum_{(K_j)_{C_3}} \mathbb{Z} \cdot [C_3/K_j] \cong \mathbb{Z}^2$$

$$\begin{array}{ccc} A(\mathfrak{F}) & \xrightarrow{\quad f \quad} & A(D_4) \times A(C_3) \\ & \searrow^{g} & \wr\| \\ & & \mathbb{Z}^5 \end{array}$$

Proof of Theorem 2

$$\begin{aligned} g(\omega) &= [\alpha_1, \alpha_2, \alpha_2, \alpha_2, \alpha_3, \alpha_1 + \alpha_3 - 3\alpha_4, 2\alpha_2 + \alpha_3 + \alpha_4] \in \mathbb{Z}^7 \\ &\Rightarrow \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z} \\ g(\omega') &= g(\omega) - g(\alpha_1 \boldsymbol{v_1} + \alpha_2 \boldsymbol{v_2} + \alpha_3 \boldsymbol{v_3}) \\ &= [0, 0, 0, 0, 0, -3\alpha_4, \alpha_4] \in \mathbb{Z}^7 \\ &\Rightarrow \alpha_4 \in \mathbb{Z} \end{aligned}$$

Proof of Theorem 2

$$
\begin{aligned}
g(\omega) &= [\alpha_1, \alpha_2, \alpha_2, \alpha_2, \alpha_3, \alpha_1 + \alpha_3 - 3\alpha_4, 2\alpha_2 + \alpha_3 + \alpha_4] \in \mathbb{Z}^7 \\
&\Rightarrow \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z} \\
g(\omega') &= g(\omega) - g(\alpha_1 \boldsymbol{v_1} + \alpha_2 \boldsymbol{v_2} + \alpha_3 \boldsymbol{v_3}) \\
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\end{aligned}
$$

Therefore,

Proof of Theorem 2

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&\Rightarrow \alpha_4 \in \mathbb{Z}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\omega' \in Im(res_{\mathcal{M}}) &\Rightarrow \omega \in Im(res_{\mathcal{M}}) \\
&\Rightarrow A(\mathfrak{F}) \subset Im(res_{\mathcal{M}}) \\
&\Rightarrow A(\mathfrak{F}) = A(G)|_{\mathcal{F}} \qquad \square
\end{aligned}
$$

Remark

$$B(A_4) := \{y_1[A_4/D_4] + y_2[A_4/C_3] + y_3[A_4/C_2] + y_4[A_4/E] \mid y_i \in \mathbb{Z}\}$$

$$\begin{array}{ccc} A(A_4) & \supset & B(A_4) \\ & {\scriptstyle res_{\mathcal{F}}} \searrow \quad \swarrow {\scriptstyle res'_{\mathcal{F}}} & \\ & A(\mathfrak{F}) = \lim_{\leftarrow \mathcal{F}} A(H) & \end{array}$$

Remark

$$B(A_4) := \{y_1[A_4/D_4] + y_2[A_4/C_3] + y_3[A_4/C_2] + y_4[A_4/E] \mid y_i \in \mathbb{Z}\}$$

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Question

Set $B(G)|_{\mathcal{F}} = Im(res'_{\mathcal{F}})$. **Does** $B(G)|_{\mathcal{F}}$ **coincide with** $A(\mathfrak{F})$ **?**

Remark

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 & 0 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & -1 \end{bmatrix}$$

Hence, $res'_{\mathcal{F}}$ isn't surjective.We then have

$$B(G)|_{\mathcal{F}} = \{(3y_1[D_4/D_4] + y_2[D_2/C_2] + y_2[D_2/C_2'] + y_2[D_2/C_2''] + y_3[D_4/E],$$
$$(y_3 + 3y_4)[C_3/C_3] + (y_1 + 2y_2 + y_3 - y_4)[C_3/E]) \in A(D_4) \times A(C_3) \mid y_i \in \mathbb{Z}\}.$$

$$\therefore \quad A(\mathfrak{F})/B(G)|_{\mathcal{F}} \cong \mathbb{Z}_3.$$

Remark

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 & 0 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & -1 \end{bmatrix}$$

Hence, $res'_{\mathcal{F}}$ isn't surjective.We then have

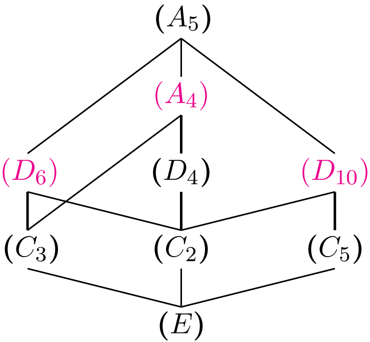
$$B(G)|_{\mathcal{F}}=\{(3y_1[D_4/D_4]+y_2[D_2/C_2]+y_2[D_2/C_2']+y_2[D_2/C_2'']+y_3[D_4/E],$$
$$(y_3+3y_4)[C_3/C_3]+(y_1+2y_2+y_3-y_4)[C_3/E])\in A(D_4)\times A(C_3) \mid y_i\in\mathbb{Z}\}.$$

$$\therefore \quad A(\mathfrak{F})/B(G)|_{\mathcal{F}} \cong \mathbb{Z}_3.$$

$\Rightarrow$ **In the case of $G = A_5$ ...?**

The case of A_5

$G = A_5$ **: Alternating group on five letters**
Relation among conjugacy classes subgroups of A_5



$(A_4), (D_{10}), (D_6)$ **: conjugacy classes of maximal subgroups of** A_5
Heavy lines show relations of normal subgroups.

The case of $G = A_5$

$$\begin{aligned}\mathcal{F} &= \mathcal{S}(G) \setminus \{G\} \\ &= (A_4) \amalg (D_{10}) \amalg (D_6) \amalg (D_4) \amalg (C_5) \amalg (C_3) \amalg (C_2) \amalg (E) \\ \mathcal{M} &= \{A_4, D_{10}, D_6\}\end{aligned}$$

Set

$$res_{\mathcal{M}} : A(A_5) \to A(A_4) \times A(D_{10}) \times A(D_6),$$

then $\mathbf{rank}_{\mathbb{Z}}(A(\mathfrak{F})) = \mathbf{rank}_{\mathbb{Z}}(Im(res_{\mathcal{M}}))$**.**

The case of $G = A_5$

$\omega_{A_5} \in A(A_5)$ **is written in the following form.**

$$\begin{aligned}\omega_{A_5} =& x_1[A_5/A_5] + x_2[A_5/A_4] + x_3[A_5/D_{10}] + x_4[A_5/D_6] + x_5[A_5/D_4] \\ &+ x_6[A_5/C_5] + x_7[A_5/C_3] + x_8[A_5/C_2] + x_9[A_5/E] \ (x_i \in \mathbb{Z})\end{aligned}$$

($[A_5/H]$ **: basis of** $A(A_5)$ **(**$H \leq A_5$**))**

$$\omega_{A_4} = a_1[A_4/A_4] + a_2[A_4/D_4] + a_3[A_4/C_3] + a_4[A_4/C_2] + a_5[A_4/E] \ (a_i \in \mathbb{Z})$$

$$\omega_{D_{10}} = b_1[D_{10}/D_{10}] + b_2[D_{10}/C_5] + b_3[D_{10}/C_2] + b_4[D_{10}/E] \ (b_i \in \mathbb{Z})$$

$$\omega_{D_6} = c_1[D_6/D_6] + c_2[D_6/C_3] + c_3[D_6/C_2] + c_4[D_6/E] \ (c_i \in \mathbb{Z})$$

The case of $G = A_5$

$H, K \leq A_5$

Table of $|(A_5/K)^H|$

A_5/K \ H	A_4	D_{10}	D_6	D_4	C_5	C_3	C_2	E
A_5/A_4	1	0	0	1	0	2	1	5
A_5/D_{10}	0	1	0	0	1	0	2	6
A_5/D_6	0	0	1	0	0	1	2	10
A_5/D_4	0	0	0	3	0	0	3	15
A_5/C_5	0	0	0	0	2	0	0	12
A_5/C_3	0	0	0	0	0	2	0	20
A_5/C_2	0	0	0	0	0	0	2	30
A_5/E	0	0	0	0	0	0	0	60

Calculation of $|(A_5/A_4)^{D_4}|$

$A_5/A_4 = \{aA_4 \mid a \in G\}$

$(G/A_4)^{D_4} = \{gA_4 \mid g \in G, g^{-1}D_4 g \subset A_4\}$

Let gA_4 be an element in $(G/A_4)^{D_4}$.

$D_4 = \{e, g_1, g_2, g_3 \mid g_i \in A_4, |g_i| = 2\ (i = 1, 2, 3)\}$, **so**

$$\begin{aligned} g^{-1}D_4 g &= g^{-1}\{e, g_1, g_2, g_3\}g \\ &= \{g^{-1}eg, g^{-1}g_1 g, g^{-1}g_2 g, g^{-1}g_3 g\} \subset D_4. \end{aligned}$$

Calculation example

$|g^{-1}D_4g| = |D_4|$ **, then**

$$D_4 = g^{-1}D_4g.$$

Hence,

$$g \in N_G(D_4) = A_4 \quad (N_G(D_4) \textbf{ is the normalizer of } D \textbf{ in } G)$$

and then, $gA_4 = A_4$**.**

Therefore, $|(G/A_4)^{D_4}| = 1$**.**

The case of $G = A_5$

Aim : Fill the following table

	$res^{A_5}_{A_4}$					$res^{A_5}_{D_{10}}$				$res^{A_5}_{D_6}$			
	$[A_4/A_4]$	$[A_4/D_4]$	$[A_4/C_3]$	$[A_4/C_2]$	$[A_4/E]$	$[D_{10}/D_{10}]$	$[D_{10}/C_5]$	$[D_{10}/C_2]$	$[D_{10}/E]$	$[D_6/D_6]$	$[D_6/C_3]$	$[D_6/C_2]$	$[D_6/E]$
$[A_5/A_5]$													
$[A_5/A_4]$													
$[A_5/D_{10}]$													
$[A_5/D_6]$													
$[A_5/D_4]$													
$[A_5/C_5]$													
$[A_5/C_3]$													
$[A_5/C_2]$													
$[A_5/E]$													

Calculation example

$< res_{A_4}^{A_5}[A_5/A_4] >$

$res_{A_4}^{A_5}[A_5/A_4] = a_1[A_4/A_4]+a_2[A_4/D_4]+a_3[A_4/C_3]+a_4[A_4/C_2]+a_5[A_4/E]$

First, we remark that $|A_5/A_4| = 5$.

$|(A_5/A_4)^{A_4}| = 1 \Longrightarrow a_1 = 1$

$|(A_5/A_4)^{D_4}| = 1$**, hence, fixed point of $(A_5/A_4)^{A_4}$ coincides with the fixed point of $(A_5/A_4)^{D_4}$, so $a_2 = 0$.**

$$|(A_5/A_4)^{C_3}| = 2, |A_4/C_3| = 4 \Longrightarrow a_3 = 1$$

$$\Longrightarrow a_4 = a_5 = 0$$

Calculation example

$< res_{A_4}^{A_5}[A_5/A_4] >$

$res_{A_4}^{A_5}[A_5/A_4] = a_1[A_4/A_4]+a_2[A_4/D_4]+a_3[A_4/C_3]+a_4[A_4/C_2]+a_5[A_4/E]$

First, we remark that $|A_5/A_4| = 5$.

$|(A_5/A_4)^{A_4}| = 1 \Longrightarrow a_1 = 1$

$|(A_5/A_4)^{D_4}| = 1$, **hence, fixed point of $(A_5/A_4)^{A_4}$ coincides with the fixed point of $(A_5/A_4)^{D_4}$, so $a_2 = 0$.**

$$\begin{aligned} |(A_5/A_4)^{C_3}| = 2, |A_4/C_3| = 4 &\Longrightarrow a_3 = 1 \\ &\Longrightarrow a_4 = a_5 = 0 \end{aligned}$$

Therefore, we have

$$res_{A_4}^{A_5}[A_5/A_4] = [A_4/A_4] + [A_4/C_3].$$

Result of table calculation

	$res^{A_5}_{A_4}$					$res^{A_5}_{D_{10}}$				$res^{A_5}_{D_6}$			
	$[A_4/A_4]$	$[A_4/D_4]$	$[A_4/C_3]$	$[A_4/C_2]$	$[A_4/E]$	$[D_{10}/D_{10}]$	$[D_{10}/C_5]$	$[D_{10}/C_2]$	$[D_{10}/E]$	$[D_6/D_6]$	$[D_6/C_3]$	$[D_6/C_2]$	$[D_6/E]$
$[A_5/A_5]$	1	0	0	0	0	1	0	0	0	1	0	0	0
$[A_5/A_4]$	1	0	1	0	0	0	0	1	0	0	1	1	0
$[A_5/D_{10}]$	0	0	0	1	0	1	0	1	0	0	0	2	0
$[A_5/D_6]$	0	0	1	1	0	0	0	2	0	1	0	1	1
$[A_5/D_4]$	0	1	0	0	1	0	0	3	0	0	0	3	1
$[A_5/C_5]$	0	0	0	0	1	0	1	0	1	0	0	0	2
$[A_5/C_3]$	0	0	2	0	1	0	0	0	2	0	1	0	3
$[A_5/C_2]$	0	0	0	1	2	0	0	2	2	0	0	2	4
$[A_5/E]$	0	0	0	0	5	0	0	0	6	0	0	0	10

$A(G)_{\mathcal{F}} = A(\mathfrak{F})$?

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\wr$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$A(G)_{\mathcal{F}} = A(\mathfrak{F})$?

Set

$$\begin{cases} \boldsymbol{v_1} &= [1,0,0,0,0,0,-2,1,0,1,0,0,0], \\ \boldsymbol{v_2} &= [0,1,0,0,0,0,-1,3,-1,0,0,3,-1], \\ \boldsymbol{v_3} &= [0,0,1,0,0,0,2,0,0,-1,1,1,0], \\ \boldsymbol{v_4} &= [0,0,0,1,0,0,-2,2,0,0,0,2,0], \\ \boldsymbol{v_5} &= [0,0,0,0,1,0,1,0,1,0,0,0,2], \\ \boldsymbol{v_6} &= [0,0,0,0,0,1,2,-1,0,0,0,0,0], \\ \boldsymbol{v_7} &= [0,0,0,0,0,0,5,0,-1,0,0,0,0], \\ \boldsymbol{v_8} &= [0,0,0,0,0,0,0,0,0,2,-1,-2,1], \end{cases}$$

then $\omega \in A(\mathfrak{F})$ **is written in the following form,**

$$\omega = \alpha_1\boldsymbol{v_1} + \alpha_2\boldsymbol{v_2} + \alpha_3\boldsymbol{v_3} + \alpha_4\boldsymbol{v_4} + \alpha_5\boldsymbol{v_5} + \alpha_6\boldsymbol{v_6} + \alpha_7\boldsymbol{v_7} + \alpha_8\boldsymbol{v_8}$$

$$(\alpha_i \in \mathbb{Q}).$$

$A(G)_{\mathcal{F}} = A(\mathfrak{F})$?

$$A(\mathfrak{F}) = (< \boldsymbol{v_1}, \cdots, \boldsymbol{v_8} >_{\mathbb{Q}}) \cap (A(A_4) \times A(D_{10}) \times A(D_6)),$$

if $\alpha_1, \cdots, \alpha_8 \in \mathbb{Z}$**, we can see that** $A(\mathfrak{F}) \subset Im(res_{\mathcal{M}})$**.**

$$A(\mathfrak{F}) \begin{array}{l} \nearrow \; A(A_4) = \sum_{(K_j)_{A_4}} \mathbb{Z} \cdot [A_4/K_j] \cong \mathbb{Z}^5 \\ \longrightarrow \; A(D_{10}) = \sum_{(K_j)_{D_{10}}} \mathbb{Z} \cdot [D_{10}/K_j] \cong \mathbb{Z}^4 \\ \searrow \; A(D_6) = \sum_{(K_j)_{D_6}} \mathbb{Z} \cdot [D_6/K_j] \cong \mathbb{Z}^4 \end{array}$$

$$\begin{array}{ccc} A(\mathfrak{F}) & \xrightarrow{\quad f \quad} & A(A_4) \times A(D_{10}) \times A(D_6) \\ & \searrow^{g} & \wr\| \\ & & \mathbb{Z}^{13} \end{array}$$

$A(G)_{\mathcal{F}} = A(\mathfrak{F})$

$$
\begin{aligned}
g(\omega) = [&\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, -2\alpha_1 - 2\alpha_2 + 2\alpha_3 - 2\alpha_4 + \alpha_5 + 2\alpha_6 + 5\alpha_7, \\
&\alpha_1 + 3\alpha_2 + 2\alpha_4 - \alpha_6, -2\alpha_2 + \alpha_5 - \alpha_7, \alpha_1 - \alpha_3 + 2\alpha_8, \alpha_3 - \alpha_8, \\
&3\alpha_2 + \alpha_3 + 2\alpha_4 - 2\alpha_8, -\alpha_2 + 2\alpha_5 + \alpha_8] \in \mathbb{Z}^{13} \\
&\Rightarrow \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \mathbb{Z} \\
g(\omega') = &\, g(\omega) - g(\alpha_1 \boldsymbol{v_1} + \alpha_2 \boldsymbol{v_2} + \alpha_3 \boldsymbol{v_3} + \alpha_4 \boldsymbol{v_4} + \alpha_5 \boldsymbol{v_5} + \alpha_6 \boldsymbol{v_6}) \\
&= [0, 0, 0, 0, 0, 0, 5\alpha_7, 0, -\alpha_7, \alpha_8, -\alpha_8, -2\alpha_8, \alpha_8] \in \mathbb{Z}^{13} \\
&\Rightarrow \alpha_7, \alpha_8 \in \mathbb{Z}
\end{aligned}
$$

$A(G)_{\mathcal{F}} = A(\mathfrak{F})$

$$
\begin{aligned}
g(\omega) &= [\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, -2\alpha_1 - 2\alpha_2 + 2\alpha_3 - 2\alpha_4 + \alpha_5 + 2\alpha_6 + 5\alpha_7, \\
&\quad \alpha_1 + 3\alpha_2 + 2\alpha_4 - \alpha_6, -2\alpha_2 + \alpha_5 - \alpha_7, \alpha_1 - \alpha_3 + 2\alpha_8, \alpha_3 - \alpha_8, \\
&\quad 3\alpha_2 + \alpha_3 + 2\alpha_4 - 2\alpha_8, -\alpha_2 + 2\alpha_5 + \alpha_8] \in \mathbb{Z}^{13} \\
&\quad \Rightarrow \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \mathbb{Z} \\
g(\omega') &= g(\omega) - g(\alpha_1 \boldsymbol{v_1} + \alpha_2 \boldsymbol{v_2} + \alpha_3 \boldsymbol{v_3} + \alpha_4 \boldsymbol{v_4} + \alpha_5 \boldsymbol{v_5} + \alpha_6 \boldsymbol{v_6}) \\
&= [0, 0, 0, 0, 0, 0, 5\alpha_7, 0, -\alpha_7, \alpha_8, -\alpha_8, -2\alpha_8, \alpha_8] \in \mathbb{Z}^{13} \\
&\quad \Rightarrow \alpha_7, \alpha_8 \in \mathbb{Z}
\end{aligned}
$$

Therefore,

$A(G)_{\mathcal{F}} = A(\mathfrak{F})$

$$\begin{aligned} g(\omega) &= [\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, -2\alpha_1 - 2\alpha_2 + 2\alpha_3 - 2\alpha_4 + \alpha_5 + 2\alpha_6 + 5\alpha_7, \\ &\quad \alpha_1 + 3\alpha_2 + 2\alpha_4 - \alpha_6, -2\alpha_2 + \alpha_5 - \alpha_7, \alpha_1 - \alpha_3 + 2\alpha_8, \alpha_3 - \alpha_8, \\ &\quad 3\alpha_2 + \alpha_3 + 2\alpha_4 - 2\alpha_8, -\alpha_2 + 2\alpha_5 + \alpha_8] \in \mathbb{Z}^{13} \\ &\Rightarrow \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \mathbb{Z} \\ g(\omega') &= g(\omega) - g(\alpha_1 \boldsymbol{v_1} + \alpha_2 \boldsymbol{v_2} + \alpha_3 \boldsymbol{v_3} + \alpha_4 \boldsymbol{v_4} + \alpha_5 \boldsymbol{v_5} + \alpha_6 \boldsymbol{v_6}) \\ &= [0, 0, 0, 0, 0, 0, 5\alpha_7, 0, -\alpha_7, \alpha_8, -\alpha_8, -2\alpha_8, \alpha_8] \in \mathbb{Z}^{13} \\ &\Rightarrow \alpha_7, \alpha_8 \in \mathbb{Z} \end{aligned}$$

Therefore,

$$\begin{aligned} \omega' \in Im(res_{\mathcal{M}}) &\Rightarrow \omega \in Im(res_{\mathcal{M}}) \\ &\Rightarrow A(\mathfrak{F}) \subset Im(res_{\mathcal{M}}) \\ &\Rightarrow A(\mathfrak{F}) = A(G)|_{\mathcal{F}} \end{aligned}$$

Remark

$$B(A_5) := \{y_1[A_5/A_4] + y_2[A_5/D_{10}] + y_3[A_5/D_6] + y_4[A_5/D_4] \\ + y_5[A_5/C_5] + y_6[A_5/C_3] + y_7[A_5/C_2] + y_8[A_5/E] \mid y_i \in \mathbb{Z}\}$$

$$\begin{array}{ccc} A(A_5) & \supset & B(A_5) \\ res_{\mathcal{F}} \searrow & & \swarrow res'_{\mathcal{F}} \\ & A(\mathfrak{F}) = \lim_{\leftarrow \mathcal{F}} A(H) & \end{array}$$

Remark

$$B(A_5) := \{y_1[A_5/A_4] + y_2[A_5/D_{10}] + y_3[A_5/D_6] + y_4[A_5/D_4] + y_5[A_5/C_5] + y_6[A_5/C_3] + y_7[A_5/C_2] + y_8[A_5/E] \mid y_i \in \mathbb{Z}\}$$

$$\begin{array}{ccc} A(A_5) & \supset & B(A_5) \\ & {\scriptstyle res_{\mathcal{F}}} \searrow \quad \swarrow {\scriptstyle res'_{\mathcal{F}}} & \\ & A(\mathfrak{F}) = \lim_{\leftarrow \mathcal{F}} A(H) & \end{array}$$

Question

Set $B(G)|_{\mathcal{F}} = Im(res'_{\mathcal{F}})$. $\Rightarrow B(G)|_{\mathcal{F}} \stackrel{?}{=} A(\mathfrak{F})$

Remark

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 \\
0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 3 \\
0 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 4 \\
0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 10
\end{bmatrix}
$$

$$\rightsquigarrow$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -1 & 1 & 2 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 3 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 4 & -1
\end{bmatrix}
$$

Hence, $res'_{\mathcal{F}}$ is surjective.We then have the result that

$$B(G)|_{\mathcal{F}} \cong A(\mathfrak{F}).$$

Result

Theorem 3

Set $\mathcal{F} = \mathcal{S}(G) \setminus \{G\}$*. For* $G = A_5$*, the alternating group on five letters,* $A(G)|_{\mathcal{F}} = A(\mathfrak{F})$*.*

Theorem 4

For $G = A_5$*, the alternating group on five letters,* $B(G)|_{\mathcal{F}} = A(\mathfrak{F})$*.*