Paramodular forms and compact twist

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In the early eighties, the author proposed several explicit conjectures on correspondence between (global) Siegel modular forms of the split $Sp(4)$ and automorphic forms of the compact twist of $Sp(4)$, both belonging to parahoric subgroups locally. Actually we treated two cases, one is for paramodular type subgroups, and the other is for the minimal parahoric subgroups. These two conjectures were based on the evidence of global dimensional equality and numerical examples supporting the conjectures. (cf. [5], [4], [3]). This was an attempt to generalize the classical but neatly described Eichler correspondence between modular forms of $SL(2)$ and $SU(2)$, rather than the general Jacquet-Langlands correspondence. The project was abandoned for many years but now I restarted this with young mathematician S. Wakatsuki, in particular in the case of paramodular groups treated in [4]. Here we would like to report some new results as well as our old thoughts. We give our conjecture in section 4 and evidence on global dimensional equality in section 5. A new explicit result on the dimension of vector valued Siegel paramodular forms is announced in section 6.

1 Paramodular groups, Siegel modular forms, and new forms

We fix a prime p throughout the paper, and denote by $K(p)$ the so called (global) paramodular group defined as follows.

$$K(p) = Sp(4,\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

Locally this is one of the parahoric subgroups of $Sp(4,\mathbb{Q}_p)$. We denote by $\rho_{k,j}$ the irreducible rational representation of $GL(2)$ defined by $\rho_{k,j}(g) = \det(g)^k Sym_j(g)$ where Sym_j is the j-th symmetric tensor representation.

For any function F on the Siegel upper half space H_2 of degree two and $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(4,\mathbb{R})$, we write

$$(F|_{k,j})[g] = \rho(CZ+D)^{-1}F(Z).$$

Let Γ be any discrete (arithmetic) subgroup of $Sp(4,\mathbb{R})$. A holomorphic cusp form of Γ of weight $\rho_{k,j}$ is a $\mathbb{C}^{j+1}$-valued holomorphic function $F(Z)$ on H_2 such that $F|_{k,j}[\gamma] = F$ for all $\gamma \in \Gamma$ which vanishes at the boundaries of the Satake compactification of $\Gamma\backslash H_2$. We denote by $S_{k,j}(\Gamma)$ the space of cusp forms of weight $\rho_{k,j}$ belonging to Γ.

We now define new forms of $S_{k,j}(K(p))$. We put $\rho = \begin{pmatrix} 0 & 1_2 \\ p1_2 & 0 \end{pmatrix}$. Three groups $Sp(4,\mathbb{Z})$, $\rho^{-1}Sp(4,\mathbb{Z})\rho$ and $K(p)$ contains the same Iwahori subgroup

$$B(p) = Sp(4,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

For $F \in S_{k,j}(Sp(4,\mathbb{Z})) + S_{k,j}(\rho^{-1}Sp(4,\mathbb{Z})\rho) \subset S_{k,j}(B(p))$, we put

$$Tr(F) = \sum_{\gamma \in B(p)\backslash K(p)} F|_{k,j}[\gamma].$$

In [4], we defined the space of old forms of $S_{k,j}(K(p))$ as $Tr(S_{k,j}(Sp(4,\mathbb{Z})) + S_{k,j}(Sp(4,\mathbb{Z}))$ and the space of new forms as the subspace of $S_{k,j}(K(p))$ orthogonal to the old forms. (In [4], we treated only the scalar valued case, i.e. the case of $j = 0$, but we need no essential change for $j > 0$.)

Locally at p there are three maximal compact subgroups of $Sp(4,\mathbb{Q}_p)$ up to conjugation, i.e. the completion of $K(p)$, $Sp(4,\mathbb{Z}_p)$ and $\rho^{-1}Sp(4,\mathbb{Z}_p)\rho$, and we are regarding in the above that those representation whose local component at p has a fixed vector by $Sp(4,\mathbb{Z}_p)$ or $\rho^{-1}Sp(4,\mathbb{Z}_p)\rho$ are "old forms". Recently we have more general theory by Roberts and Schmidt [8] for local theory of new vectors for paramodular groups of level p^n. Our old (global) definition is essentially the same as their local definition for the level p case.

We denote by $S_{k,j}^{new}(K(p))$ the space of new forms.

2 Compact twist

Let D be the division quaternion algebra ramified only at p and ∞. We put

$$G = \{g \in M_2(D); g\,{}^t\overline{g} = n(g)1_2, n(g) \in \mathbb{Q}_{>0}^{\times}\}$$

This is a $\mathbb{Q}$-form of $GSp(4)$ and we can expect that there should exist a good correspondence between automorphic forms of $GSp(4, Q_A)$ and G_A. Here we give necessary definitions. Let G_A be the adelization and G_q or G_∞ its local components. Let O be a maximal order of D and D_q or O_q the local completion at a prime q. We fix a prime element π of O_p. For any prime $q \neq p$, we put $U_q = G_q \cap M_2(O_q)^\times$. To describe the local group at p, we change the model. We put

$$\begin{aligned} G_p^* &= \left\{ g \in M_2(D_p); g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t\overline{g} = n(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, n(g) \in \mathbb{Q}_p^\times \right\}, \\ U_p &= G_p^* \cap \begin{pmatrix} O_p & \pi^{-1}O_p \\ \pi O_p & O_p \end{pmatrix}^\times . \end{aligned}$$

Since $G_p^* \cong G_p$, we identify these two groups from now on. We put $U = G_\infty \prod_{v:prime} U_v$. We take a rational representation (ρ_{f_1,f_2}, V) of $USp(4) = \{g \in M_2(\mathbb{H}); g\,{}^t\overline{g} = 1_2\}$ ($\mathbb{H}$: the Hamilton quaternions) corresponding to the Young diagram parameter $f_1 \geq f_2 \geq 0$. We assume that $f_1 \equiv f_2 \bmod 2$. Then ρ_{f_1,f_2} factors through $USp(4)/\{\pm 1_2\}$. We define a representation of G_A by

$$G_A \rightarrow G_\infty \rightarrow G_\infty/\mathbb{R}^\times 1_2 \cong USp(4)/\{\pm 1_2\} \stackrel{\rho_{f_1,f_2}}{\longrightarrow} GL(V)$$

and denote this also by ρ_{f_1,f_2}. A V-valued function $f(g)$ on G_A is defined to be an automorphic form of weight ρ_{f_1,f_2} belonging to U if $f(uga) = \rho_{f_1,f_2}(u)f(g)$ for any $a \in G$, $u \in U$, and $g \in G_A$. We denote the space of these automorphic forms by $M_{f_1,f_2}(U)$. We interpret $M_{f_1,f_2}(U)$ more concretely as follows. We take a double coset decomposition $G_A = \cup_{i=1}^H U g_i G$ and put $\Gamma_i = G \cap g_i^{-1} U g_i$. We put $V^{\Gamma_i} = \{v \in V; \rho_{f_1,f_2}(\gamma)v = v \text{ for all } \gamma \in \Gamma_i\}$. Then we have $M_{f_1,f_2}(U) \cong \oplus_{i=1}^H V^{\Gamma_i}$ (cf. [1]). The constant function on G_A is an automorphic form of weight $\rho_{0,0}$. Since we sometimes want to exclude the constant function, we denote by $S_{0,0}(U)$ the orthogonal complement of the space of constant functions in $M_{0,0}(U)$. When $(f_1, f_2) \neq (0, 0)$, we just put $S_{f_1,f_2}(U) = M_{f_1,f_2}(U)$.

3 Ihara lifting and old forms

To define old forms of $M_{f_1,f_2}(U)$, we must explain Ihara lifting from $USp(2) \times SL(2)$ to G_A (cf. [7], [6]). This is a kind of compact version of Saito-Kurokawa lifting or Yoshida lifting. We put $f_1 + f_2 = 2\nu$ (ν: a non-negative integer). We take the space $\mathcal{H}_{2\nu}$ of (real) harmonic polynomials of 8 variables. Since $\mathbb{H}^2 \cong \mathbb{R}^8$ where $\mathbb{H}$ is the Hamilton quaternions, we can regard $P \in \mathcal{H}_{2\nu}$

as a function on $\mathbb{H}^2$, so we write $P = P(x, y)$ $(x,\ y \in \mathbb{H})$. The compact orthogonal group $SO(8)$ is acting naturally on $\mathcal{H}_{2\nu}$. We put $USp(n) = \{g \in M_n(\mathbb{H}); g\,{}^t\overline{g} = 1_n\}$. Then $(\alpha, g) \in USp(2) \times USp(4)$ acts on $\mathcal{H}_{2\nu}$ by $P(x, y) \to P((\overline{\alpha}x, \overline{\alpha}y)g)$. For any integers a and b with $a \geq b \geq 0$, we denote by (σ_{a-b}, V_{a-b}) the symmetric tensor representation of $USp(2)$ of degree $a-b$ and by $(\rho_{a,b}, V_{a,b})$ the representation of $USp(4)$ corresponding to the Young diagram parameter (a, b). The space $\mathcal{H}_{2\nu}$ can be decomposed into irreducible representations of $USp(2) \times USp(4)$ as

$$\mathcal{H}_{2\nu} \cong \sum_{a+b=2\nu, a\geq b\geq 0} V_{a,b}$$

where $V_{a,b}$ is the representation space of $\sigma_{a-b} \otimes \rho_{a,b}$ ($V_{a,b}$ is more explicitly given but we omit the details here.) We take a double coset decomposition of the adelization $D_A^\times$: $D_A^\times = \cup_{i=1}^h D^\times h_i O_A^\times$ where $O_A^\times = \mathbb{H}^\times \prod_{v;primes} O_v^\times$. For i with $1 \leq i \leq h$, we write $E_i = D^\times \cap h_i^{-1} O_A^\times h_i$. For any (i, κ) with $1 \leq i \leq h$ and $1 \leq \kappa \leq H$, we denote by $V_{a,b}^{E_i \times \Gamma_\kappa}$ the space of $E_i \times \Gamma_\kappa$ invariant vectors in $V_{a,b}$. Then the space

$$W_{a,b} = \oplus_{1\leq i\leq h, 1\leq\kappa\leq H} V_{a,b}^{E_i\times\Gamma_\kappa}$$

is the tensor product of automorphic forms of $D_A^\times$ of weight σ_{a-b} with respect to $O_A^\times$ and $M_{a,b}(U)$. We take $F = (F_{i\kappa}) \in W$ where $F_{i\kappa} \in V_{a,b}^{E_i\times\Gamma_\kappa}$.

There exists a maximal left O-lattice L in D^2 such that $U = \{g \in G_A; Lg = L\}$, where we put $Lg = \cap_v (L_v g_v \cap D^2)$, $L_v = O_v \otimes L$. This is so called a maximal lattice in the non-principal genus. We put $L_{i\kappa} = \overline{h_i} L g_\kappa$. For $F = (F_{i\kappa}) \in W$, we put

$$\vartheta_{i\kappa}(\tau) = \sum_{x\in L_{i\kappa}} F_{i\kappa}(x) e^{2\pi i\, n(x)\tau/n(L_{i\kappa})}$$

where τ is the variable of the upper half space and $n(L_{i\kappa})$ the positive generator of the $\mathbb{Z}$-ideal generated by the norms of $z \in L_{i\kappa}$. We also put

$$\vartheta_F(\tau) = \sum_{i=1}^{h} \sum_{\kappa=1}^{H} \frac{1}{\#(E_i)\#(\Gamma_\kappa)} \vartheta_{i\kappa}(\tau).$$

Since it is easy to see that L_{ik} is equivalent to E_8 for our choice of U, and since $F_{i\kappa}$ is a harmonic polynomial of degree $2\nu = a + b$, we see that $\vartheta(\tau) \in A_{a+b+4}(SL_2(\mathbb{Z}))$, the space of elliptic modular forms of weight $a + b + 4$ of $SL_2(\mathbb{Z})$, and if $a + b \neq 0$, this is a cusp form.

We can naturally define the action of Hecke operators of G_A and $D_A^\times$ on F. We assume that F is the common eigenform of all the Hecke operators $S(m)$

of $(D_A^\times, O_A^\times)$ and $T(m)$ of (G_A, U). Or in other word, $F = F_1 \otimes F_2$ where F_1 or F_2 is a common eigenform of $D_A^\times$ or G_A. We put $S(m)F_1 = s(m)F_1$ and $T(m)F_2 = \tau(m)F_2$, where $S(m)$ and $T(m)$ are the Hecke operators consisting of those elements of similitude norm m. We write $s_0(m) = m^{-b}s(m)$.

Under the assumption that F is a common eigenform, we can show that $\vartheta_F(\tau) \neq 0$ if and only if the coefficient of $\vartheta_F(\tau)$ at $e^{2\pi i\tau}$ does not vanish.

Theorem 3.1 ([7],[6]) *Assumptions and notation being the same as above, we assume that $\vartheta_F \neq 0$. Then we have*

$$L(s, F_2) = L(s-b-1, F_1)L(s, \vartheta_F).$$

Here we define

$$L(s, \vartheta_F) = \prod_{q;any prime} (1 - c(q)q^{-s} + q^{a+b+3-2s})^{-1}$$

as usual where $c(m)$ is the Fourier coefficient of ϑ_F, and

$$L(s, F_1) = (1 - s_0(p)p^{-s})^{-1} \prod_{q \neq p} (1 - s_0(q)q^{-s} + q^{a-b+1-2s})^{-1}.$$

The definition of $L(s, F_2)$ is given in the next section

We define the space of old forms to be the subspace of $S_{a,b}(U)$ generated by those F_2 such that $\vartheta_{F_1 \otimes F_2} \neq 0$ for some automorphic eigenform F_1 of $D_A^\times$ with respect to $O_A^\times$ of weight σ_{a-b}. We denote this space by $S_{a,b}^{new}(U)$.

We note that by Eicher's theorem, F_1 corresponds to an elliptic modular new form of weight $a-b+2$ belonging to $\Gamma_0^{(1)}(p)$ where

$$\Gamma_0^{(1)}(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); c \equiv 0 \bmod p \right\}.$$

4 Local L functions at good and bad primes

For a prime $q \neq p$, we see that $U_q \cong GSp(2, \mathbb{Z}_q)$, so for each common eigenform $F \in S_{k,j}(K(p))$ or $F \in M_{k+j-3,k-3}(U)$, we define

$$L_q(s, F) = (1 - \lambda(q)q^{-s} + (\lambda(q)^2 - \lambda(q^2) - q^{2k+j-4})q^{-2s} - \lambda(q)q^{2k+j-3-3s} + q^{4k+2j-6-4s})^{-1},$$

where $T(q)F = \lambda(q)F$ and $T(q^2)F = \lambda(q^2)F$. Here $T(q^n)$ is the $Sp(4, \mathbb{Z}_q)$ double coset consisting of $g \in GSp(4, \mathbb{Q}_q) \cap M_4(\mathbb{Z}_q)$ with $gJ\,{}^tg = q^nJ$.

Now we consider the L function at the bad prime p. The Hecke algebra for the pair of G_p^* and U_p is generated by $T(1,p) = U_p \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} U_p$, $R(p) = U_p \pi$, and $U_p \pi^{-1}$. We put

$$T(p^\nu) = \left\{ g \in G_p^* \cap \begin{pmatrix} O_p & \pi^{-1} O_p \\ \pi O_p & O_p \end{pmatrix} ; g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t\overline{g} = p^\nu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Then we can show that

$$\sum_{\nu=0}^{\infty} T(p^\nu) u^\nu = \frac{1 + pR(p)u}{1 - (T(p) - pR(p))u + p^3 R(p^2) u^2}.$$

Let $F \in M_{f_1,f_2}(U)$ be a common eigenform of $T(p)$ and $R(p)$ with $T(p)F = \tau(p)F$ and $R(p)F = r(p)F$. We note that $r(p^2) = p^{f_1+f_2}$ and $r(p) = \pm p^{(f_1+f_2)/2}$. Although the denominator in the above series is of degree two with respect to p^{-s}, we define the local L function of F at p to be the degree three one with respect to p^{-s} given by

$$L_p(s,F) = \frac{1}{(1 - r(p)p^{1-s})(1 - (\tau(p) - pr(p))p^s + r(p^2)p^{-2s})}$$

The reason of this definition is as follows.

For the split $GSp(4)$, for a local new vector ϕ_π of the local admissible representation π of (paramodular) level p, Roberts and Schmidt gave the definition of local L-function by

$$L_p(s,\phi_\pi) = \frac{1}{1 - q^{-3/2}(\lambda_\pi + \epsilon_\pi)q^{-s} + (q^{-2}\mu_\pi + 1)q^{-2s} + \epsilon_\pi q^{-1/2} q^{-3s}}$$

where the notation is as in [8]. But actually we can show that this L function decomposes into the product of degree one and two. That is, we can show the following theorem (obtained after the conference).

Theorem 4.1 *In the above, we have*

$$\mu_\pi + \epsilon_\pi \lambda_\pi + q + 1 = 0.$$

In particular, we get

$$L(s,\phi_\pi) = \frac{1}{(1 + q^{-1/2}\epsilon_\pi q^{-s})(1 - q^{-3/2}(\lambda_\pi + (q+1)\epsilon_\pi)q^{-s} + q^{-2s})}.$$

Sketch of the proof. We can show that the action on ϕ_π of the double coset corresponding to $\mu_\pi+\epsilon_\pi\lambda_\pi+q+1$ induces $Sp(4,\mathbb{Z}_p)$ fixed vector which should be zero by our assumption that ϕ_π is a new vector of level p. The details is omitted here. (Roberts and Schmidt informed me later that this can be shown also by using their table of the classification of the local representation in [8].)

Judging from this split $GSp(4)$ case, it is also natural that we took the degree three local L function for the compact twist defined as above.

In both split and compact case, the global L function is defined to be the product of the local L functions at all primes, including the bad prime.

5 Conjectures and comparison of dimensions

The following conjecture was first proposed for the scalar valued case in [4].

Conjecture 5.1 *For any integer $k \geq 3$ and even $j \geq 0$, we have an linear isomorphism*

$$\psi : S_{k,j}^{new}(K(p)) \cong S_{k+j-3,k-3}^{new}(U)$$

such that $L(s,F) = L(s,\psi(F))$ (including the bad Euler factors) for any common eigenforms $F \in S_{k,j}(K(p))$ of Hecke operators.

We can show the following relation between dimensions which gives a good evidence for the above conjecture.

Theorem 5.2 (cf. [4] when $j=0$. The case $j>0$ is new.)
For $k>4$ and even $j \geq 0$, we have

$$\begin{aligned} & \dim S_{k,j}(K(p)) - 2\dim S_{k,j}(Sp(4,\mathbb{Z})) \\ = \; & \dim S_{k+j-3,k-3}(U) - \dim S_{2k+j-2}(SL_2(\mathbb{Z})) \times A_{j+2}^{new}(\Gamma_0^{(1)}(p)) \end{aligned}$$

Here $A_2^{new}(\Gamma_0^1(p))$ is the space of modular forms (not necessarily a cusp form) of $\Gamma_0^{(1)}(p)$ of weight 2, and for $j>0$, $A_{j+2}^{new}(\Gamma_0^{(1)}(p))$ is the space of new cusp forms of weight $j+2$,

Proof. The first term of the right hand side is known in [2] II for any $k \geq 3$. The first term of the left hand side for $j=0$ and $k>4$ is in [4] and the case $j>0$ and $k>4$ is a new result jointly obtained with S. Wakatsuki and given in the next section.

Remark. We have also several partial results on comparison of traces of Hecke operators, but not completed yet.

Based on the above relation, we give conjectures on the dimension of the space of new forms.

Conjecture 5.3 *We assume that j is even with $j \geq 0$.*
For even $k \geq 3$, we should have

$$\begin{aligned}\dim S_{k,0}^{new}(K(p)) &= \dim S_{k,0}(K(p)) - 2\dim S_{k,0}(Sp(4,\mathbb{Z})) + \dim S_{2k-2}(SL_2(\mathbb{Z})).\\ \dim S_{k-3,k-3}^{new}(U) &= \dim S_{k-3,k-3}(U) - \dim S_{2k-2}(SL_2(\mathbb{Z})) \times \dim S_2(\Gamma_0^{(1)}(p)).\end{aligned}$$

For odd k or positive j, we should have

$$\begin{aligned}\dim S_{k,j}^{new}(K(p)) &= \dim S_{k,j}(K(p)) - 2\dim S_{k,j}(Sp(4,\mathbb{Z})),\\ \dim S_{k+j-3,k-3}^{new}(U) &= \dim S_{k+j-3,k-3}(U) - \dim S_{2k+j-2}(SL_2(\mathbb{Z})) \times S_{j+2}^{new}(\Gamma_0^{(1)}(p))\end{aligned}$$

Of course this conjecture for $S_{k+j-3,k-3}^{new}(U)$ implies a certain kind of non-vanishing theorem of the Ihara lifting, with slight modification when k is even and $j = 0$. We omit the description in detail.

6 An explicit dimension formula of $S_{k,j}(K(p))$

Theorem 6.1 (See [4] when $j = 0$. Joint with S. Wakatsuki when $j > 0$)

For $k > 4$ and even $j \geq 0$, we have

$$\dim S_{k,j}(K(p)) = \sum_{i=1}^{12} H_i + \sum_{i=1}^{7} I_i$$

where H_i and I_i are given belowe.

Here the condition $k > 4$ in the theorem comes from the condition on the convergence of the Godement's trace formula.

We use the following notation. For natural number m and n, we mean by $[a_1, \ldots, a_r : m]_n$ that it is a_i if $n \equiv i \bmod m$.

$$H_1 = \frac{p^2+1}{2880} \times \frac{1}{6}(j+1)(k-2)(j+k-1)(j+2k-3).$$

$$H_2 = (-1)^k (j+k-1)(k-2) \times \begin{cases} 7/576 & \text{if } p \neq 2, \\ 11/1152 & \text{if } p = 2. \end{cases}$$

$$H_3 = \begin{cases} [(-1)^{j/2}(k-2), -(j+k-1), -(-1)^{j/2}(k-2), (j+k-1); 4]_k / 2^4 \cdot 3 \\ \qquad \text{if } p \neq 2, \\ 5[(-1)^{j/2}(k-2), -(j+k-1), -(-1)^{j/2}(k-2), (j+k-1); 4]_k / 2^5 \cdot 3 \\ \qquad \text{if } p = 2 \end{cases}$$

$$H_4 = ([j+k-1, -(j+k-1), 0; 3]_k + [k-2, 0, -(k-2); 3]_{j+k}) \times \begin{cases} 1/2^2 \cdot 3^3 & \text{if } p \neq 3 \\ 5/2^2 \cdot 3^3 & \text{if } p = 3 \end{cases}$$

$$H_5 = [-(j+k-1), -(j+k-1), 0, (j+k-1), (j+k-1), 0; 6]_k / 2^2 \cdot 3^2$$

$$H_6 = \begin{cases} \dfrac{5(p+1)(-1)^{j/2}(2k+j-3)}{2^7 3} + \dfrac{(p+1)(-1)^{k+j/2}(j+1)}{2^7} \\ \qquad \text{if } p \equiv 1 \bmod 4 \\ \dfrac{(p-1)(-1)^{j/2}(2k+j-3)}{2^7} + \dfrac{5(p-1)(-1)^{k+j/2}(j+1)}{2^7 3} \\ \qquad \text{if } p \equiv 3 \bmod 4 \\ \dfrac{3(-1)^{j/2}(2k+j-3)}{2^7} + \dfrac{7(-1)^{k+j/2}(j+1)}{2^7 3} \\ \qquad \text{if } p = 2 \end{cases}$$

$$H_7 = \begin{cases} \dfrac{p+1}{2\cdot 3^3}(2k+j-3)[-1,0,1;3]_{j+2} + \dfrac{p+1}{2^2 3^3}(j+1)[-1,0,1;3]_{2k+j-2} \\ \text{if } p \equiv 1 \bmod 3 \\ \dfrac{p-1}{2^2 3^3}(2k+j-3)[-1,0,1:3]_{j+2} + \dfrac{p-1}{2\cdot 3^3}(j+1)[-1,0,1;3]_{2k+j-2} \\ \text{if } p \equiv 2 \bmod 3 \\ \dfrac{5}{2^3 3^3}(2k+j-3)[-1,0,1;3]_{j+2} + \dfrac{1}{3^3}(j+1)[-1,0,1;3]_{2k+j-2} \\ \text{if } p = 3. \end{cases}$$

$H_8 = 2^{-1}3^{-1}C_1$,

where

$$C_1 = \begin{cases} [1,0,0,-1,-1,-1,-1,0,0,1,1,1;12]_k & \text{if } j \equiv 0 \bmod 12 \\ [-1,1,0,1,1,0,1,-1,0,-1,-1,0;12]_k & \text{if } j \equiv 2 \bmod 12 \\ [1,-1,0,1,1,0,1,-1,0,-1,-1,0;12]_k & \text{if } j \equiv 4 \bmod 12 \\ [-1,0,0,-1,1,-1,1,0,0,1,-1,1;12]_k & \text{if } j \equiv 6 \bmod 12 \\ [1,1,0,1,-1,0,-1,-1,0,-1,1,0;12]_k & \text{if } j \equiv 8 \bmod 12 \\ [-1,-1,0,0,1,1,1,1,0,0,-1,-1;12]_k & \text{if } j \equiv 10 \bmod 12 \end{cases}$$

$$H_9 = \begin{cases} \dfrac{2}{3^2} C_2 & \text{if } p \neq 2 \\ \dfrac{1}{2\cdot 3^2} C_2 & \text{if } p = 2 \end{cases}$$

where

$$C_2 = \begin{cases} [1,0,0,-1,0,0,6]_k & (j \equiv 0 \bmod 6) \\ [-1,1,0,1,-1,0;6]_k & (j \equiv 2 \bmod 6) \\ [0,-1,0,0,1,0;6]_k & (j \equiv 4 \bmod 6) \end{cases}$$

$$H_{10} = C_3 \times \begin{cases} 2/5 & \text{if } p \equiv \pm 1 \bmod 5 \\ 1/5 & \text{if } p = 5 \\ 0 & \text{if } p \equiv 2,3 \bmod 5 \end{cases}$$

where

$$C_3 = \begin{cases} [1,0,0,-1,0;5]_k & \text{if } j \equiv 0 \bmod 10 \\ [-1,1,0,0,0;5]_k & \text{if } j \equiv 2 \bmod 10 \\ 0 & \text{if } j \equiv 4 \bmod 10 \\ [0,0,0,1,-1;5]_k & \text{if } j \equiv 6 \bmod 10 \\ [0,-1,0,0,1;5]_k & \text{if } j \equiv 8 \bmod 10 \end{cases}$$

$$H_{11} = C_4 \times \begin{cases} 1/4 & \text{if } p \equiv 1,-1 \bmod 8 \\ 0 & \text{if } p \equiv 3,5 \bmod 8 \\ 1/8 & \text{if } p = 2 \end{cases}$$

where

$$C_4 = \begin{cases} [1,0,0,-1;4]_k & j \equiv 0 \bmod 8 \\ [-1,1,0,0;4]_k & j \equiv 2 \bmod 8 \\ [-1,0,0,1;4]_k & j \equiv 4 \bmod 8 \\ [1,-1,0,0;4]_k & j \equiv 6 \bmod 8 \end{cases}$$

$$H_{12} = \begin{cases} -\frac{1}{6}(-1)^{(j+2)/2}[1,0,-1;3]_{2k+j-2} & \text{if } p \equiv 1 \bmod 12 \\ \frac{1}{6}(-1)^{(2k+j-2)/2}[1,0,-1;3]_{j+2} & \text{if } p \equiv 11 \bmod 12 \\ 0 & \text{if } p \equiv 5,7 \bmod 12 \\ \frac{1}{12}(-1)^{(2k+j-2)/2}[1,0,-1;3]_{j+2} & \text{if } p = 2 \text{ or } 3. \end{cases}$$

$$\begin{aligned} I_1 &= -p(j+1)(2k+j-3)/2^4 3^2 + \frac{(p+1)(j+1)}{2^3 3} - (j+1)/24 \\ I_2 &= \frac{1}{2^4}(-1)^k \left(4 - \left(\frac{-1}{p}\right)\right) - (-1)^k (j+2k-3)/24. \\ I_3 &= -2^{-2}[(-1)^{j/2}, -1, (-1)^{j/2+1}, 1; 4]_k \end{aligned}$$

$$I_4 = \frac{1}{18}A + \begin{cases} \frac{1}{9}A & \text{if } p = 3\ , \\ \frac{1}{9}C & \text{if } p \equiv 1 \bmod 3\ , \\ \frac{1}{9}B & \text{if } p \equiv 2 \bmod 3\ . \end{cases}$$

where

$$\begin{aligned} A &= -[1,-1,0;3]_k - [1,0,-1;3]_{j+k} \\ C &= -2[1,0,1;3]_k - 2[0,-1,-1;3]_{j+k} \\ B &= 2A - C \end{aligned}$$

$$\begin{aligned} I_5 &= -\frac{1}{6} \times ([-1,-1,0,1,1,0;6]_k + [1,0,-1,-1,0,1;6]_{j+k})\,. \\ I_6 &= (-1)^{(j+2)/2} \times \frac{1}{8}\left(1 + \left(\frac{-1}{p}\right)\right). \\ I_7 &= -\frac{1}{6}\left(1 + \left(\frac{-3}{p}\right)\right)[-1,0,1;3]_{j+2} \end{aligned}$$

By several reasons, we propose the following coinjecture.

Conjecture 6.2 *For $k \geq 4$ or $k = 3$ and $j > 0$, the formula for* $\dim S_{k,j}(K(p))$ *is the same as above. For $k = 3$ and $j = 0$, the formula for* $\dim S_{3,0}(K(p))$ *should be the above formula plus one.*

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