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Conjectures of Shimura Type and of Harder Type Revisited

by

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Dedicated to Professor Fumihiro Sato on the occasion of his 65th birthday.

Abstract. We propose a conjecture of Shimura type that the level one part of the space of vector valued Siegel modular forms of degree two of half integral weight without character (Haupt type) corresponds bijectively, up to liftings, to the space of vector valued Siegel modular forms of integral weight of degree two of level one. This is a generalization of our previous conjecture for Neben type (with character). Together with the previous conjecture, this means that Siegel modular forms of degree two of half integral weight with character and without character should correspond bijectively and Hecke equivariantly up to liftings. The Harder conjecture on congruences for vector valued Siegel cusp forms of integral weight is now interpreted as a half-integral weight version which means the congruence between eigenvalues of Siegel cusp forms and non-cusp forms of half-integral weight of the same group. We give a concrete example that this congruence really holds.

In the previous paper [12], we gave a conjecture on bijective correspondence between vector valued holomorphic Siegel cusp forms of integral weight of degree two of level one and those of half-integral weight belonging to the plus subspace in level 4 with character (of *Neben* type), preserving L functions. As a by-product, we stated there a half-integral interpretation of Harder's conjecture on congruences as a congruence between a Siegel cusp form of half integral weight of *Neben* type and the Klingen type Eisenstein series of half integral weight of *Haupt* type. This is interesting since in the original Harder's conjecture, the congruence is stated as a congruence between a Siegel modular form and an elliptic cusp form and not between Siegel modular forms. But this new version has an unsatisfactory point, that is, two Siegel modular forms in question belong apparently to different discrete subgroups, and this caused difficulty to imagine a general proof. In March in 2012, Neil Dummigan wrote me an email on his guess that the case without character (Haupt type) and with character (Neben type) might not be so different. His guess is based on the following observation. Some lifting conjecture for half integral weight is known in [9] for Haupt type (as a special case of [15]). On the other hand, for integral weight, there is no

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lifting to holomorphic vector valued Siegel modular forms of level one, so maybe liftings would appear in real analytic ones. This would mean that real analytic ones correspond with half-integral weight of Haupt type. But these real analytic ones and holomorphic ones for integral weight would correspond up to liftings by some L packet conjecture, so maybe Haupt type and Neben type might not be so different. By reading this, I started to think of evidence of this problem and was fully convinced that his guess is right. So, in this paper, we give a similar conjecture as the previous one with good evidence that the subspace of vector valued Siegel modular forms of half-integral weight without character (of *Haupt* type), which is orthogonal to the certain subspace of lifts, corresponds bijectively to those of integral weights. As a result of this conjecture, we can interpret Harder's conjecture on congruences as a congruence between a holomorphic Siegel cusp form and a holomorphic Siegel non-cusp form, both of the same half-integral weight belonging to the same discrete group, and we can give a concrete example of this type of congruences. The conjecture mentioned above are based on coincidence of (conjectural) dimension formulas and numerical examples. We state our main conjectures in section 1. In section 2, we give precise definitions and notations we used in section 1. In section 3, we give a comparison of general dimensions of the spaces of modular forms in question as very strong evidence of our conjectures. In section 4, first we explain relations between the spaces of Siegel modular forms of half-integral weight and the spaces of holomorphic and skew holomorphic Jacobi forms. Then we give a concrete numerical example of the correspondence and the congruence in the conjectures, together with some general explanation how to obtain such examples.

Acknowledgements. I would like to thank Neil Dummigan very much for drawing my attention to compare the Haupt type and Neben type. Without his email, this study would not have been done. (See his preprint [1] for discussion of related matters.)

1. Main conjectures

We will explain more details now. We denote by $\Gamma_n = Sp(n, \mathbb{Z}) \subset M_{2n}(\mathbb{Z})$ the Siegel modular group of degree n. For any integer $j \geq 0$, we denote by $\rho_j = Sym_j$ the j-th symmetric tensor representation of GL_n. For any integer $k, j \geq 0$, we denote by $A_{k,j}(\Gamma_n)$ the space of Siegel modular forms of weight $\det^k Sym_j$ belonging to Γ_n, and by $S_{k,j}(\Gamma_n)$ the subspace of cusp forms in $A_{k,j}(\Gamma_n)$. We note that if j is odd and $n = 2$, then we have $A_{k,j}(\Gamma_2) = 0$.

We define a congruence subgroup $\Gamma_0^{(n)}(4)$ of Γ_n by

$$\Gamma_0^{(n)}(4) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n; \quad C \equiv 0 \bmod 4 \right\}.$$

We denote by ψ the character of $\Gamma_0^{(n)}(4)$ defined by $\psi(\gamma) = \left(\frac{-4}{\det(D)}\right)$ for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(4)$, where $(-4/*)$ is the primitive Dirichlet character modulo 4. For any character χ of $\Gamma_0^{(n)}(4)$, we denote by $S_{k-1/2,j}(\Gamma_0^{(n)}(4), \chi)$ the space of Siegel cusp form of weight $\det^{k-1/2} Sym_j$ with character χ. When χ is ψ or the trivial character, we denote by

$S^{+}_{k-1/2,j}(\Gamma_0^{(n)}(4),\chi)$ the *plus subspace* of $S_{k-1/2,j}(\Gamma_0^{(n)}(4),\chi)$, which is a kind of level one part. (All the precise definitions above will be given in the next section.) When χ is trivial, we omit χ in the above notation.

We first explain a lifting conjecture in [15] to the Haupt type. This is a generalization of our conjecture in [9] on the scalar valued case to the vector valued case.

CONJECTURE 1.1 ([9],[15]). *For any pair of an integer $k \geq 0$ and an even integer $j \geq 0$, there exists an injective linear map $\mathcal{L}$ from $(f,g) \in S_{2k-4}(\Gamma_1) \times S_{2k+2j-2}(\Gamma_1)$ to $\mathcal{L}(f,g) \in S^{+}_{k-1/2,j}(\Gamma_0^{(2)}(4))$. Besides, if f and g are Hecke eigenforms, then $\mathcal{L}(f,g)$ is also a Hecke eigenform and satisfies the relation*

$$L(s,\mathcal{L}(f,g)) = L(s-j-1,f)L(s,g)\,.$$

Here the L function in the left hand side is defined as in Zhuravlev [26], [27] (See also [9], [12]) and the right hand side is the usual Hecke L function (classically normalized).

For any pair of *even* integers k and $j \geq 0$, we can define a concrete lifting map $\mathcal{L}$ from $S_{2k-4}(\Gamma_1) \times S_{2k+2j-2}(\Gamma_1)$ into $S^{+}_{k-1/2,j}(\Gamma_0^{(2)}(4))$. (See Hayashida [8] for $j=0$ and [15] for $j>0$.) In this case, we already know that if $f \in S_{2k-4}(\Gamma_1)$ and $g \in S_{2k+2j-2}(\Gamma_1)$ are Hecke eigenforms and $\mathcal{L}(f,g) \neq 0$ for this concrete $\mathcal{L}$, then $\mathcal{L}(f,g)$ is also a Hecke eigenform and

$$L(s,\mathcal{L}(f,g)) = L(s,g)L(s-j-1,f)\,.$$

Our conjecture claims that this lifting map $\mathcal{L}$ is an injective mapping. For odd k, we do not know how to construct $\mathcal{L}$.

We denote by $S^{+,0}_{k-1/2,j}(\Gamma_0^{(2)}(4))$ the orthogonal complement of the image of this map $\mathcal{L}$ (conjectural in general) in $S^{+}_{k-1/2,j}(\Gamma_0^{(2)}(4))$.

CONJECTURE 1.2. *For any integer $k \geq 3$ and any even integer $j \geq 0$, there exists a linear isomorphism*

$$\sigma : S^{+,0}_{k-1/2,j}(\Gamma_0^{(2)}(4)) \cong S_{j+3,2k-6}(\Gamma_2)$$

such that for a Hecke eigenform $F \in S^{+,0}_{k-1/2,j}(\Gamma_0^{(2)}(4))$, the image $\sigma(F)$ is also a Hecke eigenform and

$$L(s,F) = L(s,\sigma(F),Sp)\,.$$

Here $L(s,\sigma(F),Sp)$ denotes the spinor L function of $\sigma(F)$. Together with our old conjecture in [12], we should have

CONJECTURE 1.3. *Notations and assumptions being the same as before, there exists a linear isomorphism*

$$S^{+,0}_{k-1/2,j}(\Gamma_0^{(2)}(4)) \cong S^{+}_{k-1/2,j}(\Gamma_0^{(2)}(4),\psi)$$

which preserves L functions.

These conjectures lead us to an interesting conjecture of Harder type and now we explain this. Harder gave the following conjecture in [3]. Let g be a Hecke eigenform in $S_{2k+j-2}(\Gamma_1)$ and assume that $\mathfrak{l}$ is a *big prime* ideal dividing the algebraic part $L_{alg}(k+j,g)$

of L values of g at $k+j$. For each prime p, we denote by $a_g(p)$ the eigenvalue of the Hecke operator $T(p)$ of g. Then there should exist a Hecke eigenform $F \in S_{k,j}(\Gamma_2)$ such that

$$\begin{aligned}1 - \lambda(p)u + (\lambda(p)^2 - \lambda(p^2) - p^{2k+j-4})u^2 - \lambda(p)p^{2k+j-3}u^3 + p^{4k+2j-6}u^4 \\ \equiv (1 - p^{k-2}u)(1 - p^{k+j-1}u)(1 - a_g(p)u + p^{2k+j-3}u^2) \bmod \mathfrak{l}\end{aligned}$$

for all primes p, where $\lambda(p^i)$ is the eigenvalue of the Hecke operator $T(p^i)$ of F, u is an indeterminant, and the congruence means that all the coefficients as polynomials in u are congruent. Here the left hand side is the Euler p-factor of the spinor L function $L(s, F, Sp)$ if we replace u by p^{-s}. We note that the word "big prime" is not a rigorous mathematical word. Here we understand this in the meaning that the ideal $\mathfrak{l}$ seems generic enough excluding exceptions of all *small* primes.

For a Hecke eigen cusp form $h \in S^{+}_{k+j-1/2}(\Gamma_0^{(1)}(4))$, let g be an elliptic (primitive) cusp form in $S_{2k+2j-2}(\Gamma_1)$ corresponding to h in the sense of Shimura. Then for $k > 5$, there exists a Klingen type Siegel Eisenstein series $E(h) \in A^{+}_{k-1/2,j}(\Gamma_0^{(2)}(4))$ such that

$$L(s, E(h)) = \zeta(s - j - 1)\zeta(s - 2k - j + 4)L(s, g) .$$

The existence of such form $E(h)$ is explained in [12] in terms of Jacobi forms.

For $A_{k-1/2,j}(\Gamma_0^{(2)}(4))$, for each odd prime p, there are two fundamental Hecke operators $T_1(p)$ and $T_2(p)$ of the metaplectic double coset explained in the next section. Since $A^{+}_{k-1/2,j}(\Gamma_0^{(2)}(4))$ is isomorphic to a space of certain Jacobi forms of level one (see §4), we can define also $T_1(2)$ and $T_2(2)$ from the action of Hecke operators on Jacobi forms (see [9] and [12]). If we denote by $\lambda^*(p)$ and $\omega(p)$ the eigenvalues of $G \in A^{+}_{k-1/2,j}(\Gamma_0^{(2)}(4))$ for $T_1(p)$ and $T_2(p)$ respectively, then for any prime p, the Euler p-factors of $L(s, G)$ are defined by $H_p(p^{-s}, G)$, where

$$\begin{aligned}H_p(u, G) = 1 - \lambda^*(p)u + (p\omega(p) + p^{2k+2j-5}(1 + p^2))u^2 \\ - \lambda^*(p)p^{2k+2j-3}u^3 + p^{4k+4j-6}u^4 .\end{aligned}$$

In particular, for the Klingen type Eisenstein series $E(h)$, we have

$$H_p(u, E(h)) = (1 - p^{j+1}u)(1 - p^{2k+j-4}u)(1 - a_g(p)u + p^{2k+2j-3}u^2) ,$$

where $a_g(p)$ is the eigenvalue of g at p.

By taking Conjecture 1.2 into account, we can interpret Harder's conjecture on congruences for $S_{j+3,2k-6}(\Gamma_2)$ to that of $S^{+}_{k-1/2,j}(\Gamma_0^{(2)}(4))$ as follows.

CONJECTURE 1.4 (Half integral version of Harder's conjecture). *Notation being as above, assume that $\mathfrak{l}$ is a big prime ideal dividing $L_{alg}(2k + j - 3, g)$. Then there exists $G \in S^{+,0}_{k-1/2,j}(\Gamma_0^{(2)}(4))$ such that*

$$H_p(u, G) \equiv H_p(u, E(h)) \bmod \mathfrak{l}$$

for any prime p. In particular, we have

$$\lambda^*(p) \equiv p^{j+1} + p^{2k+j-4} + a_g(p) \bmod \mathfrak{l} ,$$

where $\lambda^*(p)$ is the eigenvalue of $T_1(p)$ of G.

We had this type of conjecture already in [12] on congruences between an element of $S^{+}_{k-1/2,j}(\Gamma_0^{(2)}(4),\psi)$ and $E(h)$, but there was an inconvenience that $E(h)$ belongs to $\Gamma_0^{(2)}(4)$ without character while G with character. The above conjecture has an advantage that both G and $E(h)$ belong to the same discrete group. This enables us to prove the congruence modulo a fixed congruence prime for all Euler p-factors for a concrete example. Also there are several known general strategy to prove this kind of congruence and our conjecture will give a key to the general proof.

2. Review on definitions and notations

We write the Siegel upper half space of degree n by

$$H_n = \{Z = X + iY \in M_n(\mathbb{C});\ {}^tX = X, {}^tY = Y \in M_n(\mathbb{R}), Y > 0\},$$

where $Y > 0$ means that Y is positive definite. We denote by $Sp(n,\mathbb{R})$ the split real symplectic group of size $2n$. Let (ρ_0, V) be an irreducible representation of $GL_n(\mathbb{C})$ which does not contain the determinant part (i.e. the depth of the corresponding Young diagram is less than n). For any positive integer k, any $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$, and any V-valued function F, we write

$$F|_{k,\rho_0}[g] = \det(CZ+D)^{-k}\rho_0(CZ+D)^{-1}F(gZ).$$

We say that a V-valued holomorphic function F of H_n is a Siegel modular form of weight $\det^k \otimes \rho_0$ if F satisfies

$$F|_{k,\rho_0}[\gamma] = F \quad \text{for all } \gamma \in \Gamma_n = Sp(n, Z)$$

(and with extra boundedness condition of F on the boundary when $n = 1$). We denote this space by $A_{k,\rho_0}(\Gamma_n)$, We say that $F \in A_{k,\rho_0}(\Gamma_n)$ is a cusp form when F vanishes on the boundary of the Satake compactification of $\Gamma_n\backslash H_n$ (i.e. if $\Phi(F) = 0$ for the Siegel Φ operator), and the subspace of cusp forms is denoted by $S_{k,\rho_0}(\Gamma_n)$. When ρ_0 is the j-th symmetric tensor representation $\rho_j = Sym_j$, we write $A_{k,\rho_0}(\Gamma_n) = A_{k,j}(\Gamma_n)$ and $S_{k,\rho_0}(\Gamma_n) = S_{k,j}(\Gamma_n)$. If $j = 0$ besides, we simply write these as $A_k(\Gamma_n)$ and $S_k(\Gamma_n)$. When $n = 2$ and $\rho_0 = Sym(j)$, the Euler p-factor of the Spinor L function of $F \in A_{k,j}(\Gamma_2)$ is given by

$$1 - \lambda(p)p^{-s} + (\lambda(p)^2 - \lambda(p^2) - p^{2k+j-4})p^{-2s} - \lambda(p)p^{2k+j-3-3s} + p^{4k+2j-6-4s},$$

where each $\lambda(p^i)$ is the eigenvalue of the Hecke operator $T(p^i)$ of F.

To define modular forms of half-integral weight and the action of Hecke operators, we introduce the metaplectic group. We write

$$GSp^+(n,\mathbb{R}) = \left\{g \in GL_{2n}(\mathbb{R});\ {}^tgJg = n(g)J,\ n(g) \in \mathbb{R}^\times_+\right\},$$

where $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. The metaplectic group $\widetilde{GSp}^+(n,\mathbb{R})$ consists of elements $(g, \phi(Z))$, where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp^+(n,\mathbb{R})$ and $\phi(Z)$ is a holomorphic function such that $|\phi(Z)| =$

$\det(g)^{-1/4}|\det(CZ+D)|^{1/2}$. The multiplication of elements of $\widetilde{GSp}^+(n,\mathbb{R})$ is defined by

$$(g_1,\phi_1(Z))(g_2,\phi_2(Z)) = (g_1g_2,\phi_1(g_2Z)\phi_2(Z))\,.$$

We put $\vartheta_n(Z) = \sum_{p\in\mathbb{Z}^n} e(pZ\,{}^tp)$ for $Z \in H_n$, where we write $e(x) = exp(2\pi ix)$ for any x. Then we can define an injective homomorphism of $\Gamma_0^{(n)}(4)$ into $\widetilde{GSp}^+(n,\mathbb{R})$ by

$$\Gamma_0^{(n)}(4) \ni \gamma \to (\gamma, \vartheta_n(\gamma Z)/\vartheta_n(Z)) \in \widetilde{GSp}^+(n,\mathbb{R})\,.$$

We denote by $\widetilde{\Gamma}_0^{(n)}(4)$ the image of $\Gamma_0^{(n)}(4)$ by this map. For any V-valued function F, any element $\gamma \in \Gamma_0^{(n)}(4)$, and any $\widetilde{g} = (g,\phi(Z)) \in GSp^+(n,\mathbb{R})$ with $n(g) = m^2$, we put $m^{-1}g = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ and define

$$F|_{k-1/2,\rho_0}[\widetilde{g}] = \phi(Z)^{-2k+1}\rho_0(C_1Z+D_1)^{-1}F(gZ)\,.$$

Let χ be a character of $\Gamma_0^{(n)}(4)$. We say that F is a Siegel modular form of weight $\det^{k-1/2}\otimes\rho_0$ of $\Gamma_0^{(n)}(4)$ with character χ if $F|_{k-1/2,\rho_0}[\gamma] = \chi(\gamma)F$ for all $\gamma \in \Gamma_0^{(n)}(4)$ and besides if F satisfies the boundedness condition at cusps of $\Gamma_0^{(n)}(4)$ when $n = 1$. The space of such forms is denoted by $A_{k-1/2,\rho_0}(\Gamma_0^{(n)}(4),\chi)$ where χ is omitted if χ is trivial. If $\rho_0 = Sym(j)$ (the j-th symmetric tensor representation), we write $A_{k-1/2,\rho_0} = A_{k-1/2,j}$. The form F is said to be a cusp form if it vanishes on the cusps of $\Gamma_0^{(n)}(4)$ and this space is denoted by $S_{k-1/2,\rho_0}$ or $S_{k-1/2,j}$. To extract the level one part of these spaces, we define the plus subspace, originally defined by Kohnen for $n = 1$ and generalized for general n in [10], [9], [4], We write the Fourier expansion of $F \in A_{k-1/2,\rho_0}(\Gamma_0^{(n)}(4),\psi^l)$ ($l = 0$ or 1) by

$$F(Z) = \sum_{T\in L_n^*} a(T)e(\mathrm{Tr}(TZ))\,,$$

where $a(T) \in V$ and L_n^* is the space of $n\times n$ half-integral symmetric matrices. We say that F belongs to the plus subspace, if $a(T) = 0$ unless $T - (-1)^{k+l-1}\,{}^t\mu\mu \in 4L_n^*$ for some $\mu \in \mathbb{Z}^n$ (row vectors). The plus subspace is denoted by $A^+_{k-1/2,\rho_0}(\Gamma_0^{(n)}(4),\psi^l)$ and we put $S^+_{k-1/2,\rho_0}(\Gamma_0^{(2)}(4),\psi^l) = S_{k-1/2,\rho_0}(\Gamma_0^{(n)}(4),\psi^l) \cap A^+_{k-1/2,\rho_0}(\Gamma_0^{(n)}(4),\psi^l)$. When $l = 0$, we omit ψ^l in the above notation. When $n = 2$, there are two fundamental Hecke operators $T_1(p)$ and $T_2(p)$ for each prime p. When p is odd, these are defined as follows. We define elements $K_1(p^2)$ and $K_2(p^2)$ of $\widetilde{GSp}^+(2,\mathbb{R})$ by

$$K_1(p^2) = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, p^{1/2}\right), \quad K_2(p^2) = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix}, p\right).$$

For the double cosets

$$T_i(p) = \widetilde{\Gamma}_0^{(2)}(4)K_i(p^2)\widetilde{\Gamma}_0^{(2)}(4) = \bigcup_\nu \widetilde{\Gamma}_0^{(2)}(4)\widetilde{g}_\nu\,.$$

and $F \in A_{k-1/2,j}(\Gamma_0^{(2)}(4))$, we define

$$F|_{k-1/2,j}T_i(p) = p^{i(k+j-7/2)} \sum_{\nu} F|_{k-1/2,j}[\widetilde{g}_\nu] \,.$$

When $p = 2$, we define $T_i(2)$ from the corresponding action on Jacobi forms (see §4). The Euler factors explained in §1 was defined by using these.

3. Comparison of dimensions

Tsushima calculated certain Euler Poincare characteristics (for $k \geq 5$), which should be equal to the true dimension of the plus space of degree two under some standard conjectural vanishing theorem of cohomology. He expressed this formula by a sum of a large number of combinatorial arithmetic quantities depending on k and j, and it is not so easy to see. Here we change his formula to a generating function of dimensions. We note that it is needed considerable efforts to change it into the generating function written later. It is a routine but length calculation. He stated the conjecture only for $k \geq 5$ but there is a good reason to expect this is also true for $k \geq 3$. Actually, as we state later, the half of the cases of this conjecture is now proved.

The result is given below.

CONJECTURE 3.1 ([25]). *For $k \geq 3$, the following equality holds.*

$$\sum_{\substack{k=3,j=0\\ j:even}}^{\infty} \dim S^+_{k-1/2,j}(\Gamma_0(4))t^k s^j$$
$$= \frac{h(t,s)}{(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-s^4)(1-s^6)(1-s^{10})(1-s^{12})}$$

$$\sum_{\substack{k=3,j=0\\ j:even}}^{\infty} \dim S^+_{k-1/2,j}(\Gamma_0(4),\psi)t^k s^j$$
$$= \frac{h_\psi(t,s)}{(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-s^4)(1-s^6)(1-s^{10})(1-s^{12})}$$

Here we define

$$\begin{aligned}
&h(t,s) = \\
&s^{32}t^3 + (s^{18}+s^{20}+s^{24}+s^{26}-s^{30})t^4 \\
&+ (s^{12}+s^{14}+s^{16}+s^{18}+s^{20})t^5 \\
&+ (s^8+s^{10}+s^{12}+s^{14}+s^{16}+s^{18}+s^{20}-s^{32})t^6 \\
&+ (s^6+s^8+s^{10}+2s^{12}+2s^{14}+s^{16}-s^{18}-s^{24}-s^{26}+s^{30}-s^{32})t^7 \\
&+ (s^2+s^4+s^6+s^8+s^{10}+s^{12}-s^{18}-s^{20}-s^{24}-s^{26}+s^{30}-s^{32})t^8 \\
&+ (s^4+2s^6+2s^8+s^{10}-s^{16}-3s^{18}-3s^{20}-s^{24}-s^{26}+s^{30}-s^{32})t^9
\end{aligned}$$

$$+ (1 + s^2 + s^4 + s^6 + s^8 - 2s^{12} - 2s^{14} - 2s^{16} - 3s^{18}$$
$$- 4s^{20} - s^{24} - s^{26} + s^{30} + s^{32})t^{10}$$
$$+ (1 + s^2 + s^4 - 4s^{12} - 4s^{14} - 3s^{16} - s^{18} - 3s^{20} - s^{22} + s^{24} + s^{26} - s^{30} + s^{32})t^{11}$$
$$+ (1 + s^2 + s^4 - 2s^8 - 2s^{10} - 4s^{12} - 4s^{14} - 3s^{16}$$
$$- s^{18} - s^{20} + s^{24} + s^{26} - s^{30} + 2s^{32})t^{12}$$
$$+ (1 + s^2 - 2s^6 - 2s^8 - 2s^{10} - 4s^{12} - 4s^{14} - 2s^{16}$$
$$+ 2s^{18} + s^{20} + 2s^{24} + 2s^{26} - 2s^{30} + s^{32})t^{13}$$
$$+ (-s^6 - 2s^8 - 2s^{10} - 2s^{12} - 2s^{14} - s^{16} + s^{18} + 2s^{20} + 2s^{24} + 2s^{26} - s^{30} + s^{32})t^{14}$$
$$+ (-1 - s^6 - 2s^8 - s^{10} + s^{12} - s^{14} - s^{16} + 2s^{18}$$
$$+ 4s^{20} + s^{24} + 2s^{26} + s^{28} - s^{30} - s^{32})t^{15}$$
$$+ (-1 - s^2 - s^4 + 2s^{12} + 2s^{14} + s^{16} + 2s^{20} + s^{28} + s^{30} - s^{32})t^{16}$$
$$+ (-1 + 2s^{12} + s^{14} + s^{16} + 2s^{20} + s^{22} + s^{30} - s^{32})t^{17}$$
$$+ (-1 - s^2 + s^6 + s^8 + s^{10} + 3s^{12} + 2s^{14} - s^{18} - s^{24} + s^{28} + s^{30} - s^{32})t^{18}$$
$$+ (s^6 + s^8 + s^{14} + s^{16} - s^{18} - s^{20} + s^{22} - s^{26} + s^{30})t^{19}$$
$$+ (s^8 + s^{10} - s^{20})t^{20} + (1 - s^{12} - s^{20} + s^{32})t^{21}$$

$$h_\psi(t, s) =$$
$$s^{32}t^3 + (s^{18} + s^{20} + s^{24} + s^{26} - s^{30})t^4$$
$$+ (s^{12} + s^{14} + s^{16} + s^{18} + s^{20})t^5$$
$$+ (s^8 + s^{10} + s^{12} + s^{14} + s^{16} + s^{18} + s^{20} - s^{32})t^6$$
$$+ (s^6 + s^8 + s^{10} + 2s^{12} + 2s^{14} + s^{16} - s^{18} - s^{24} - s^{26} + s^{30} - s^{32})t^7$$
$$+ (s^6 + s^8 + s^{10} + 2s^{12} + 2s^{14} + s^{16} - s^{18} - s^{20} - 2s^{24} - 2s^{26} + s^{30} - s^{32})t^8$$
$$+ (s^4 + 2s^6 + 2s^8 + s^{10} - s^{16} - 3s^{18} - 3s^{20} - s^{24} - s^{26} + s^{30} - s^{32})t^9$$
$$+ (s^4 + s^6 + s^8 + s^{10} - s^{14} - 2s^{16} - 3s^{18} - 4s^{20}$$
$$- s^{22} - 2s^{24} - s^{26} + s^{30} + s^{32})t^{10}$$
$$+ (s^4 + s^6 + s^8 + s^{10} - 2s^{12} - 3s^{14} - 4s^{16} - 3s^{18} - 4s^{20} - 2s^{22}$$
$$+ s^{26} + s^{28} + s^{32})t^{11}$$
$$+ (s^2 + s^4 - s^8 - s^{10} - 3s^{12} - 4s^{14} - 3s^{16}$$
$$- 2s^{18} - 2s^{20} - s^{22} + s^{24} + s^{26} + 2s^{32})t^{12}$$
$$+ (s^2 + s^4 - s^6 - s^8 - s^{10} - 3s^{12} - 5s^{14} - 4s^{16}$$
$$- s^{22} + 2s^{24} + 3s^{26} + s^{28} - s^{30} + s^{32})t^{13}$$

$$+ (-s^8 - 2s^{10} - 2s^{12} - 2s^{14} - 2s^{16} - s^{18} + s^{20} + 2s^{24} + 2s^{26} + s^{28} + s^{32})t^{14}$$
$$+ (s^2 - s^6 - 2s^8 - 2s^{10} - s^{12} - 2s^{14} - s^{16} + 2s^{18}$$
$$+ 4s^{20} + s^{22} + 2s^{24} + 2s^{26} + s^{28} - s^{30} - s^{32})t^{15}$$
$$+ (-s^8 - s^{10} + s^{18} + 3s^{20} + s^{22} + s^{24} + s^{26} + s^{28} - s^{32})t^{16}$$
$$+ (-s^8 - s^{10} + s^{12} + s^{14} + s^{16} + s^{18} + 3s^{20} + 2s^{22} - s^{32})t^{17}$$
$$+ (s^{12} + s^{14} + s^{16} + s^{18} + s^{20} + s^{22} - s^{32})t^{18}$$
$$+ (s^{14} + 2s^{16} + s^{18} + s^{22} - s^{26} - s^{28})t^{19}$$
$$+ (s^8 + s^{10} - s^{20})t^{20}$$
$$+ (1 - s^4 - s^6 - s^{12} + s^{14} + 2s^{16} + s^{18} - s^{20} - s^{26} - s^{28} + s^{32})t^{21}$$

The numerator $h_\psi(t,s)$ of the generating function is also written in [12], but we reproduce it here for the readers convenience. Now about Tsushima's conjecture above, we can prove the following results.

THEOREM 3.2 ([16]). *The above formula for* $S_{k-1/2,j}(\Gamma_0^{(2)}(4))$ *is true for even* $k \geq 8$, *and the formula for* $S_{k-1/2,j}(\Gamma_0^{(2)}(4), \psi)$ *is true for odd* $k \geq 9$.

In order to evaluate the part for liftings, we need the generating function of $\dim S_{2k-4}(\Gamma_1) \times \dim S_{2k+2j-2}(\Gamma_1)$. By the classical formula of dimensions of elliptic cusp forms, for even $k \geq 0$, we have

$$\dim S_k(\Gamma_1) = \frac{k-1}{12} + \frac{1}{4}(-1)^{k/2} + \frac{1}{3}[1,0,-1;3]_k - \frac{1}{2} + \delta_{k2}\,,$$

where $\delta_{k2} = 1$ if $k = 2$ and $= 0$ otherwise, and the notation $[1,0,-1;3]_k$ means that if $k \equiv 0$, 1 and 2 mod 3, then it takes values 1, 0 and -1, respectively. By using this, by routine but lengthy calculation, we can show

$$\sum_{\substack{k,j=0\\ j:even}}^{\infty} \dim S_{2k-4}(\Gamma_1) \times \dim S_{2k+2j-2}(\Gamma_1) t^k s^j$$
$$= \frac{t^{10} + s^2(t^8 - t^9) + s^4(t^{12} - t^{13}) - s^6 t^{11}}{(1-t)(1-t^3)(1-t^6)(1-s^2)(1-s^6)}$$

Here we note that j is assumed to be even. If j is odd, it is easy to see that $S_{k,j}(\Gamma_2) = 0$ and $S_{k-1/2,j}(\Gamma_0(4)) = S_{k-1/2,j}(\Gamma_0(4), \psi) = 0$. When j is even, the dimension of $\dim S_{k,j}(\Gamma_2)$ was obtained by Igusa for $j = 0$ and Tsushima for $k > 4$ for any even $j > 0$ (See [18], [19], [22]). The conjecture that the same formula for $S_{k,j}(\Gamma_2)$ should be true even for $k \geq 3$ was given in [17], and I heard recently that Dan Petersen proved this conjecture([21]).

THEOREM 3.3. *Assume that* k *and* j *are integers such that* j *is even*, $j \geq 0$, *and* $k \geq 3$. *Assuming the above conjectural formulas of Tsushima on dimensions*, *we have*

$$\dim S^+_{k-1/2,j}(\Gamma_0^{(2)}(4)) - \dim S_{2k-4}(\Gamma_1) \times \dim S_{2k+2j-2}(\Gamma_1)$$
$$= \dim S^+_{k-1/2,j}(\Gamma_0^{(2)}(4), \psi) = \dim S_{j+3,2k-6}(\Gamma_2)\,.$$

This claim surely supports our Conjecture 1.1, 1.2, 1.3 and 1.4 strongly.

4. Numerical examples on L functions and congruences

In this section, we give numerical examples of liftings, correspondences and congruences in the conjectures we already mentioned.

4.1. General set up

The structures of $S_{k-1/2,j}(\Gamma_0^{(2)}(4),\chi)$ for $\chi=\psi$ or χ trivial are known for small j (see [24], [14]), but the subspace $S^+_{k-1/2,j}(\Gamma_0^{(2)}(4),\chi)$ is much more smaller and it is very hard to determine elements of this subspace from $S_{k-1/2,j}(\Gamma_0^{(2)}(4),\chi)$, as we can see in [12]. But now we have a better strategy in case when the weight and the character have good parity since the structure of holomorphic Jacobi forms of degree two of index one are now known in [16] and these correspond Siegel modular forms of half integral weight in the plus subspace of Haupt type when k is even. For readers convenience and to clarify the meaning of the plus space, we review here the isomorphisms between Siegel modular forms of half integral weight and Jacobi forms. First we define Jacobi forms of general degree. We define the Jacobi modular group Γ_n^J by the following subgroup of Γ_{n+1}:

$$\Gamma_n^J=\left\{\begin{pmatrix}A&0&B&0\\0&1&0&0\\C&0&D&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1_n&0&0&{}^t\mu\\\lambda&1&\mu&\kappa\\0&0&1_n&-{}^t\lambda\\0&0&0&1\end{pmatrix};\right.$$
$$\left.\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Gamma_n,\ \lambda,\mu\in\mathbb{Z}^n,\ \kappa\in\mathbb{Z}\right\}.$$

We identify Γ_n as a subgroup of Γ_n^J by taking the first factor of the above definition. Let (ρ_0,V) be an irreducible representation of $GL(n,\mathbb{C})$ without determinant factor. Let $F(\tau,z)$ be a holomorphic function $F:H_n\times\mathbb{C}^n\to V$. When $n\geq 2$, we say that F is a holomorphic Jacobi form of weight $\det^k\otimes\rho_0$ of index one of Γ_n^J if it satisfies the following conditions (1) and (2);

$$\begin{aligned}(1)\quad F(\gamma\tau,z(C\tau+D)^{-1})&=e(z\,{}^t(C\tau+d)^{-1}Cz)\det(CZ+D)^k\\&\qquad\times\rho_0(C\tau+D)F(\tau,z),\\(2)\qquad F(\tau,z+\lambda\tau+\mu)&=e(-\lambda\tau\lambda-2\lambda\,{}^tz)F(\tau,z)\,,\end{aligned}$$

for any $\gamma=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Gamma_n$ and $\lambda,\mu\in\mathbb{Z}^n$. The Fourier expansion of F is written as

$$F(\tau,z)=\sum_{(N,r)\in L_n^*\times\mathbb{Z}^n}C(N,r)e(\mathrm{Tr}(N\tau))e(r\,{}^tz)$$

For the definition when $n=1$, we need the extra condition that $C(N,r)=0$ unless $4N-{}^trr\geq 0$ for the definition. (When $n\geq 2$, this condition is satisfied always by the Koecher principle proved by Ziegler.) For general n, if $C(N,r)=0$ unless $4N-{}^trr$ is positive definite, we say that F is a Jacobi cusp form. We denote this space of holomorphic Jacobi forms of index one by $J_{(k,\rho_0),1}(\Gamma_n^J)$ and Jacobi cusp forms of index one by $J_{(k,\rho_0),1}^{cusp}(\Gamma_n^J)$. For a real analytic function $F : H_n \times \mathbb{C}^n \to V$, we say that F is a skew holomorphic Jacobi form of index one if it satisfies the condition (2) above and the following relation

$$(1^*)\ F(\gamma\tau, z(C\tau+D)^{-1}) = e(z(C\tau+D)^{-1}C\,{}^tz) \\ \times \overline{\det(C\tau+D)}^{k-1} |\det(C\tau+D)|\, \overline{\rho_0(C\tau+D)} F(\tau, z),$$

for any $\gamma \in \Gamma_n$, where the bar is the complex conjugation, and besides the condition (3) that F has the Fourier expansion of the shape

$$F = \sum_{(N,r)\in L_n^* \times \mathbb{Z}^n} C(N,r) e\left(\mathrm{Tr}\left(N\tau - \frac{1}{2} i (N - {}^t rr) Y\right)\right) e(r\,{}^tz)\,,$$

where (N, r) runs over $L_n^* \times \mathbb{Z}^n$ such that ${}^trr - 4N$ is positive semi-definite. If $C(N,r) = 0$ unless ${}^trr - 4N$ is positive definite, we say that F is a skew holomorphic Jacobi cusp form. We denote by $J_{(k,\rho_0),1}^{skew}(\Gamma_n^J)$ and $J_{(k,\rho_0),1}^{skew,cusp}(\Gamma_n^J)$ the space of skew holomorphic Jacobi forms of index one and skew holomorphic Jacobi cusp forms of index one, respectively. The following theorem is known.

THEOREM 4.1 ([10],[4], [9], [20]). *We have the following Hecke equivariant isomorphism.*

$$A_{k-1/2,\rho_0}^{+}(\Gamma_0^{(n)}(4), \psi^l) \cong \begin{cases} J_{(k,\rho_0),1}(\Gamma_n^J) & \text{if } k+l \text{ is even}\,, \\ J_{(k,\rho_0),1}^{skew}(\Gamma_n^J) & \text{if } k+l \text{ is odd}\,, \end{cases}$$

$$S_{k-1/2,\rho_0}^{+}(\Gamma_0^{(n)}(4), \psi^l) \cong \begin{cases} J_{(k,\rho_0),1}^{cusp}(\Gamma_n^J) & \text{if } k+l \text{ is even}\,, \\ J_{(k,\rho_0),1}^{skew,cusp}(\Gamma_n^J) & \text{if } k+l \text{ is odd}\,. \end{cases}$$

Throught these isomorphisms, we defined the Hecke operators at 2.

In case $\rho_0 = Sym(j)$, we write $(k, \rho_0) = (k, j)$. Together with the conjecture 1.3, we should have

CONJECTURE 4.2. *Let j be an even integer with $j \geq 0$. For any even integer $k \geq 3$, there exists an injective linear isomorphism*

$$J_{(k,j),1}^{skew}(\Gamma_2^J) \to J_{(k,j),1}(\Gamma_2^J)\,,$$

and for any odd integer $k \geq 3$, there exists an injective linear isomorphism

$$J_{(k,j),1}(\Gamma_2^J) \to J_{(k,j),1}^{skew}(\Gamma_2^J)\,,$$

which commute with the action of Hecke operators. The kernels correspond with the lifting parts from $S_{2k-4}(\Gamma_1) \times S_{2k+2j-2}(\Gamma_1)$.

Since we do not know how to describe the structure of skew holomorphic Jacobi forms directly, we consider here only the case of holomorphic Jacobi forms. For simplicity, we assume that k is even from now on until the end of the paper. Then the above theorem means in this case that

$$A^+_{k-1/2,j}(\Gamma_0^{(2)}(4)) \cong J_{(k,j),1}(\Gamma_2^J)\,,$$
$$S^+_{k-1/2,j}(\Gamma_0^{(2)}(4)) \cong J^{cusp}_{(k,j),1}(\Gamma_2^J)\,.$$

We note that for $k > 5$, we have

$$\dim J_{(k,j),1}(\Gamma_2^J) = \dim J^{cusp}_{(k,j),1}(\Gamma_2^J) + \dim S_{2k+2j-2}(\Gamma_1)\,,$$

where $S_{2k+2j-2}(\Gamma_1) \cong S^+_{k+j-1/2}(\Gamma_0^{(1)}(4)) \cong J^{cusp}_{k+j,1}(\Gamma_1^J)$. (We use the characterization of Jacobi cusp forms of index 1 by Jacobi-Siegel Φ^J operator and the surjectivity of Φ^J for $k > 5$ to $J^{cusp}_{k+j,1}(\Gamma_1^J)$. The details will be explained in [16].)

Now we explain first how to construct elements of $A^+_{k-1/2,j}(\Gamma_0^{(2)}(4))$ from Jacobi forms in $J_{(k,j),1}(\Gamma_2^J)$ for even k. For $m = (m_1, m_2) \in (\mathbb{Z}/2\mathbb{Z})^2$ and $(\tau, z) \in H_2 \times \mathbb{C}^2$, we define

$$\vartheta_m(\tau, z) = \sum_{p\in\mathbb{Z}^2} e\left(\left(p + \frac{m}{2}\right)\tau\, {}^t\!\left(p + \frac{m}{2}\right) + 2\left(p + \frac{m}{2}\right){}^t z\right).$$

Then for $F \in J_{(k,j),1}(\Gamma_2^J)$, there exist holomorphic functions $c_{ij}(\tau)$ $(0 \le i, j \le 1)$, uniquely determined by F, such that

$$F(\tau, z) = c_{00}(\tau)\vartheta_{00}(\tau, z) + c_{01}(\tau)\vartheta_{01}(\tau, z) + c_{10}(\tau)\vartheta_{10}(\tau, z) + c_{11}(\tau)\vartheta_{11}(\tau, z)\,.$$

We call this expression the theta expansion. If we put

$$h(\tau) = c_{00}(\tau) + c_{01}(\tau) + c_{10}(\tau) + c_{11}(\tau)\,,$$

then we have $h(4\tau) \in A^+_{k-1/2,j}(\Gamma_0^{(2)}(4))$ and this gives an isomorphism between $J_{(k,j),1}(\Gamma_2^J)$ and $A^+_{k-1/2,j}(\Gamma_0^{(2)}(4))$. By this map, Jacobi cusp forms correspond to cusp forms. Now we can describe elements in $J_{(k,j),1}(\Gamma_2^J)$ by using the coefficients of the Taylor expansion along $z = 0$, and we can describe those coefficients by vector valued Siegel modular forms as we shall see in [16]. From that paper, we extract here the necessary result only in the case $j = 2$ under some special assumptions since the general theory is much more complicated. We omit all the proofs here. The Taylor expansion of $F(\tau, z) \in J_{(k,j),1}(\Gamma_2^J)$ along $z = (z_1, z_2) = (0, 0)$ has no terms of odd degrees as we can see by the action of -1_4, and it is written as

$$F(\tau, z) = f_0(\tau) + f_{20}(\tau)z_1^2 + f_{11}(\tau)z_1z_2 + f_{02}(\tau)z_2^2 + O(z^4)\,.$$

We can show that the mapping of F to $(f_0, f_{20}, f_{11}, f_{02})$ is injective. The reason is as follows. We write $\partial_{ij} = \frac{1}{2\pi i}\frac{\partial}{\partial \tau_{ij}}$ where we write $\tau = (\tau_{ij})$ for $\tau \in H_2$. We denote by $\vartheta_{ij}(\tau)$ the theta constants $\vartheta_{ij}(\tau) = \vartheta_{ij}(\tau, 0)$. It is easy to see that $2(1 + \delta_{ij})\partial_{ij}\vartheta_m(\tau) = \frac{1}{(2\pi i)^2}\frac{\partial^2\vartheta_m(\tau,z)}{\partial z_i \partial z_j}\Big|_{z=0}$ for any $m \in (\mathbb{Z}/2\mathbb{Z})^2$, where δ_{ij} is the Kronecker delta. If we put

$$\Theta(\tau)=\begin{pmatrix}\vartheta_{00}(\tau) & \vartheta_{01}(\tau) & \vartheta_{10}(\tau) & \vartheta_{11}(\tau)\\ \partial_{11}\vartheta_{00}(\tau) & \partial_{11}\vartheta_{01}(\tau) & \partial_{11}\vartheta_{10}(\tau) & \partial_{11}\vartheta_{11}(\tau)\\ \partial_{12}\vartheta_{00}(\tau) & \partial_{12}\vartheta_{01}(\tau) & \partial_{12}\vartheta_{10}(\tau) & \partial_{12}\vartheta_{11}(\tau)\\ \partial_{22}\vartheta_{00}(\tau) & \partial_{22}\vartheta_{01}(\tau) & \partial_{22}\vartheta_{10}(\tau) & \partial_{22}\vartheta_{11}(\tau)\end{pmatrix}, \tag{1}$$

then we have

$$\Theta(\tau)\begin{pmatrix}c_{00}(\tau)\\ c_{01}(\tau)\\ c_{10}(\tau)\\ c_{11}(\tau)\end{pmatrix}=\begin{pmatrix}f_0(\tau)\\ \frac{1}{2(2\pi i)^2}f_{20}(\tau)\\ \frac{1}{2(2\pi i)^2}f_{11}(\tau)\\ \frac{1}{2(2\pi i)^2}f_{02}(\tau)\end{pmatrix} \tag{2}$$

and since we can show that $\det(\Theta(\tau))\neq 0$, the functions $c_{ij}(\tau)$ are uniquely determined by f_0, f_{20}, f_{11} and f_{02}. In fact, if we put $\chi_5(\tau)=\det(\Theta(\tau))$, then χ_5 is the unique nonzero cusp form of weight 5 (up to constant) of Γ_2 with character $sgn : Sp(2,\mathbb{F}_2)\cong S_6\to S_6/A_6\cong\{\pm 1\}$, where S_6 and A_6 are the symmetric group and the alternating group on six letters, respectively. Now we see necessary conditions on $f_0(\tau)$ and $f_{ij}(\tau)$. It is easy to see that $f_0(\tau)\in A_{k,j}(\Gamma_2)$. Now for the sake of simplicity, we write

$$f_2(\tau,z)=\frac{1}{2(2\pi i)^2}(f_{20}(\tau)z_1^2+f_{11}(\tau,z)z_1z_2+f_{02}(\tau)z_2^2)\,.$$

Then f_2 takes values in

$$V_{j,2}=\mathbb{C}[u_1,u_2]_j\otimes(\mathbb{C}z_1^2+\mathbb{C}z_1z_2+\mathbb{C}z_2^2)\,,$$

that is, the space of polynomials in u_i and z_i of degree j with respect to u and of degree 2 with respect to z. On $V_{j,2}$, the group $GL(2)$ acts naturally by $P(u,z)\to P(uU,zU)$ for polynomials $P(u,z)$ and $U\in GL_2(\mathbb{C})$, but this action is not irreducible. The irreducible decomposition of $V_{j,2}$ is given by

$$V_{j,2}=Sym(j+2)\oplus\det Sym(j)\oplus\det{}^2\,Sym(j-2)\,,$$

if $j\geq 2$. We can show that if we add to $f_2(\tau,z)$ a certain polynomials in z_i of degree two whose coefficients are certain derivatives of $f_0(\tau)$ by τ_{ij}, which are polynomials in u of degree j, then the result is regarded as an element of $A_{k,j+2}(\Gamma_1)\oplus A_{k+1,j}(\Gamma_2)\oplus A_{k+2,j-2}(\Gamma_2)$. Since the general case is slightly complicated (and will be explained in [16] in details), here, for simplicity, we assume that $j=2$ and $A_{k,2}(\Gamma_2)=A_{k+1,2}(\Gamma_2)=0$. This is satisfied when $k=8$ and 12, for example. Under this assumption, we have $f_0=0$, so there is no correction term from f_0, and hence f_2 itself is written as follows. We write an element $A(\tau)\in A_{k,4}(\Gamma_2)$ by $\sum_{i=0}^4 a_i(\tau)u_1^{4-i}u_2^i\in A_{k,4}(\Gamma_2)$, where we identify the representation space of $Sym(4)$ by polynomials in u_1, u_2 of degree 4. For $A(\tau)$, we put

$$\begin{aligned}A(\tau,u,z)=\;&6a_0(\tau)u_1^2z_1^2+3a_1(\tau)(u_1u_2z_1^2+u_1^2z_1z_2)\\&+a_2(\tau)(u_2z_1^2+4u_1u_2z_1z_2+u_1^2z_2^2)\\&+3a_3(\tau)(u_2^2z_1z_2+u_1u_2z_2^2)+6a_4(\tau)u_2^2z_2^2)\,.\end{aligned}$$

Then we have

$$f_2(\tau,z)=A(\tau,u,z)+(u_1z_2-u_2z_1)^2c(\tau)$$

for some $A(\tau)\in A_{k,4}(\Gamma_2)$ and $c(\tau)\in A_{k+2}(\Gamma_2)$. On the other hand, put $f_0=0$ and define f_{20}, f_{11}, f_{02} for any $A(\tau)\in A_{k,4}(\Gamma_2)$ and $c(\tau)\in A_{k+2}(\Gamma_2)$ by the above relation, that is,

$$\begin{aligned}
f_{20}(\tau)/2(2\pi i)^2 =&(6a_0(\tau)u_1^2+3a_1(\tau)u_1u_2+a_2(\tau)u_2^2)+c(\tau)u_2^2\,,\\
f_{11}(\tau)/2(2\pi i)^2 =&(3a_1(\tau)zu_1^2+4a_2(\tau)u_1u_2+3a_3(\tau)u_2^2)-2c(\tau)u_1u_2\,,\\
f_{02}(\tau)/2(2\pi i)^2 =&(a_2(\tau)u_1^1+3a_3(\tau)u_1u_2+6a_4(\tau)u_2^2)+c(\tau)u_1^2\,.
\end{aligned}$$

For these $f_0(\tau)$ and $f_{ij}(\tau)$, we can show that the solution $c_{ij}(\tau)$ of the simultaneous equation (2) are holomorphic if and only if $W(f_{11})=0$, where W is the Witt operator which means

$$W(f)(\tau_{11},\tau_{22})=f\begin{pmatrix}\tau_{11} & 0\\ 0 & \tau_{22}\end{pmatrix}.$$

Since $W(a_1)$ and $W(a_3)$ are products of elliptic cusp forms of odd weights, we have $W(a_1)=W(a_3)=0$, so the only condition is

$$W(4a_2-2c)=0\,. \tag{3}$$

As a whole, we have

PROPOSITION 4.3. *Let k be an even positive integer and assume that $A_{k,2}(\Gamma_2)=A_{k+1,2}(\Gamma_2)=0$. We write*

$$\begin{aligned}
A^T_{k,2}(\Gamma_2)&=\big\{(A(\tau),c(\tau))\in A_{k,4}(\Gamma_2)\times A_{k+2}(\Gamma_2);\ 2W(a_2)=W(c).\big\}\\
S^T_{k,2}(\Gamma_2)&=\{(A(\tau),c(\tau))\in A^T_{k,2}(\Gamma_2);\ A(\tau)\in S_{k,4}(\Gamma_2)\}\,.
\end{aligned}$$

Then $J_{(k,2),1}(\Gamma_2^J)$ can be identified with the space $A^T_{k,2}(\Gamma_2)$, and $J^{cusp}_{(k,2),1}(\Gamma_2^J)$ with $S^T_{k,2}(\Gamma_2)$.

Let $\Theta(\tau)$ be as in (1) and denote by $B_{ij}(\tau)$ the (i,j)-cofactor of $\Theta(\tau)$, that is, $(-1)^{i+j}$ times the determinant of the matrix obtained by removing the i-th row and the j-th column of $\Theta(\tau)$. We write

$$B_i(\tau)=B_{i1}(\tau)+B_{i2}(\tau)+B_{i3}(\tau)+B_{i4}(\tau)\,.$$

Then the solution $(c_{ij}(\tau))$ of (2) is given by

$$\begin{aligned}
\chi_5(\tau)c_{00}(\tau)&=B_{11}(\tau)f_0(\tau)+(B_{21}(\tau)f_{20}(\tau)+B_{31}(\tau)f_{11}(\tau)+B_{41}(\tau)f_{02}(\tau))/2(2\pi i)^2\,,\\
\chi_5(\tau)c_{01}(\tau)&=B_{12}(\tau)f_0(\tau)+(B_{22}(\tau)f_{20}(\tau)+B_{32}(\tau)f_{11}(\tau)+B_{42}(\tau)f_{02}(\tau))/2(2\pi i)^2\\
\chi_5(\tau)c_{10}(\tau)&=B_{13}(\tau)f_0(\tau)+(B_{23}(\tau)f_{20}(\tau)+B_{33}(\tau)f_{11}(\tau)+B_{43}(\tau)f_{02}(\tau))/2(2\pi i)^2\\
\chi_5(\tau)c_{11}(\tau)&=B_{14}(\tau)f_0(\tau)+(B_{24}(\tau)f_{20}(\tau)+B_{34}(\tau)f_{11}(\tau)+B_{44}(\tau)f_{02}(\tau))/2(2\pi i)^2
\end{aligned}$$

So, for each element $(A(\tau),c(\tau))\in A^T_{k,2}(\Gamma_2)$ with $A(\tau)=\sum_{i=0}^4 a_i(\tau)u_1^{4-i}u_2^i$, the associated element $h(4\tau)=\sum_{0\le i,j\le 1}c_{ij}(4\tau)\in S^+_{k-1/2,2}(\Gamma_0^{(2)}(4))$ is given by

$$\begin{aligned}
\Big(&\big(6a_0(4\tau)B_2(4\tau)+3a_1(4\tau)B_3(4\tau)+a_2(4\tau)B_4(4\tau)\big)u_1^2\\
&+\big(3a_1(4\tau)B_2(4\tau)+4a_2(4\tau)B_3(4\tau)+3a_3(4\tau)B_4(4\tau)\big)u_1u_2
\end{aligned} \tag{4}$$

$$+ \big(a_2(4\tau)B_2(4\tau) + 3a_3(4\tau)B_3(4\tau) + 6a_4(4\tau)B_4(4\tau)\big)u_2^2$$
$$+ c(4\tau)\big(B_4(4\tau)u_1^2 - 2B_3(4\tau)u_1u_2 + B_2(4\tau)u_2^2\big)\Big)\Big/\chi_5(4\tau)\,.$$

4.2. The eigenvalues in the case $k = 8$ and $k = 12$

For $j = 2$, we give a table of dimensions for small k below.

k	2	4	6	8	10	12	14	16	18	20
$\dim J_{(k,2),1}(\Gamma_2^J)$	0	0	0	2	2	3	6	8	10	16
$\dim J_{(k,2),1}^{cusp}(\Gamma_2^J)$	0	0	0	1	1	2	4	6	8	13
$\dim S_{2k-4}(\Gamma_1) \times \dim S_{2k+2}(\Gamma_1)$	0	0	0	1	1	1	4	4	4	9
$\dim S_{2k+2}(\Gamma_1)$	0	0	0	1	1	1	2	2	2	3

In this section, we describe the space $J_{(8,2),1}^{cusp}(\Gamma_2^J) \cong S_{15/2,2}^+(\Gamma_0^{(2)}(4))$ and $J_{(12,2),1}(\Gamma_2^J) \cong A_{23/2,2}^+(\Gamma_0^{(2)}(4))$ and give the Euler 2 factors explicitly. We sometimes denote an element $T = \begin{pmatrix} t_1 & t_{12}/2 \\ t_{12}/2 & t_2 \end{pmatrix} \in L_2^*$ by (t_1, t_2, t_{12}) and the Fourier coefficient $a(T)$ of a form F by $a(t_1, t_2, t_{12}) = a(t_1, t_2, t_{12}; F)$. First we assume that $k = 8$. Then we have $\dim A_{8,4}(\Gamma_2) = 1$, $\dim S_{8,4}(\Gamma_2) = 0$ and $\dim A_{10}(\Gamma_2) = 2$. We denote by ϕ_l the Siegel Eisenstein series of weight l normalized so that the constant term of the Fourier expansion is 1. For any $A(\tau) = \sum_{i=0}^4 a_i(\tau)u_1^{4-i}u_2^i \in A_{8,4}(\Gamma_2)$, we know that $W(a_2)$ should be a product of elliptic cusp form of weight 10, but since $S_{10}(\Gamma_1) = 0$, we have $W(a_2) = 0$. On the other hand $A_{10}(\Gamma_2)$ is spanned by χ_{10} and $\phi_4\phi_6$. where $\chi_{10} = \chi_5^2$ is the unique cusp form in $S_{10}(\Gamma_2)$ such that the Fourier coefficient at $(1, 1, 1)$ is 1. Since $W(\chi_{10}) = 0$ and $W(\phi_4\phi_6) \neq 0$, we have $A_{8,2}^T(\Gamma_2) = A_{8,4}(\Gamma_2) \times S_{10}(\Gamma_2)$ and $S_{8,2}^T(\Gamma_2) = \{0\} \times S_{10}(\Gamma_2)$. Let $F_{8-1/2} = F_{15/2}$ be the element in $S_{15/2,2}^+(\Gamma_0^{(2)}(4))$ corresponding to $(0, \chi_{10}) \in S_{8,2}^T(\Gamma_2)$. Then since $\chi_{10} = \chi_5^2$, we have

$$F_{15/2} = B_4(4\tau)\chi_5(4\tau)u_1^2 - 2B_3(4\tau)\chi_5(4\tau)u_1u_2 + B_2(4\tau)\chi_5(4\tau)u_2^2\,.$$

Then by using the concrete Fourier coefficients of $F_{15/2}$ (which we omit here), we can show that the Euler 2 factor of $L(s, F_{15/2})$ is given by

$$(1 - 24 \cdot 2^{3-s} + 2^{17-2s})(1 + 528 \cdot 2^{-s} + 2^{17-2s})\,.$$

Actually in this case we can show directly that the concrete lifting map $\mathcal{L}$ from $S_{12}(\Gamma_1) \times S_{18}(\Gamma_1)$ to $S_{15/2,2}^+(\Gamma_0^{(2)}(4))$ in [15] does not vanish and we can prove that

$$L(s, F_{15/2}) = L(s-3, \Delta_{12})L(s, \Delta_{18})\,,$$

where Δ_{12} and Δ_{18} are primitive elliptic eigenforms of weight 12 and 18, respectively.

Next we see the case $k = 12$. Then we have $\dim A_{12,4}(\Gamma_2) = 2$, $\dim A_{14}(\Gamma_2) = 2$. For $A(\tau) = \sum_{i=0}^4 a_i(\tau)u_1^{4-i}u_2^i \in A_{12,4}(\Gamma_2)$, $W(a_2)$ is a product of elliptic cusp forms of weight 14, which is zero. So the condition that $(A(\tau), c(\tau)) \in A_{12,2}^T(\Gamma_2)$ is that $W(c(\tau)) = 0$. Since $A_{14}(\Gamma_2)$ is spanned by $\phi_4^2\phi_6$ and $\phi_4\chi_{10}$, and $W(\phi_4^2\phi_6) \neq 0$, $W(\phi_4\chi_{10}) = 0$, we have

$$A^T_{12,2}(\Gamma_2) = \{(A(\tau), c(\tau)); A(\tau) \in A_{12,4}(\Gamma_2),\ c(\tau) \in \mathbb{C}\phi_4\chi_{10}\},$$
$$S^T_{12,2}(\Gamma_2) = \{(A(\tau), c(\tau)); A(\tau) \in S_{12,4}(\Gamma_2),\ c(\tau) \in \mathbb{C}\phi_4\chi_{10}\}.$$

Now, we have $\dim A_{12,4}(\Gamma_2) = 2$ and $\dim S_{12,4}(\Gamma_2) = 1$. The space $S_{12,4}(\Gamma_2)$ is spanned by $\{\phi_4, \phi_6\}_{\det{}^2 Sym(4)}$, which is a kind of Rankin-Cohen bracket explicitly defined as follows (See [13]). We define differential operators m and Δ by

$$m = u_1^2 \frac{\partial}{\partial\tau_{11}} + u_1u_2\frac{\partial}{\partial\tau_{12}} + u_2^2\frac{\partial}{\partial\tau_{22}},$$

$$\Delta = \begin{vmatrix} \dfrac{\partial}{\partial\tau_{11}} & \dfrac{1}{2}\dfrac{\partial}{\partial\tau_{12}} \\ \dfrac{1}{2}\dfrac{\partial}{\partial\tau_{12}} & \dfrac{\partial}{\partial\tau_{22}} \end{vmatrix}.$$

Then

$$\begin{aligned}\{\phi_4, \phi_6\}_{\det^2 Sym(4)} =& -924(\Delta\phi_4)(m^2\phi_6) + 2156((\Delta m)\phi_4)(m\phi_6) - 924(\Delta\phi_4)(m^2\phi_6) \\ & - 210(\phi_4)(\Delta m^2\phi_6) + 945(m\phi_4)(\Delta m\phi_6) - 756(m^2\phi_4)(\Delta\phi_6) \\ & + \frac{1155}{4}\left(\frac{\partial\phi_4}{\partial\tau_{22}}\frac{\partial(m^2\phi_6)}{\partial\tau_{11}} - \frac{1}{2}\frac{\partial\phi_4}{\partial\tau_{12}}\frac{\partial(m^2\phi_6)}{\partial\tau_{12}} + \frac{\partial\phi_4}{\partial\tau_{11}}\frac{\partial(m^2\phi_6)}{\partial\tau_{22}}\right) \\ & - 924\left(\frac{\partial(m\phi_4)}{\partial\tau_{22}}\frac{\partial(m\phi_6)}{\partial\tau_{11}} - \frac{1}{2}\frac{\partial(m\phi_4)}{\partial\tau_{12}}\frac{\partial(m\phi_6)}{\partial\tau_{12}} + \frac{\partial(m\phi_4)}{\partial\tau_{11}}\frac{\partial(m\phi_6)}{\partial\tau_{22}}\right) \\ & + 539\left(\frac{\partial(m^2\phi_4)}{\partial\tau_{22}}\frac{\partial\phi_6}{\partial\tau_{11}} - \frac{1}{2}\frac{\partial(m^2\phi_4)}{\partial\tau_{12}}\frac{\partial\phi_6}{\partial\tau_{12}} + \frac{\partial(m^2\phi_4)}{\partial\tau_{11}}\frac{\partial\phi_6}{\partial\tau_{22}}\right).\end{aligned}$$

We denote by $F_{12-1/2} = F_{23/2} \in S^+_{23/2,2}(\Gamma_0^{(2)}(4))$ the Siegel cusp form corresponding to $\left(\{\phi_4, \phi_6\}_{\det^2 Sym(4)}, 0\right) \in S^T_{12,2}(\Gamma_2)$. We also denote by $G_{23/2} \in S^+_{23/2,2}(\Gamma_0^{(2)}(4))$ the Siegel cusp form corresponding to $(0, \phi_4\chi_{10}) \in S^T_{12,2}(\Gamma_2)$. We will calculate the Euler 2 factors of eigenforms in $S^+_{23/2,2}(\Gamma_0^{(2)}(4))$. Now we review how to calculate the eigenvalues of the Hecke operators. For $F(\tau) \in A^+_{k-1/2,j}(\Gamma_0^{(2)}(4))$ for even k, let $F(\tau) = \sum_{T\in L_2^*} a(T)e(\mathrm{Tr}(T\tau))$ be the Fourier expansion. We denote by $a(T_i(p); T) = a(T_i(p); (t_1, t_2, t_{12}))$ the Fourier coefficient of $T_i(p)F$ at (t_1, t_2, t_{12}). Then we have

$$\begin{aligned}a(T_1(2); (3,3,2)) =& \rho_j\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}a(3,12,4) + \rho_j\begin{pmatrix}2 & -1\\ 0 & 1\end{pmatrix}a(8,12,16) \\ & + \rho_j\begin{pmatrix}0 & 1\\ 2 & 0\end{pmatrix}a(3,12,4) - 2^{k+j-2}a(3,3,2),\end{aligned}$$

$$a(T_2(2); (3,3,2)) = 2^j a(12,12,8)$$

$$
\begin{aligned}
&- 2^{k+j-3}\left(\rho_j\begin{pmatrix}2&0\\0&1\end{pmatrix}a(3,12,4)+\rho_j\begin{pmatrix}0&1\\2&0\end{pmatrix}a(3,12,4)\right)\\
&+ 2^{2k+2j-6}a(3,3,2)\,,\\
a(T_1(2);(3,4,0)) =&\rho_j\begin{pmatrix}2&0\\0&1\end{pmatrix}a(3,16,0)+\rho_j\begin{pmatrix}2&-1\\0&1\end{pmatrix}a(7,16,16)\\
&+\rho_j\begin{pmatrix}0&-1\\2&0\end{pmatrix}a(4,12,0)\,,\\
a(T_2(2);(3,4,0)) =&2^{2k+j-5}\rho_j\begin{pmatrix}0&1\\4&-2\end{pmatrix}a(4,12,12)+2^j a(12,16,0)\\
&+2^{k+j-3}\left(-\rho_j\begin{pmatrix}2&0\\0&1\end{pmatrix}a(3,16,0)+\rho_j\begin{pmatrix}2&-1\\0&1\end{pmatrix}a(7,16,16)\right)\\
&-2^{2k+2j-6}a(3,4,0)\,.
\end{aligned}
$$

In the following explanation, we identify the representation space $\mathbb{C}^{j+1}$ of $\rho_j = Sym(j)$ with the space $\mathbb{C}[u_1,u_2]_j$ of homogeneous polynomials in u_1, u_2 of degree j. Then the Fourier coefficients $a(T)$ are polynomials $a(T)(u_1,u_2)\in\mathbb{C}[u_1,u_2]_j$ in u_1, u_2 and the action of $\rho_j = Sym(j)$ is given by $\rho_j(U)a(T)(u_1,u_2) = a(T)((u_1,u_2)U)$ for $U\in GL_2(\mathbb{R})$. We denote the Fourier coefficient of $F_{23/2}$ and $G_{23/2}$ by $A(T)$ and $B(T)$, respectively. Then by computer calculation we have

$$
\begin{aligned}
A(3,3,2) &= -u_1^2-2u_1u_2-u_2^2 = -(u_1+u_2)^2\\
A(3,4,0) &= -18u_1^2+4u_2^2\\
A(3,12,4) &= -1328u_1^2-3872u_1u_2-1664u_2^2\\
A(3,16,0) &= -5472u_1^2-11264u_2^2\\
A(4,12,0) &= 1280u_1^2-9600u_2^2\\
A(4,12,12) &= 4u_1^2+12u_1u_2+12u_2^2\\
A(7,16,16) &= 19488u_1^2-5376u_1u_2-5376u_2^2\\
A(8,12,16) &= 6144u_1^2+12288u_1u_2+5760u_2^2\\
A(12,12,8) &= 6017024u_1^2+11247616u_1u_2+6017024u_2^2\\
A(12,16,0) &= 74686464u_1^2-41369600u_2^2.
\end{aligned}
$$

$$
\begin{aligned}
B(3,3,2) &= -\frac{1}{2}u_1u_2\\
B(3,4,0) &= -u_2^2\\
B(3,12,4) &= -788u_1u_2-564u_2^2
\end{aligned}
$$

$$
\begin{aligned}
B(3,16,0) &= 4328u_2^2 \\
B(4,12,0) &= 112u_1^2 + 24u_2^2 \\
B(4,12,12) &= -u_1^2 - 3u_1u_2 - 3u_2^2 \\
B(7,16,16) &= 240u_1^2 - 4056u_1u_2 - 4056u_2^2 \\
B(8,12,16) &= -12u_1^2 - 24u_1u_2 - 36u_2^2 \\
B(12,12,8) &= 359232u_1^2 - 1355456u_1u_2 + 359232u_2^2 \\
B(12,16,0) &= 10668672u_1^2 - 374656u_2^2.
\end{aligned}
$$

By these Fourier coefficients, we see that

$$
\begin{aligned}
&A(T_1(2);(3,3,0)) = -2880(-u_1^2 - 2u_1u_2 - u_2^2) = -2880A(3,3,2)\,, \\
&A(T_1(2);(3,4,0)) = -2880(-18u_1^2 + 4u_2^2) = -2880A(3,4,0)\,, \\
&B(T_1(2);(3,3,0)) = 360(u_1^2 - 3u_1u_2 + u_2^2) = 3600B(3,3,2) - 360A(3,3,2)\,, \\
&B(T_1(2);(3,4,0)) = 720(9u_1^2 - 7u_2^2) = 3600B(3,4,0) - 360A(3,4,0)\,.
\end{aligned}
$$

So we have

$$
(T_1(2)F, T_1(2)G) = (F, G)\begin{pmatrix} -2880 & -360 \\ 0 & 3600 \end{pmatrix}.
$$

So $F_{23/2}$ and $F_{23/2} - 18G_{23/2}$ are eigenforms, and eigenvalues of $T_1(2)$ are given by $\lambda^*(2, F_{23/2}) = -2880$ and $\lambda^*(2, F_{23/2} - 18G_{23/2}) = 3600$, respectively. Actually again we can show that the concrete lift $\mathcal{L}$ in [15] from $S_{20}(\Gamma_1) \times S_{26}(\Gamma_1)$ into $S^+_{23/2,2}(\Gamma_0^{(2)}(4))$ does not vanish and the image is spanned by $F_{23/2} - 18G_{23/2}$. We have

$$
L(s, F_{23/2} - 18G_{23/2}) = L(s-3, \Delta_{20})L(s, \Delta_{26})\,,
$$

where Δ_{20} and Δ_{26} are the unique primitive cusp form of weight 20 and 26 of Γ_1, respectively. Calculating the action of $T_2(2)$ on $F_{23/2}$ in the same way, we see that $\omega(2) = -34160640$ and the Euler 2-factor of $L(s, F_{23/2})$ is given by

$$
H_2(F_{23/2}, u) = 1 + 2880u - 26378240u^2 + 2880 \cdot 2^{25}u^3 + 2^{50}u^4
$$

where $u = 2^{-s}$. We have $\dim S_{23/2,2}(\Gamma_0^{(2)}(4), \psi) = \dim S_{5,18}(\Gamma_2) = 1$, and we denote each basis by $F^{\psi}_{23/2}$ and $F_{5,18}$. Then by the calculation in [12] p. 123, we see that

$$
H_2(F_{23/2}, u) = H_2(F^{\psi}_{23/2}, u) = H_2(F_{5,18}, u)\,.
$$

Similarly we can show the same equaliy for the Euler 3 factors:

$$
H_3(F_{23/2}, u) = H_3(F^{\psi}_{23/2}, u) = H_3(F_{5,18}, u)\,.
$$

These support Conjecture 1.3 and 1.2. (Note the correction of the typo in the previous paper at the end of this paper.) The calculation for the Euler 3-factors can be done by using the following Fourier coefficients and the relations for either (3, 3, 2) or (3, 4, 0). We omit the details.

$$\begin{aligned}
A(3,27,6) &= 115182u_1^2 + 230364u_1u_2 - 295002u_2^2\\
A(3,27,-6) &= 115182u_1^2 - 230364u_1u_2 - 295002u_2^2\\
A(4,3,4) &= A(4,3,-4) = -u_2^2\\
A(8,11,8) &= 227988u_1^2 + 227988u_1u_2 - 51219u_2^2\\
A(19,4,4) &= -681201u_1^2 + 36864u_1u_2 + 36864u_2^2\\
A(27,27,18) &= 15501347379u_1^2 - 21240081306u_1u_2 + 15501347379u_2^2\\
A(3,36,0) &= 1010394u_1^2 + 3204684u_2^2\\
A(7,36,24) &= -660492u_1^2 - 1820880u_1u_2 - 2011608u_2^2\\
A(19,36,48) &= -851220u_1^2 - 2202336u_1u_2 - 2011608u_2^2\\
A(4,27,0) &= 87624u_1^2 - 3972132u_2^2\\
A(27,36,0) &= 440392060944u_1^2 - 200717867808u_2^2
\end{aligned}$$

$$\begin{aligned}
A(T_1(3);(3,3,2)) = {} & \rho_j\begin{pmatrix}2 & -1\\ 1 & 1\end{pmatrix}A(8,11,8) + \rho_j\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}A(3,27,6)\\
& + \rho_j\begin{pmatrix}0 & -1\\ 3 & 0\end{pmatrix}A(3,27,-6) + \rho_j\begin{pmatrix}1 & -2\\ 1 & 1\end{pmatrix}A(19,4,4)
\end{aligned}$$

$$\begin{aligned}
A(T_2(3);(3,3,2)) = {} & 3^jA(27,27,18)\\
& + 3^{2k+j-5}\left(\rho_j\begin{pmatrix}3 & -3\\ 2 & 1\end{pmatrix}A(4,3,4) + \rho_j\begin{pmatrix}2 & -1\\ 3 & 3\end{pmatrix}A(4,3,-4)\right)\\
& + 3^{k+j-3}\left(\rho_j\begin{pmatrix}2 & -1\\ 1 & 1\end{pmatrix}A(8,11,8) - \rho_j\begin{pmatrix}1 & -2\\ 1 & 1\end{pmatrix}A(19,4,4)\right)\\
& - 3^{2k+2j-6}A(3,3,2)
\end{aligned}$$

$$\begin{aligned}
A(T_1(3),(3,4,0)) = {} & \rho_j\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}A(3,36,0) + \rho_j\begin{pmatrix}3 & -1\\ 0 & 1\end{pmatrix}A(7,36,24)\\
& + \rho_j\begin{pmatrix}3 & -2\\ 0 & 1\end{pmatrix}A(19,36,48) + \rho_j\begin{pmatrix}0 & -1\\ 3 & 0\end{pmatrix}A(4,27,0)\\
& - 3^{k+j-2}A(3,4,0)\,,\\
A(T_2(3),(3,4,0)) = {} & 3^jA(27,36,0) - 3^{k+j-3}\big(\rho_j\begin{pmatrix}3 & -1\\ 0 & 1\end{pmatrix}A(7,36,24)
\end{aligned}$$

$$+ \rho_j \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix} A(19, 36, 48) + \rho_j \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix} A(4, 27, 0)\Big)$$
$$+ 2 \cdot 3^{2k+2j-6} A(3, 4, 0)\,.$$

4.3. The congruence

Now we explain an example of the half-integral version of Harder's congruences. For any prime p, let $\lambda^*(p)$ and $\omega(p)$ be the eigenvalues of $T_1(p)$ and $T_2(p)$ of $F_{23/2}$ respectively, and $a_{26}(p)$ the eigenvalue at p of the unique primitive elliptic cusp form Δ_{26} of level 1.

THEOREM 4.4. *Notations being as above, for any prime p, we have the following congruence.*

$$\begin{aligned} &1 - \lambda^*(p)u + (p\omega(p) + p^{23}(p^2+1))u^2 + \lambda^*(p)p^{25}u^3 + p^{50}u^4 \\ &\quad \equiv (1 - p^3u)(1 - p^{22}u)(1 - a_{26}(p)u + p^{25}u^2) \bmod 43\,. \end{aligned}$$

In particular, we have

$$\lambda^*(p) \equiv p^3 + p^{22} + a_{26}(p) \bmod 43$$

for all primes p.

It seems to the author that, even in the case of integral weight, there had been no known example such that such congruence holds for all the Euler p factors for a fixed congruence prime, since this type of theorem for all p cannot be checked just by calculating the examples of eigenvalues at finitely many p. By the way, if we believe Conjecture 1.2, this congruence means the congruence between eigenvalues of the cusp form of weight $\det^5 Sym(18)$ and the quantity as above coming from Δ_{26}, which is the original Harder's conjecture.

We note that $43 | L_{alg}(23, \Delta_{26})$ (see [3]) and that there exists the Klingen type Eisenstein series $E_{23/2} \in A^+_{23/2,2}(\Gamma_0^{(2)}(4))$ such that

$$L(s, E_{23/2}) = \zeta(s-3)\zeta(s-22)L(s, \Delta_{26})$$

and the right hand side of the first congruence in the above theorem gives the Euler p factor of $L(s, E_{23/2})$. So this theorem gives an example of Conjecture 1.4.

Now in order to give $E_{23/2}$ explicitly and prove the above theorem, we consider a non-cusp form in $A^+_{23/2,2}(\Gamma_0^{(2)}(4))$. First we define a non-zero non-cusp form $H_{15/2} \in A^+_{15/2,2}(\Gamma_0^{(2)}(4))$. We define $H_{15/2}$ from a certain Jacobi form in $J_{(8,2),1}(\Gamma_2^J)$ corresponding to $(A(\tau), 0) \in A^T_{8,2}(\Gamma_2)$ with $A(\tau) \in A_{8,4}(\Gamma_2)$. The space $A_{8,4}(\Gamma_2)$ is spanned by $\{\phi_4, \phi_4\}_{Sym(4)}$. (See [13].) Here for any $f \in A_k(\Gamma_2)$ and $g \in A_l(\Gamma_2)$, the form $\{f, g\}_{Sym(4)} \in A_{k,4}(\Gamma_2)$ is the Rankin-Cohen type bracket defined by

$$(2\pi i)^2 \{f, g\}_{Sym(4)} =$$

$$\Big(\frac{l(l+1)}{2}\frac{\partial^2 f}{\partial \tau_{11}^2} g - (l+1)(k+1)\frac{\partial f}{\partial \tau_{11}}\frac{\partial g}{\partial \tau_{11}} + \frac{k(k+1)}{2} f \frac{\partial^2 g}{\partial \tau_{11}^2}\Big) u_1^4$$

$$
\begin{aligned}
&+\Bigg(l(l+1)\frac{\partial^2 f}{\partial\tau_{11}\partial\tau_{12}}g-(k+1)(l+1)\left(\frac{\partial f}{\partial\tau_{12}}\frac{\partial g}{\partial\tau_{11}}+\frac{\partial f}{\partial\tau_{11}}\frac{\partial g}{\partial\tau_{12}}\right)\\
&+k(k+1)f\frac{\partial^2 g}{\partial\tau_{11}\partial\tau_{12}}\Bigg)u_1^3u_2+\Bigg(\frac{l(l+1)}{2}\frac{\partial^2 f}{\partial\tau_{12}^2}g+l(l+1)\frac{\partial^2 f}{\partial\tau_{11}\partial\tau_{22}}g\\
&-(k+1)(l+1)\frac{\partial f}{\partial\tau_{22}}\frac{\partial g}{\partial\tau_{11}}-(k+1)(l+1)\frac{\partial f}{\partial\tau_{12}}\frac{\partial g}{\partial\tau_{12}}\\
&+\frac{k(k+1)}{2}f\frac{\partial^2 g}{\partial\tau_{12}^2}-(k+1)(l+1)\frac{\partial f}{\partial\tau_{11}}\frac{\partial g}{\partial\tau_{22}}+k(k+1)f\frac{\partial^2 g}{\partial\tau_{11}\partial\tau_{22}}\Bigg)u_1^2u_2^2\\
&+\Bigg(l(l+1)\frac{\partial^2 f}{\partial\tau_{12}\partial\tau_{22}}g-(k+1)(l+1)\left(\frac{\partial f}{\partial\tau_{22}}\frac{\partial g}{\partial\tau_{12}}+\frac{\partial f}{\partial\tau_{12}}\frac{\partial g}{\partial\tau_{22}}\right)\\
&+k(k+1)f\frac{\partial^2 g}{\partial\tau_{12}\partial\tau_{22}}\Bigg)u_1u_2^3\\
&+\left(\frac{l(l+1)}{2}\frac{\partial^2 f}{\partial\tau_{22}^2}g-(k+1)(l+1)\frac{\partial f}{\partial\tau_{22}}\frac{\partial g}{\partial\tau_{22}}+\frac{k(k+1)}{2}f\frac{\partial^2 g}{\partial\tau_{22}^2}\right)u_2^4.
\end{aligned}
$$

For $\{\phi_4,\phi_4\}_{Sym(4)}/14400=\sum_{i=0}^4 a_i(\tau)u_1^{4-i}u_2^i$, the form $H_{15/2}\in A^+_{15/2,2}(\Gamma_0^{(2)}(4))$ is defined as in (4) with $c(\tau)=0$. This is a non-cusp form. Examples of the Fourier coefficients of $H_{15/2}$ are given by

$$
\begin{aligned}
A(3,0,0;H_{15/2})&=u_1^2\,,\\
A(4,0,0;H_{15/2})&=-2u_1^2\,,\\
A(3,3,2;H_{15/2})&=(-1/2)u_1u_2\,,\\
A(3,4,0;H_{15/2})&=-u_2^2\,.
\end{aligned}
$$

Now we put $H_{23/2}(\tau)=\phi_4(4\tau)H_{15/2}(\tau)$. Then it is clear that $H_{23/2}$ is also a non-cusp form in $A^+_{23/2,2}(\Gamma_0^{(2)}(4))$, though this is not an eigenform. Next we determine a Hecke eigen non-cusp form. Since there exists the Klingen-type Eisenstein series $E_{23/2}$ such that

$$L(s,E_{23/2})=\zeta(s-3)\zeta(s-22)L(s,\Delta_{26})\,,$$

and we have $\dim A^+_{23/2,2}(\Gamma_0^{(2)}(4))-\dim S^+_{23/2,2}(\Gamma_0^{(2)}(4))=3-2=1$, we see that $T_1(2)H_{23/2}-c(2)H_{23/2}\in S^+_{23/2,2}(\Gamma_0^{(2)}(4))$ where $c(2)=2^3+2^{22}+a_{26}(2)=4194264$, and $a_{26}(2)=-48$ is the eigenvalue of Δ_{26} at 2. We denote the Fourier coefficients of $H_{23/2}$ by $C(T)$. Then by computer calculation we have

$$
\begin{aligned}
C(3,3,2)&=27u_1^2+58u_1u_2+27u_2^2\\
C(3,4,0)&=552u_1^2-104u_2^2\\
C(3,12,4)&=1775979u_1^2+2431012u_1u_2+541596u_2^2\\
C(3,16,0)&=100608456u_1^2-24084608u_2^2\\
C(4,12,0)&=-9827008u_1^2+1756896u_2^2
\end{aligned}
$$

$$C(7, 16, 16) = 2559648u_1^2 + 7944384u_1u_2 + 7944384u_2^2$$
$$C(8, 12, 16) = -167688u_1^2 - 335376u_1u_2 - 331392u_2^2$$

By this table we can calculate the coefficient of $T_1(2)H_{23/2}$ as follows.

$$C(T_1(2); (3, 3, 2)) = 72(100049u_1^2 + 131646u_1u_2 + 100049u_2^2)\,,$$
$$C(T_1(2); (3, 4, 0)) = 192(2117109u_1^2 - 288793u_2^2)\,.$$

Then the coefficients of $T_1(2)H_{23/2} - c(2)H_{23/2}$ at (3, 3, 2) and (3, 4, 0) are given respectively as follows.

$$(3, 3, 2) \quad -67200(1578u_1^2 + 3479u_1u_2 + 1578u_2^2)\,,$$

$$(3, 4, 0) \quad -134400(14202u_1^2 - 2833u_2^2)\,.$$

So by comparing the coefficients at (3, 3, 2) and (3, 4, 0), we easily see that

$$T_1(2)H_{23/2} - c(2)H_{23/2} = 134400(789F_{23/2} + 323G_{23/2})\,.$$

So we have

$$\begin{pmatrix} T_1(2)H_{23/2} \\ T_1(2)G_{23/2} \\ T_1(2)F_{23/2} \end{pmatrix} = \begin{pmatrix} 2^3 \cdot (2^{19}+1) - 48 & (134400)\cdot(323) & (134400)\cdot(789) \\ 0 & 3600 & -360 \\ 0 & 0 & -2880 \end{pmatrix} \begin{pmatrix} H_{23/2} \\ G_{23/2} \\ F_{23/2} \end{pmatrix}$$

and the eigenform $E_{23/2}$ corresponding to the eigenvalue $c(2) = 2^3(2^{19}+1) - 48$, that is, the Klingen type Eisenstein series, is given by

$$E_{23/2} = 7508273H_{23/2} + 77778400G_{23/2} + 189691200F_{23/2}\,.$$

We know already that $E_{23/2}$ and $F_{23/2}$ are eigenforms for any Hecke operators. We denote by $\lambda(T_i(p), F)$ the eigenvalue of $T_i(p)$ for any eigenform F for each $i = 1, 2$. Then we have

$$\lambda(T_1(p), E_{23/2}) = p^3(1 + p^{19}) - a_{26}(p)\,,$$
$$\lambda(T_2(p), E_{23/2}) = p^2(1 + p^{19})a_{26}(p) + p^{22} + p^{24}\,,$$

and these are integers. We have

$$\begin{aligned} T_i(p)E_{23/2} =& 7508273T_i(p)H_{23/2} + 77778400T_i(p)G_{23/2} \\ &+ 189691200T_i(p)F_{23/2}\,, \\ \lambda(T_i(p), E_{23/2})E_{23/2} =& 7508273\lambda(T_i(p), E_{23/2})H_{23/2} + \lambda(T_i(p), E_{23/2})G_{23/2} \\ &+ 189691200\lambda(T_i(p), E_{23/2})F_{23/2}\,. \end{aligned}$$

Since we have $T_i(p)E_{23/2} = \lambda(T_i(p), E_{23/2})E_{23/2}$, subtracting both sides of the above equalities, we have

$$(5) \quad 189691200\Big(\lambda(T_i(p), F_{23/2}) - \lambda(T_i(p), E_{23/2})\Big)F_{23/2} =$$

$$7508273\Big(\lambda(T_i(p), E_{23/2})H_{23/2} - T_i(p)H_{23/2}\Big) \\ + 77778400\Big(\lambda(T_i(p), E_{23/2})G_{23/2} - T_i(p)G_{23/2}\Big).$$

We have $7508273 = 43 \cdot 283 \cdot 617$ and $77778400 = 2^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 19 \cdot 43$, both divisible by 43, but $189691200 = 2^6 \cdot 3^3 \cdot 5^2 \cdot 4391$, which is not divisible by 43. Now we see that the Fourier coefficients of $H_{23/2}$, $G_{23/2}$ (and $F_{23/2}$) are 43-adically integral. First we see that the Fourier coefficients of ϕ_4, ϕ_6 and χ_{10} are all integral since these are obtained by the Maass lift (or the Saito-Kurokawa lift) of Jacobi forms with integral coefficients given by $E_{4,1}$, $E_{6,1}$, $\chi_{10,1}$ of degree one of weight 4, 6, 10 (See [2]). Then Rankin-Cohen operators are integral except for the power of 2. The normalizing constants $14400 = 2^6 \cdot 3^2 \cdot 5^2$ and $24449040 = 2^4 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11$ used as the denominator in the definition of Jacobi forms corresponding with $H_{15/2}$ and $F_{23/2}$ does not contain 43 as a prime factor. Now the cofactors $B_{ij}(\tau)$ have integral coefficients, since all ϑ_{ij} do so, and all $\chi_5(\tau)c_{ij}(\tau)$ have 43-integral Fourier coefficients. Now in the final stage, we must divide $\chi_5(4\tau)c_{ij}(4\tau)$ by $\chi_5(4\tau)$. This calculation can be done by using the Fourier expansion by induction with respect to the orders of $e(\tau_{11})$ and $e(\tau_{22})$. By definition, we see that the Fourier coefficients of χ_5 are integral, and since we have

$$\chi_5(4\tau) = e(2\tau_{11})e(2\tau_{22})(e(2\tau_{12}) - e(-2\tau_{12})) + \text{ higher terms},$$

and the lowest order term has coefficient 1, the quotients by this have also 43-integral Fourier coefficients. By the formula of Fourier coefficients of $T_i(p)F$ for $F \in A^+_{k-1/2,j}(\Gamma_0^{(2)}(4))$ (see [12] pp.127–128), we see that if F has l-adically integral Fourier coefficients for a prime l, then $T_i(p)F$ also. So we see that the Fourier coefficients of the right hand side of (5) are all divisible by 43 locally, and hence the Fourier coefficients of $(\lambda(T_i(p), F_{23/2}) - \lambda(T_i(p), E_{23/2}))F/43$ are all 43-adically integral. So, seeing the Fourier coefficient $-(u_1 + u_2)^2$ at $(3, 3, 2)$ of $F_{23/2}$, we have

$$\lambda(T_i(p), F_{23/2}) \equiv \lambda(T_i(p), E_{23/2}) \bmod 43$$

for $i = 1$ and 2 for any prime p. This proves Theorem 4.4.

Correction. There are following typos in [12].
In p. 111, l.5, $|\phi(Z)| = |\det(CZ + D)|^{1/2}$ should read
$|\phi(Z)| = (\det(g))^{-1/4}|\det(CZ + D)|^{1/2}$.
In p. 123, l.6 from the bottom, two 538970 should read 538920.
In p. 129, l. 10, $\rho_j\begin{pmatrix}0 & -1\\ 0 & 3\end{pmatrix}$ should read $\rho_j\begin{pmatrix}0 & -1\\ 3 & 0\end{pmatrix}$.
In p. 134, l.7 from the bottom, "holomorphic Jacobi form $\phi(\tau_1, z_1) \in J_{k+j,1}$" should read "holomorphic Jacobi cusp form $\phi(\tau_1, z_1) \in J^{cusp}_{k+j,1}$. In p. 134, l.2 from the bottom, "For $\phi \in J^{skew}_{k+j,1}$" should read "For $\phi \in J^{skew,cusp}_{k+j,1}$".

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