

Harder’s conjecture I

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Abstract. Let f be a primitive form with respect to $\mathrm{SL}_2(\mathbb{Z})$. Then, we propose a conjecture on the congruence between the Klingen–Eisenstein lift of the Duke–Imamoglu–Ikeda lift of f and a certain lift of a vector valued Hecke eigenform with respect to $\mathrm{Sp}_2(\mathbb{Z})$. This conjecture implies Harder’s conjecture. We prove the above conjecture in some cases.

1. Introduction.

Harder’s conjecture is one of the most fascinating conjectures in the arithmetic of automorphic forms. It plays an important role in constructing nontrivial elements of the Bloch–Kato Selmer group attached to a modular form (cf. [15], [17]). Harder’s conjecture predicts that the Hecke eigenvalues of a primitive form for $\mathrm{SL}_2(\mathbb{Z})$ are related with those of a certain Hecke eigenform for $\mathrm{Sp}_2(\mathbb{Z})$ modulo some prime ideal. We explain it more precisely. For a non-increasing sequence $\mathbf{k} = (k_1, \dots, k_n)$ of non-negative integers we denote by $M_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z}))$ and $S_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z}))$ the spaces of modular forms and cusp forms of weight \mathbf{k} (or, weight k , if $\mathbf{k} = \overbrace{(k, \dots, k)}^n$) for $\mathrm{Sp}_n(\mathbb{Z})$, respectively. (For the definition of modular forms, see Section 2.) Let $f(z) = \sum_{m=1}^{\infty} a(m, f) \exp(2\pi\sqrt{-1} mz)$ be a primitive form in $S_{2k+j-2}(\mathrm{SL}_2(\mathbb{Z}))$, and suppose that a ‘big prime’ \mathfrak{p} divides the algebraic part of $L(k+j, f)$. Then, Harder [17] conjectured that there exists a Hecke eigenform F in $S_{(k+j, k)}(\mathrm{Sp}_2(\mathbb{Z}))$ such that

$$\lambda_F(T(p)) \equiv a(p, f) + p^{k-2} + p^{j+k-1} \pmod{\mathfrak{p}'}$$

for any prime number p , where $\lambda_F(T(p))$ is the eigenvalue of the Hecke operator $T(p)$ on F , and \mathfrak{p}' is a prime ideal of $\mathbb{Q}(f) \cdot \mathbb{Q}(F)$ lying above \mathfrak{p} . One of main difficulties in treating this congruence arises from the fact that this is not concerning the congruence between Hecke eigenvalues of two Hecke eigenforms of the same weight. Indeed, the right-hand side of the above is not the Hecke eigenvalue of a Hecke eigenform if $j > 0$. Several attempts have been made to overcome this obstacle. Ibukiyama [21], [23] proposed a half-integral weight version of Harder’s conjecture given as congruences of Hecke

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eigenforms and related it to the original Harder’s conjecture through his conjectural Shimura type correspondence for vector valued Siegel modular forms of degree two (and this Shimura type conjecture was now proved by Ishimoto [34]). This explains the Harder conjecture for odd k ([26]) and the proved example of congruence in [23] means the Harder conjecture for $(k, j) = (5, 18)$. In [5], Bergström and Dummigan, among other things, reformulated Harder’s conjecture as congruence between a certain induced representation of π_f and a cuspidal automorphic representation of $\mathrm{GSp}(2)$. In [12], Chenevier and Lannes gave several congruences between theta series of even unimodular lattices, and using Arthur’s endoscopic classification and Galois representation theoretic method, they, among other things, proved Harder’s conjecture for $(k, j) = (10, 4)$. In this paper we consider a conjecture concerning the congruence between two liftings to higher degree of Hecke eigenforms (of integral weight) of degree two. More precisely, for the f above, let $J_n(f)$ be the Duke–Imamoglu–Ikeda lift of f to the space of cusp forms of weight $\frac{j}{2} + k + \frac{n}{2} - 1$ for $\mathrm{Sp}_n(\mathbb{Z})$ with n even. For a sequence

$$\mathbf{k} = \left(\overbrace{\left(\frac{j}{2} + k + \frac{n}{2} - 1, \dots, \frac{j}{2} + k + \frac{n}{2} - 1 \right)}^n, \overbrace{\left(\frac{j}{2} + \frac{3n}{2} + 1, \dots, \frac{j}{2} + \frac{3n}{2} + 1 \right)}^n \right)$$

with $k \geq n + 2$, let $[J_n(f)]^{\mathbf{k}}$ be the Klingen–Eisenstein lift of $J_n(f)$ to $M_{\mathbf{k}}(\mathrm{Sp}_{2n}(\mathbb{Z}))$. Then, we propose the following conjecture:

CONJECTURE 1.1 (Conjecture 4.5). *Let k, j and \mathbf{k} be as above. Let $f(z) \in S_{2k+j-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a primitive form and \mathfrak{p} a prime ideal of $\mathbb{Q}(f)$. Then under certain assumptions, there exists a Hecke eigenform F in $S_{(k+j,k)}(\mathrm{Sp}_2(\mathbb{Z}))$ such that*

$$\lambda_{\mathcal{A}_{2n}^{(I)}(F)}(T) \equiv \lambda_{[J_n(f)]^{\mathbf{k}}}(T) \pmod{\mathfrak{p}'}$$

for any integral Hecke operator T . Here, $\mathcal{A}_{2n}^{(I)}(F)$ is the lift of F to $S_{\mathbf{k}}(\mathrm{Sp}_{2n}(\mathbb{Z}))$, called the lift of type $\mathcal{A}^{(I)}$, which will be defined in Theorem 4.2. (As for the definition of integral Hecke operators, see Section 3.)

This conjecture implies Harder’s conjecture (cf. Theorem 4.8).

The advantage of this formulation is that one can compare the Hecke eigenvalues of two Hecke eigenforms. Indeed, by using the same argument as in Katsurada–Mizumoto [39], under the above assumption, we can prove that there exists a Hecke eigenform $G \in M_{\mathbf{k}}(\mathrm{Sp}_{2n}(\mathbb{Z}))$ such that G is not a constant multiple of $[J_n(f)]^{\mathbf{k}}$ and

$$\lambda_G(T) \equiv \lambda_{[J_n(f)]^{\mathbf{k}}}(T) \pmod{\mathfrak{p}'}$$

for any integral Hecke operator T . Therefore, to prove the above conjecture, it suffices to show that G is a lift of type $\mathcal{A}^{(I)}$. Here we expect that G can be taken as $\mathcal{A}^{(I)}$ and indeed we will see that in the cases $(k, j) = (10, 4), (14, 4)$ and $(4, 24)$ using the dimension formula due to [57] and the numerical tables of Hecke eigenvalues due to Poor–Ryan–Yuen [52] and Ibukiyama–Katsurada–Poor–Yuen [30]. As a result, we prove Conjecture 4.5 and so Harder’s conjecture in those cases.

This paper is organized as follows. In Section 2, we give a brief summary of Siegel modular forms, especially about their \mathbb{Q} -structures or \mathbb{Z} -structures. In Section 3, after giving a summary of several L -values, we state Harder’s conjecture. In Section 4, we introduce several lifts, and among other things define the lift of $\mathcal{A}^{(l)}$ -type of a vector valued modular form in $S_{(k+j,k)}(\mathrm{Sp}_2(\mathbb{Z}))$, and propose a conjecture and explain how this conjecture implies Harder’s conjecture. In Section 5, we consider the pullback formula of the Siegel–Eisenstein series with differential operators. In Section 6, we consider the congruence for vector valued Klingen–Eisenstein series, which is a generalization of [39], and explain how the assumption that \mathfrak{p} divides the algebraic part of $L(k+j, f)$ for $f \in S_{2k+j-2}(\mathrm{SL}_2(\mathbb{Z}))$ gives the congruence between $[J_n(f)]^{\mathfrak{k}}$ and another Hecke eigenform in $M_{\mathfrak{k}}(\mathrm{Sp}_{2n}(\mathbb{Z}))$. In Section 7, we give a formula for the Fourier coefficients of the Klingen–Eisenstein series, from which we can confirm some assumption in our main results. In Section 8, we state our main results, which confirm our conjecture, and so Harder’s.

In a subsequent paper, we will prove Conjecture 4.5 and so Harder’s in more general setting, that is, in the case k is even and $j \equiv 0 \pmod{4}$, that is, we will prove these conjectures without using the dimension formula or the computation of Hecke eigenvalues of Siegel modular forms (cf. [4]).

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NOTATION. Let R be a commutative ring. We denote by R^\times the unit group of R . We denote by $M_{mn}(R)$ the set of $m \times n$ -matrices with entries in R . In particular put $M_n(R) = M_{nn}(R)$. Put $\mathrm{GL}_m(R) = \{A \in M_m(R) \mid \det A \in R^\times\}$, where $\det A$ denotes the determinant of a square matrix A . For an $m \times n$ -matrix X and an $m \times m$ -matrix A , we write $A[X] = {}^tXAX$, where tX denotes the transpose of X . Let $\mathrm{Sym}_n(R)$ denote the set of symmetric matrices of degree n with entries in R . Furthermore, if R is an integral domain of characteristic different from 2, let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree n over R , that is, $\mathcal{H}_n(R)$ is the subset of symmetric matrices of degree n with entries in the field of fractions of R whose (i, j) -component belongs to R or $\frac{1}{2}R$ according as $i = j$ or not. We say that an element A of $M_n(R)$ is non-degenerate if $\det A \neq 0$. For a subset S of $M_n(R)$ we denote by S^{nd} the subset of S consisting of non-degenerate matrices. If S is a subset of $\mathrm{Sym}_n(\mathbb{R})$ with \mathbb{R} the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. The group $\mathrm{GL}_n(R)$ acts on the set $\mathrm{Sym}_n(R)$ by

$$\mathrm{GL}_n(R) \times \mathrm{Sym}_n(R) \ni (g, A) \longmapsto A[g] \in \mathrm{Sym}_n(R).$$

Let G be a subgroup of $\mathrm{GL}_n(R)$. For a G -stable subset \mathcal{B} of $\mathrm{Sym}_n(R)$ we denote by \mathcal{B}/G the set of equivalence classes of \mathcal{B} under the action of G . We sometimes use the same symbol \mathcal{B}/G to denote a complete set of representatives of \mathcal{B}/G . We abbreviate $\mathcal{B}/\mathrm{GL}_n(R)$ as \mathcal{B}/\sim if there is no fear of confusion. Let R' be a subring of R . Then two symmetric matrices A and A' with entries in R are said to be equivalent over R' with

each other and write $A \sim_{R'} A'$ if there is an element X of $GL_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices X and Y we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

For an integer $D \in \mathbb{Z}$ such that $D \equiv 0$ or $\equiv 1 \pmod{4}$, let \mathfrak{d}_D be the discriminant of $\mathbb{Q}(\sqrt{D})$, and put $\mathfrak{f}_D = \sqrt{\frac{D}{\mathfrak{d}_D}}$. We call an integer D a fundamental discriminant if it is the discriminant of some quadratic extension of \mathbb{Q} or 1. For a fundamental discriminant D , let $(\frac{D}{*})$ be the character corresponding to $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$. Here we make the convention that $(\frac{D}{*}) = 1$ if $D = 1$. For an integer D such that $D \equiv 0$ or $\equiv 1 \pmod{4}$, we define $(\frac{D}{*}) = (\frac{\mathfrak{d}_D}{*})$. We put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$ for $x \in \mathbb{C}$, and for a prime number p we denote by $\mathbf{e}_p(*)$ the continuous additive character of \mathbb{Q}_p such that $\mathbf{e}_p(x) = \mathbf{e}(x)$ for $x \in \mathbb{Z}[p^{-1}]$.

Let K be an algebraic number field, and $\mathfrak{O} = \mathfrak{O}_K$ the ring of integers in K . For a prime ideal \mathfrak{p} we denote by $K_{\mathfrak{p}}$ and $\mathfrak{O}_{\mathfrak{p}}$ the \mathfrak{p} -adic completion of K and \mathfrak{O} , respectively, and put $\mathfrak{O}_{(\mathfrak{p})} = \mathfrak{O}_{\mathfrak{p}} \cap K$. For a prime ideal number \mathfrak{p} of \mathfrak{O} , we denote by $\text{ord}_{\mathfrak{p}}(*)$ the additive valuation of $K_{\mathfrak{p}}$ normalized so that $\text{ord}_{\mathfrak{p}}(\varpi) = 1$ for a prime element ϖ of $K_{\mathfrak{p}}$. Moreover for any element $a, b \in \mathfrak{O}_{\mathfrak{p}}$ we write $b \equiv a \pmod{\mathfrak{p}}$ if $\text{ord}_{\mathfrak{p}}(a - b) > 0$.

2. Siegel modular forms.

We denote by \mathbb{H}_n the Siegel upper half space of degree n , i.e.,

$$\mathbb{H}_n = \{Z \in M_n(\mathbb{C}) \mid Z = {}^t Z = X + \sqrt{-1}Y, X, Y \in M_n(\mathbb{R}), Y > 0\}.$$

For any ring R and any natural integer n , we define the group $\text{GSp}_n(R)$ by

$$\text{GSp}_n(R) = \{g \in M_{2n}(R) \mid gJ_n {}^t g = \nu(g)J_n \text{ with some } \nu(g) \in R^\times\},$$

where $J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. We call $\nu(g)$ the symplectic similitude of g . We also define the symplectic group of degree n over R by

$$\text{Sp}_n(R) = \{g \in \text{GSp}_n(R) \mid \nu(g) = 1\}.$$

In particular, if R is a subfield of \mathbb{R} , we define

$$\text{GSp}_n^+(R) = \{g \in \text{GSp}_n(R) \mid \nu(g) > 0\}.$$

We put $\Gamma^{(n)} = \text{Sp}_n(\mathbb{Z})$ for the sake of simplicity. Now we define vector valued Siegel modular forms of $\Gamma^{(n)}$. Let (ρ, V) be a polynomial representation of $GL_n(\mathbb{C})$ on a finite dimensional complex vector space V . We fix a Hermitian inner product $\langle *, * \rangle$ on V such that

$$\langle \rho(g)v, w \rangle = \langle v, \rho({}^t \bar{g})w \rangle \quad \text{for } g \in GL_n(\mathbb{C}), v, w \in V. \tag{H}$$

For any V -valued function F on \mathbb{H}_n , and for any $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GSp}_n^+(\mathbb{R})$, we put $J(g, Z) = CZ + D$ and

$$F|_{\rho}[g] = \rho(J(g, Z))^{-1}F(gZ).$$

For a positive integer N , we define the principal congruence subgroup $\Gamma^{(n)}(N)$ of $\Gamma^{(n)}$ of level N by

$$\Gamma^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \mid A \equiv D \equiv 1_n, B \equiv C \equiv O_n \pmod{N} \right\}.$$

A subgroup Γ of $\Gamma^{(n)}$ is said to be a congruence subgroup if Γ contains $\Gamma^{(n)}(N)$ with some N . By definition, $\Gamma^{(n)}(N)$ is a congruence subgroup. Another example of congruence subgroup is the group $\Gamma_0^{(n)}(N)$ defined by

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \mid C \equiv O_n \pmod{N} \right\}.$$

Let Γ be a congruence subgroup of $\Gamma^{(n)}$. We say that F is a holomorphic Siegel modular form of weight ρ with respect to Γ if F is holomorphic on \mathbb{H} and $F|_{\rho}[\gamma] = F$ for any $\gamma \in \Gamma$ (with the extra condition of holomorphy at all the cusps if $n = 1$). We denote by $M_{\rho}(\Gamma)$ the space of modular forms of weight ρ with respect to Γ , and by $S_{\rho}(\Gamma)$ its subspace consisting of cusp forms.

A modular form $F \in M_{\rho}(\Gamma)$ has the following Fourier expansion

$$F(Z) = \sum_{T \in S_n(\mathbb{Q})_{\geq 0}} a(T, F) \mathbf{e}(\text{tr}(TZ)) \quad \text{with } a(T, F) \in V,$$

where $\text{tr}(T)$ is the trace of a matrix T , and in particular if $\Gamma = \Gamma^{(n)}$, we have

$$F(Z) = \sum_{T \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}} a(T, F) \mathbf{e}(\text{tr}(TZ)),$$

and we have $F \in S_{\rho}(\Gamma^{(n)})$ if and only if we have $a(T, F) = 0$ unless T is positive definite. For $F, G \in M_{\rho}(\Gamma)$ the Petersson inner product is defined by

$$(F, G)_{\Gamma} = \int_{\Gamma \backslash \mathbb{H}_n} \langle \rho(\sqrt{Y})F(Z), \rho(\sqrt{Y})G(Z) \rangle \det(Y)^{-n-1} dZ, \tag{P}$$

where $Y = \text{Im}(Z)$ and \sqrt{Y} is a positive definite symmetric matrix such that $\sqrt{Y}^2 = Y$. This integral converges if either F or G belongs to $S_{\rho}(\Gamma)$. We also define (F, G) as

$$(F, G) = [\Gamma^{(n)} : \Gamma]^{-1} (F, G)_{\Gamma}.$$

Let $\lambda = (k_1, k_2, \dots)$ be a finite or an infinite sequence of non-negative integers such that $k_i \geq k_{i+1}$ for all i and $k_m = 0$ for some m . We call this a dominant integral weight (or the Young diagram). We call the biggest integer m such that $k_m \neq 0$ a depth of λ and write it by $\text{depth}(\lambda)$. It is well known that the set of dominant integral weights λ with $\text{depth}(\lambda) \leq n$ corresponds bijectively to the set of irreducible polynomial representations of the $\text{GL}_n(\mathbb{C})$. We denote this representation by $(\rho_{n,\lambda}, V_{n,\lambda})$. We also denote it by $(\rho_{\mathbf{k}}, V_{\mathbf{k}})$ with $\mathbf{k} = (k_1, \dots, k_n)$ and call it the irreducible polynomial representation of $\text{GL}_n(\mathbb{C})$ of highest weight \mathbf{k} . We then set $M_{\mathbf{k}}(\Gamma) = M_{\rho_{\mathbf{k}}}(\Gamma)$ and $S_{\mathbf{k}}(\Gamma) = S_{\rho_{\mathbf{k}}}(\Gamma)$. We

say F is a modular form of weight \mathbf{k} if it is a modular form of weight $\rho_{\mathbf{k}}$. If $\mathbf{k} = \overbrace{(k, \dots, k)}^n$, we simply write $M_{\mathbf{k}}(\Gamma) = M_k(\Gamma)$ and $S_{\mathbf{k}}(\Gamma) = S_k(\Gamma)$. We note that

$$M_{(k+j,k)}(\Gamma^{(2)}) = M_{\det^k \otimes \text{Sym}^j}(\Gamma^{(2)}) \quad \text{and} \quad S_{(k+j,k)}(\Gamma^{(2)}) = S_{\det^k \otimes \text{Sym}^j}(\Gamma^{(2)}),$$

where Sym^j is the j -th symmetric tensor representation of $\text{GL}_2(\mathbb{C})$. In general, for the $\mathbf{k} = (k_1, \dots, k_n)$ above, we write $\mathbf{k}' = (k_1 - k_n, \dots, k_{n-1} - k_n, 0)$. Then, we have $\rho_{\mathbf{k}} \cong \det^{k_n} \otimes \rho_{\mathbf{k}'}$ with $(\rho_{\mathbf{k}'}, V_{\mathbf{k}'})$ an irreducible polynomial representation of highest weight \mathbf{k}' . Here we understand that $(\rho_{\mathbf{k}'}, V_{\mathbf{k}'})$ is the trivial representation on \mathbb{C} if $k_1 = \dots = k_{n-1} = k_n$. Moreover, we may regard an element $F \in M_{\mathbf{k}}(\Gamma)$ as a $V_{\mathbf{k}'}$ -valued holomorphic function on \mathbb{H} such that

$$F|_{\det^{k_n} \otimes \rho_{\mathbf{k}'}}[\gamma] = F$$

for any $\gamma \in \Gamma$ (with the extra condition of holomorphy at all the cusps if $n = 1$). For a representation (ρ, V) of $\text{GL}_n(\mathbb{C})$, we denote by $\mathfrak{F}(\mathbb{H}_n, V)$ the set of Fourier series $F(Z)$ on \mathbb{H}_n with values in V of the following form:

$$F(Z) = \sum_{A \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}} a(A, F) \mathbf{e}(\text{tr}(AZ)) \quad (Z \in \mathbb{H}_n, a(A, F) \in V).$$

For $F(Z) \in \mathfrak{F}(\mathbb{H}_n, V)$ and a positive integer $r \leq n$ we define $\Phi(F)(Z_1) = \Phi_r^n(F)(Z_1)$ ($Z_1 \in \mathbb{H}_r$) as

$$\Phi(F)(Z_1) = \lim_{\lambda \rightarrow \infty} F \left(\begin{pmatrix} Z_1 & O \\ O & \sqrt{-1} \lambda 1_{n-r} \end{pmatrix} \right).$$

We make the convention that $\mathfrak{F}(\mathbb{H}_0, V) = V$ and $\Phi_0^n(F) = a(O_n, F)$. Then, $\Phi(F)$ belongs to $\mathfrak{F}(\mathbb{H}_r, V)$. For a representation (ρ, V) of $\text{GL}_n(\mathbb{C})$, we denote by $\tilde{\mathfrak{F}}(\mathbb{H}_n, V) = \tilde{\mathfrak{F}}(\mathbb{H}_n, (\rho, V))$ the subset of $\mathfrak{F}(\mathbb{H}_n, V)$ consisting of elements $F(Z)$ such that the following condition is satisfied:

$$(K0) \quad a(A[g], F) = \rho(g)a(A, F) \text{ for any } g \in \text{GL}_n(\mathbb{C}).$$

Now let $\boldsymbol{\ell} = (l_1, \dots, l_n)$ be a dominant integral weight of length n of depth m . Then we realize the representation space $V_{\boldsymbol{\ell}}$ in terms of bideterminants (cf. [31]). Let $U = (u_{ij})$ be an $m \times n$ matrix of variables. For a positive integer $a \leq m$ let $\mathcal{SI}_{n,a}$ denote the set of strictly increasing sequences of positive integers not greater than n of length a . For each $J = (j_1, \dots, j_a) \in \mathcal{SI}_{n,a}$ we define U_J as

$$\begin{vmatrix} u_{1,j_1} & \dots & u_{1,j_a} \\ \vdots & \ddots & \vdots \\ u_{a,j_1} & \dots & u_{a,j_a} \end{vmatrix}.$$

Then we say that a polynomial $P(U)$ in U is a bideterminant of weight $\boldsymbol{\ell}$ if $P(U)$ is of the following form:

$$P(U) = \prod_{i=1}^m \prod_{j=1}^{l_i - l_{i+1}} U_{J_{ij}},$$

where $(J_{i1}, \dots, J_{i, l_i - l_{i+1}}) \in \mathcal{ST}_{n,i}^{l_i - l_{i+1}}$. Here we understand that $\prod_{j=1}^{l_i - l_{i+1}} U_{J_{ij}} = 1$ if $l_i = l_{i+1}$. Let \mathcal{BD}_ℓ be the set of all bideterminants of weight ℓ . Here we make the convention that $\mathcal{BD}_\ell = \{1\}$ if $\ell = (0, \dots, 0)$. For a commutative ring R and an R -algebra S let $S[U]_\ell$ denote the R -module of all S -linear combinations of $P(U)$ for $P(U) \in \mathcal{BD}_\ell$. Then we can define an action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathbb{C}[U]_\ell$ as

$$\mathrm{GL}_n(\mathbb{C}) \times \mathbb{C}[U]_\ell \ni (g, P(U)) \mapsto P(Ug) \in \mathbb{C}[U]_\ell,$$

and we can take the \mathbb{C} -vector space $\mathbb{C}[U]_\ell$ as a representation space V_ℓ of ρ_ℓ under this action. Let $m \leq n - 1$ be a non-negative integer and $U = (u_{ij})$ be an $m \times n$ matrix of variables. Let $\mathbf{k} = (k_1, \dots, k_n)$ with $k_1 \geq \dots \geq k_m > k_{m+1} = \dots = k_n$ and $\mathbf{k}' = (k_1 - k_{m+1}, \dots, k_m - k_{m+1}, \overbrace{0, \dots, 0}^{n-m})$. Here we make the convention that $\mathbf{k} = (k_1, \dots, k_1)$ and $\mathbf{k}' = (0, \dots, 0)$ if $m = 0$. Then under this notation and convention, $M_{\mathbf{k}}(\Gamma^{(n)})$ can be regarded as a \mathbb{C} -sub-vector space of $\mathrm{Hol}(\mathbb{H}_n)[U]_{\mathbf{k}'}$, where $\mathrm{Hol}(\mathbb{H}_n)$ denotes the ring of holomorphic functions on \mathbb{H}_n . We sometimes write $F(Z)(U)$ for $F(Z) \in M_{\mathbf{k}}(\Gamma^{(n)})$. Moreover, the Fourier expansion of $F(Z) \in M_{\mathbf{k}}(\Gamma^{(n)})$ can be expressed as

$$F(Z) = \sum_{A \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}} a(A, F) \mathbf{e}(\mathrm{tr}(AZ)),$$

where $a(A, F) = a(A, F)(U) \in \mathbb{C}[U]_{\mathbf{k}'}$.

Let r be an integer such that $m \leq r \leq n$ and let $\mathbf{l} = (k_1, \dots, k_{r-1}, k_r)$ and $\mathbf{l}' = (k_1 - k_{m+1}, \dots, k_m - k_{m+1}, \overbrace{0, \dots, 0}^{r-m})$. For the $m \times n$ matrix U , let $U^{(r)} = (u_{ij})_{1 \leq i \leq m, 1 \leq j \leq r}$ and put $W' = \mathbb{C}[U^{(r)}]_{\mathbf{l}'}$. Then we can define a representation (τ', W') of $\mathrm{GL}_r(\mathbb{C})$. The representations $(\rho_{\mathbf{k}'}, V_{\mathbf{k}'})$ and (τ', W') satisfy the following conditions:

- (K1) $W' \subset V_{\mathbf{k}'}$.
- (K2) $\rho_{\mathbf{k}'}\left(\begin{pmatrix} g_1 & g_2 \\ O & g_4 \end{pmatrix}\right)v = \tau'(g_1)v$ for $\begin{pmatrix} g_1 & g_2 \\ O & g_4 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C})$ with $g_1 \in \mathrm{GL}_r(\mathbb{C})$ and $v \in W'$.
- (K3) If $v \in V_{\mathbf{k}'}$ satisfies the condition

$$\rho_{\mathbf{k}'}\left(\begin{pmatrix} 1_r & O \\ O & h \end{pmatrix}\right)v = v \text{ for any } h \in \mathrm{GL}_{n-r}(\mathbb{C}),$$

then v belongs to W' .

Let $F(Z) = \sum_{A \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}} a(A, F) \mathbf{e}(\mathrm{tr}(AZ)) \in \mathfrak{F}(\mathbb{H}_n, V_{\mathbf{k}'})$. Then, in a way similar to [1, (2.3.29)], we have

$$\Phi_r^n(F)(Z_1) = \sum_{A_1 \in \mathcal{H}_r(\mathbb{Z})_{\geq 0}} a\left(\begin{pmatrix} A_1 & O \\ O & O \end{pmatrix}, F\right) \mathbf{e}(\mathrm{tr}(A_1 Z_1)) \quad (Z_1 \in \mathbb{H}_r).$$

Suppose that $F(Z)$ belongs to $\tilde{\mathfrak{F}}(\mathbb{H}_n, V_{\mathbf{k}'})$. Then, by (K0),

$$\rho_{\mathbf{k}'} \left(\begin{pmatrix} 1_r & O \\ O & h \end{pmatrix} \right) \left(a \left(\begin{pmatrix} A_1 & O \\ O & O \end{pmatrix}, F \right) \right) = a \left(\begin{pmatrix} A_1 & O \\ O & O \end{pmatrix}, F \right) \quad \text{for any } h \in \text{GL}_{n-r}(\mathbb{C}).$$

Hence, by (K3), $a(\begin{pmatrix} A_1 & O \\ O & O \end{pmatrix}, F)$ belongs to W' for any $A_1 \in \mathcal{H}_r(\mathbb{Z})_{\geq 0}$. This implies that $\Phi_r^n(F)$ belongs to $\mathfrak{F}(\mathbb{H}_r, W')$. We easily see that $\Phi_r^n(F)$ belongs to $\tilde{\mathfrak{F}}(\mathbb{H}_r, W')$, and therefore Φ_r^n sends $\tilde{\mathfrak{F}}(\mathbb{H}_n, V_{\mathbf{k}'})$ to $\tilde{\mathfrak{F}}(\mathbb{H}_r, W')$. It is easily seen that it induces a mapping from $M_\rho(\Gamma^{(n)})$ to $M_\tau(\Gamma^{(r)})$, where $\rho = \det^{k_n} \otimes \rho_{\mathbf{k}'}$ and $\tau = \det^{k_n} \otimes \tau'$. Let $\Delta_{n,r}$ be the subgroup of $\Gamma^{(n)}$ defined by

$$\Delta_{n,r} := \left\{ \begin{pmatrix} * & * \\ O_{(n-r, n+r)} & * \end{pmatrix} \in \Gamma^{(n)} \right\}.$$

For $F \in S_\tau(\Gamma^{(r)})$ the Klingen–Eisenstein series $[F]_\tau^\rho(Z, s)$ of F associated to ρ is defined by

$$[F]_\tau^\rho(Z, s) := \sum_{\gamma \in \Delta_{n,r} \backslash \Gamma^{(n)}} \left(\frac{\det \text{Im}(Z)}{\det \text{Im}(\text{pr}_r^n(Z))} \right)^s F(\text{pr}_r^n(Z))|_{\rho} \gamma.$$

Here $\text{pr}_r^n(Z) = Z_1$ for $Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_2 & Z_4 \end{pmatrix} \in \mathbb{H}_n$ with $Z_1 \in \mathbb{H}_r, Z_4 \in \mathbb{H}_{n-r}, Z_2 \in M_{r, n-r}(\mathbb{C})$. We also write $[F]_\tau^\rho(Z, s)$ as $[F]_{\mathbf{k}}^{\rho}(Z, s)$ or $[F]^{\mathbf{k}}(Z, s)$.

Suppose that k_n is even and $2\text{Re}(s) + k_n > n + r + 1$. Then, $[F]_\tau^\rho(Z, s)$ converges absolutely and uniformly on \mathbb{H}_n . This is proved by [40] in the scalar valued case, and can be proved similarly in general case. If $[F]^{\mathbf{k}}(Z, s)$ can be continued holomorphically in the neighborhood of 0 as a function of s , we put $[F]_\tau^\rho(Z) = [F]_\tau^\rho(Z, 0)$. If $[F]_\tau^\rho(Z)$ is holomorphic as a function of Z , it belongs to $M_{\mathbf{k}}(\Gamma^{(n)})$, and we say that it is the Klingen–Eisenstein lift of F to $M_{\mathbf{k}}(\Gamma^{(n)})$. In particular, if $k_n > n + r + 1$, then $[F]_\tau^\rho(Z, s)$ is holomorphic at $s = 0$ as a function of s , and $[F]_\tau^\rho(Z, 0)$ belongs to $M_{\mathbf{k}}(\Gamma^{(n)})$, and $\Phi_r^\rho([F]_\tau^\rho) = F$. We note that $[F]_\tau^\rho(Z)$ is not necessarily holomorphic as a function of Z if $k_n \leq n + r + 1$.

We define $E_{n,\mathbf{k}}(Z, s)$ as

$$E_{n,\mathbf{k}}(Z, s) = \sum_{\gamma \in \Delta_{n,0} \backslash \Gamma^{(n)}} (\det \text{Im}(Z))^s |_{\rho} \gamma$$

and call it the Siegel–Eisenstein series of weight \mathbf{k} with respect to $\Gamma^{(n)}$. In particular, if

$\mathbf{k} = \overbrace{(k, \dots, k)}^n$ with k even, we write $E_{n,k}(Z, s)$ for $E_{n,\mathbf{k}}(Z, s)$. If $k > 0$, then $E_{n,k}(Z, s)$ can be continued meromorphically to the whole s -plane as a function of s . Let $\mathbf{k} =$

$\overbrace{(k+l, \dots, k+l)}^m, \overbrace{(k, \dots, k)}^{n-m}$ such that $k, l \geq 0$, and put $\rho = \det^k \otimes \rho_{\mathbf{k}'}$ and $\tau = \det^k \otimes \rho_{\mathbf{l}'}$

with $\mathbf{k}' = \overbrace{(l, \dots, l, 0, \dots, 0)}^m$ and $\mathbf{l}' = \overbrace{(l, \dots, l)}^m$. Then, for $F \in S_\tau(\Gamma^{(m)})$ we can define the Klingen–Eisenstein series $[F]_{\tau}^{\rho_{\mathbf{k}}}(Z, s)$ of F associated to $\rho_{\mathbf{k}}$ if k is even and $2\text{Re}(s) + k > n + m + 1$. We note that $\mathbb{C}[U^{(m)}]_{\mathbf{l}'}$ is a subspace of $\mathbb{C}[U]_{\mathbf{k}'}$ spanned by $(\det U^{(m)})^l$, and

hence we have a natural isomorphism

$$\iota : S_{k+l}(\Gamma^{(m)}) \ni f \mapsto \tilde{f} := f(\det U^{(m)})^l \in S_\tau(\Gamma^{(m)}).$$

We sometimes write $[f]^{\rho_{\mathbf{k}}}$ or $[f]^{\mathbf{k}}$ instead of $[\tilde{f}]_{\tau}^{\rho_{\mathbf{k}}}$ for $f \in S_{k+l}(\Gamma^{(m)})$. We state the holomorphy of the Klingen–Eisenstein series.

PROPOSITION 2.1. *Let k be an even integer.*

(1) *Suppose that $k \geq (n + 1)/2$ and that neither $k = (n + 2)/2 \equiv 2 \pmod{4}$ nor $k = (n + 3)/2 \equiv 2 \pmod{4}$. Then $E_{n,k}(Z)$ belongs to $M_k(\Gamma^{(n)})$.*

(2) *Let $\mathbf{k} = (\overbrace{k+l, \dots, k+l}^m, \overbrace{k, \dots, k}^{n-m})$ such that $l \geq 0$ and $k > 3m/2 + 1$ and let f be a Hecke eigenform in $S_{k+l}(\Gamma^{(m)})$. Then $[f]^{\mathbf{k}}(Z, s)$ can be continued meromorphically to the whole s -plane as a function of s , and holomorphic at $s = 0$. Moreover suppose that $k > (n + m + 3)/2$. Then $[f]^{\mathbf{k}}(Z)$ belongs to $M_{\mathbf{k}}(\Gamma^{(n)})$.*

PROOF. The assertion (1) follows from [55, Theorem 17.7]. The assertion (2) has been proved in the case $l = 0$ (cf. [47], [55]). The case $l > 0$ will be proved in Section 5. □

Let $\ell = (l_1, \dots, l_n)$ be a dominant integral weight of length n of depth m . Let $\tilde{V} = \tilde{V}_\ell = \mathbb{Q}[U]_\ell$. Then, $(\rho_\ell |_{\mathrm{GL}_n(\mathbb{Q})}, \tilde{V})$ is a representation of $\mathrm{GL}_n(\mathbb{Q})$, and $\tilde{V} \otimes \mathbb{C} = V_\ell$. We consider a \mathbb{Z} structure of V_ℓ . To do this, we fix a basis $\mathcal{S} = \mathcal{S}_\ell = \{P\}$ of $\mathbb{Z}[U]_\ell$. We note here that the bideterminants are not linearly independent over \mathbb{Z} and even over \mathbb{C} in general, so the set \mathcal{BD}_ℓ is not necessarily a basis of $\mathbb{Z}[U]_\ell$. Let R be a subring of \mathbb{C} . Since the set \mathcal{S} is also linearly independent over \mathbb{C} , an element a of $R[U]_\ell$ is uniquely written as

$$a = \sum_{P \in \mathcal{S}} a_P P \text{ with } a_P \in R.$$

Let K be a number field, and \mathfrak{O} the ring of integers in K . For a prime ideal \mathfrak{p} of \mathfrak{O} and $a = a(U) = \sum_{P \in \mathcal{S}} a_P P \in K[U]_\ell$ with $a_P \in K$, define

$$\mathrm{ord}_{\mathfrak{p}}(a) = \min_{P \in \mathcal{S}} \mathrm{ord}_{\mathfrak{p}}(a_P).$$

We say that \mathfrak{p} divides a if $\mathrm{ord}_{\mathfrak{p}}(a) > 0$.

REMARK 2.2. (1) The definition of $\mathrm{ord}_{\mathfrak{p}}(*)$ does not depend on the choice of a basis of $\mathbb{Z}[U]_\ell$. We note that \mathfrak{p} does not divide $a = a(U)$ if \mathfrak{p} does not divide $a(U_0)$ for some element U_0 of $M_{m,n}(\mathfrak{O})$.

(2) There is no canonical choice of a basis of \tilde{V} . But several standard choices are known. One of them is a basis associated with the semi-standard Young tableaux (cf. [16]). We note that it is also a basis of $\mathbb{Z}[U]_\ell$. This can be proved by a careful analysis of the proof of [16, (4.5a)] combined with [16, (4.6a)].

For a subring R of \mathbb{C} , we denote by $M_{\mathbf{k}}(\Gamma^{(n)})(R)$ the R -submodule of $M_{\mathbf{k}}(\Gamma^{(n)})$ consisting of all modular forms F such that $a(T, F) \in R[U]_{\mathbf{k}'}$ for all $T \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}$. Here,

$\mathbf{k}' = (k_1 - k_{m+1}, \dots, k_m - k_{m+1}, \overbrace{0, \dots, 0}^{n-m})$ for $\mathbf{k} = (k_1, \dots, k_n)$ with $k_1 \geq \dots \geq k_m > k_{m+1} = \dots = k_n$ as stated before.

We consider tensor products of modular forms, which will be used on and after Section 5. Let n_1 and n_2 be positive integers. Let $\mathbf{k}_1 = (k_1, \dots, k_m, k_{m+1}, \dots, k_{n_1})$ and $\mathbf{k}_2 = (k_1, \dots, k_m, k_{m+1}, \dots, k_{n_2})$ be non-increasing sequences of integers such that $k_m > k_{m+1} = \dots = k_{n_i} = l$ for $i = 1, 2$. Then $(\rho_{\mathbf{k}_1} \otimes \rho_{\mathbf{k}_2}, V_1 \otimes V_2)$ is a representation of

$\text{GL}_{n_1}(\mathbb{C}) \times \text{GL}_{n_2}(\mathbb{C})$. Put $\mathbf{k}'_1 = (k_1 - l, \dots, k_m - l, \overbrace{0, \dots, 0}^{n_1-m})$ and $\mathbf{k}'_2 = (k_1 - l, \dots, k_m - l, \overbrace{0, \dots, 0}^{n_2-m})$.

Then, $\rho_{\mathbf{k}_1} \otimes \rho_{\mathbf{k}_2} = (\det^l \otimes \rho_{\mathbf{k}'_1}) \otimes (\det^l \otimes \rho_{\mathbf{k}'_2})$ with $(\rho_{\mathbf{k}'_i}, V'_i)$ a polynomial representation of highest weight \mathbf{k}'_i for $i = 1, 2$. To make our formulation smooth, we sometimes regard a modular form of scalar weight k for $\Gamma^{(n)}$ as a function with values in the one-dimensional vector space spanned by $\det U^l$ with a non-negative integer $l \leq k$, where U is an $n \times n$ matrix of variables. Let U_1 and U_2 be $m \times n_1$ and $m \times n_2$ matrices, respectively, of variables and for a commutative ring R and an R -algebra S let

$$S[U_1, U_2]_{\mathbf{k}'_1, \mathbf{k}'_2} = \left\{ \sum_j P_j(U_1)P_j(U_2) \quad (\text{finite sum}) \text{ with } P_j(U_i) \in S[U_i]_{\mathbf{k}'_i} \ (i = 1, 2) \right\}.$$

Here we make the convention that $P_j(U_i) \in \langle (\det U_i)^{k_1-l} \rangle_{\mathbb{C}}$ if $n_i = m$ and $k_1 = \dots = k_m$ as stated above. Then, as a representation space $W = W_{\mathbf{k}'_1, \mathbf{k}'_2}$ of $\rho_{\mathbf{k}'_1} \otimes \rho_{\mathbf{k}'_2}$ we can take $\mathbb{C}[U_1, U_2]_{\mathbf{k}'_1, \mathbf{k}'_2}$. Let

$$\widetilde{W} = \widetilde{W}_{\mathbf{k}'_1, \mathbf{k}'_2} = \mathbb{Q}[U_1, U_2]_{\mathbf{k}'_1, \mathbf{k}'_2}.$$

Then $\widetilde{W} \cong \widetilde{V}'_1 \otimes \widetilde{V}'_2$ and $\widetilde{W} \otimes_{\mathbb{Q}} \mathbb{C} = W$. Let

$$M = M_{\mathbf{k}'_1, \mathbf{k}'_2} = \mathbb{Z}[U_1, U_2]_{\mathbf{k}'_1, \mathbf{k}'_2}.$$

We note that

$$M = \left\{ \sum_{P_{\tau_1} \in \mathcal{S}_{\mathbf{k}'_1}, P_{\tau_2} \in \mathcal{S}_{\mathbf{k}'_2}} a_{\tau_1, \tau_2} P_{\tau_1}(U_1)P_{\tau_2}(U_2) \mid a_{\tau_1, \tau_2} \in \mathbb{Z} \right\}.$$

Here we make the convention that $P_{\tau_i}(U_i) = (\det U_i)^{k_1-l}$ if $n_i = m$ and $k_1 = \dots = k_m$. Therefore, M is a lattice of \widetilde{W} and $M \cong L_1 \otimes L_2$ with $L_i = \mathbb{Z}[U_i]_{\mathbf{k}'_i}$ ($i = 1, 2$). Thus $(\rho_{\mathbf{k}_1} \otimes \rho_{\mathbf{k}_2}, V_1 \otimes V_2)$ has also a \mathbb{Q} -structure and \mathbb{Z} -structure and we can define $\text{ord}_{\mathfrak{p}}(a \otimes b)$ for $a \otimes b \in \widetilde{W}_K$. If $\dim_{\mathbb{C}} V_1 = 1$, then we identify V_1, \widetilde{V}'_1 and L_1 with \mathbb{C}, \mathbb{Q} and \mathbb{Z} , respectively, and for $a, b \in V_1$ and $w \in V_2$, we write $a \otimes b$ and $a \otimes w$ as ab and aw , respectively through the identifications $V_1 \otimes V_1 \cong V_1$ and $V_1 \otimes V_2 \cong V_2 \otimes V_1 \cong V_2$. The tensor product $M_{\mathbf{k}_1}(\Gamma^{(n_1)}) \otimes M_{\mathbf{k}_2}(\Gamma^{(n_2)})$ is regarded as a \mathbb{C} -subspace of $(\text{Hol}(\mathbb{H}_{n_1}) \otimes \text{Hol}(\mathbb{H}_{n_2}))[U_1, U_2]_{\mathbf{k}'_1, \mathbf{k}'_2}$.

3. Harder's conjecture.

In this section we review several arithmetical properties of Hecke eigenvalues and L values of modular forms, then state the original Harder's conjecture in [17]. In the later section, we will treat a generalized version of the conjecture. From now on, until the end of Proposition 3.3, let $\mathbf{k} = (k_1, \dots, k_n)$ with $k_1 \geq \dots \geq k_n \geq 0$. Let $\mathbf{L}_n = \mathbf{L}(\Gamma^{(n)}, \mathrm{GSp}_n^+(\mathbb{Q}) \cap M_{2n}(\mathbb{Z}))$ be the Hecke algebra over \mathbb{Z} associated to the Hecke pair $(\Gamma^{(n)}, \mathrm{GSp}_n^+(\mathbb{Q}) \cap M_{2n}(\mathbb{Z}))$ and for a subring R of \mathbb{C} put $\mathbf{L}_n(R) = \mathbf{L}_n \otimes_{\mathbb{Z}} R$. For an element $T = \Gamma^{(n)}g\Gamma^{(n)} \in \mathbf{L}_n(\mathbb{C})$, let

$$T = \bigsqcup_{i=1}^r \Gamma^{(n)}g_i$$

be the coset decomposition. Then, for a modular form $F \in M_{\mathbf{k}}(\Gamma^{(n)})$ we define $F|T$ as

$$F|T = \nu(g)^{k_1 + \dots + k_n - n(n+1)/2} \sum_{i=1}^r F|_{\rho_{\mathbf{k}}} g_i.$$

This defines an action of the Hecke algebra $\mathbf{L}_n(\mathbb{C})$ on $M_{\mathbf{k}}$. The operator $F \mapsto F|T$ with $T \in \mathbf{L}_n(\mathbb{C})$ is called the Hecke operator. We say that F is a Hecke eigenform if F is a common eigenfunction of all Hecke operators $T \in \mathbf{L}_n(\mathbb{C})$. Then we have

$$F|T = \lambda_F(T)F \text{ with } \lambda_F(T) \in \mathbb{C} \text{ for any } T \in \mathbf{L}_n(\mathbb{C}).$$

We call $\lambda_F(T)$ the Hecke eigenvalue of T with respect to F . For a Hecke eigenform F in $M_{\mathbf{k}}(\Gamma^{(n)})$, we denote by $\mathbb{Q}(F)$ the field generated over \mathbb{Q} by all the Hecke eigenvalues $\lambda_F(T)$ with $T \in \mathbf{L}_n(\mathbb{Q})$ and call it the Hecke field of F . For two Hecke eigenforms F and G we sometimes write $\mathbb{Q}(F, G) = \mathbb{Q}(F)\mathbb{Q}(G)$. We say that an element $T \in \mathbf{L}_n(\mathbb{Q})$ is integral with respect to $M_{\mathbf{k}}(\Gamma^{(n)})$ if $F|T \in M_{\mathbf{k}}(\Gamma^{(n)})(\mathbb{Z})$ for any $F \in M_{\mathbf{k}}(\Gamma^{(n)})(\mathbb{Z})$. We denote by $\mathbf{L}_n^{(\mathbf{k})}$ the subset of $\mathbf{L}_n(\mathbb{Q})$ consisting of all integral elements with respect to $M_{\mathbf{k}}(\Gamma^{(n)})$. The following proposition can be proved in the same manner as Proposition 4.2 of [36].

PROPOSITION 3.1. *We have $\mathbf{L}_n \subset \mathbf{L}_n^{(\mathbf{k})}$ for any $\mathbf{k} = (k_1, \dots, k_n)$ with $k_n \geq n + 1$.*

For a non-zero rational number a , we define an element $[a] = [a]_n$ of \mathbf{L}_n by $[a]_n = \Gamma^{(n)}(a1_n)\Gamma^{(n)}$. For each integer m define an element $T(m)$ of \mathbf{L}_n by

$$T(m) = \sum_{d_1, \dots, d_n, e_1, \dots, e_n} \Gamma^{(n)}(d_1 \perp \dots \perp d_n \perp e_1 \perp \dots \perp e_n)\Gamma^{(n)},$$

where $d_1, \dots, d_n, e_1, \dots, e_n$ run over all positive integer satisfying

$$d_i | d_{i+1}, e_{i+1} | e_i \ (i = 1, \dots, n - 1), d_n | e_n, d_i e_i = m \ (i = 1, \dots, n).$$

Furthermore, for $i = 1, \dots, n$ and a prime number p put

$$T_i(p^2) = \Gamma^{(n)}(1_{n-i} \perp p1_i \perp p^2 1_{n-i} \perp p1_i)\Gamma^{(n)}.$$

As is well known, $\mathbf{L}_n(\mathbb{Q})$ is generated over \mathbb{Q} by $T(p), T_i(p^2)$ ($i = 1, \dots, n$), and $[p^{-1}]_n$ for all p . We note that $T_n(p^2) = [p]_n$. We note that \mathbf{L}_n is generated over \mathbb{Z} by $T(p)$ and $T_i(p^2)$ ($i = 1, \dots, n$) for all p . Let F be a Hecke eigenform in $M_{\mathbf{k}}(\Gamma^{(n)})$. As is well known, $\mathbb{Q}(F)$ is a totally real algebraic number field of finite degree. Now, first we consider the integrality of the eigenvalues of Hecke operators. For an algebraic number field K , let \mathfrak{O}_K denote the ring of integers in K . The following assertion can be proved in the same manner as in [50]. (See also [36].)

PROPOSITION 3.2. *Let $\mathbf{k} = (k_1, \dots, k_n)$ be as above. Suppose that $k_n \geq n + 1$. Let F be a Hecke eigenform in $S_{\mathbf{k}}(\Gamma^{(n)})$. Then $\lambda_F(T)$ belongs to $\mathfrak{O}_{\mathbb{Q}(F)}$ for any $T \in \mathbf{L}_n^{(\mathbf{k})}$.*

Let $\mathbf{L}_{n,p} = \mathbf{L}(\Gamma^{(n)}, \mathrm{GSp}_n^+(\mathbb{Q}) \cap \mathrm{GL}_{2n}(\mathbb{Z}[p^{-1}]))$ be the Hecke algebra associated with the pair $(\Gamma^{(n)}, \mathrm{GSp}_n^+(\mathbb{Q}) \cap \mathrm{GL}_{2n}(\mathbb{Z}[p^{-1}]))$. $\mathbf{L}_{n,p}$ can be considered as a subalgebra of \mathbf{L}_n , and is generated over \mathbb{Q} by $T(p)$ and $T_i(p^2)$ ($i = 1, 2, \dots, n$), and $[p^{-1}]_n$. We now review the Satake p -parameters of $\mathbf{L}_{n,p}$; let $\mathbf{P}_n = \mathbb{Q}[X_0^\pm, X_1^\pm, \dots, X_n^\pm]$ be the ring of Laurent polynomials in X_0, X_1, \dots, X_n over \mathbb{Q} . Let \mathbf{W}_n be the group of \mathbb{Q} -automorphisms of \mathbf{P}_n generated by all permutations in variables X_1, \dots, X_n and by the automorphisms τ_1, \dots, τ_n defined by

$$\tau_i(X_0) = X_0 X_i, \tau_i(X_i) = X_i^{-1}, \tau_i(X_j) = X_j \quad (j \neq i).$$

Moreover, a group $\widetilde{\mathbf{W}}_n$ isomorphic to \mathbf{W}_n acts on the set $T_n = (\mathbb{C}^\times)^{n+1}$ in a way similar to the above. Then there exists a \mathbb{Q} -algebra isomorphism $\Phi_{n,p}$, called the Satake isomorphism, from $\mathbf{L}_{n,p}$ to the \mathbf{W}_n -invariant subring $\mathbf{P}_n^{\mathbf{W}_n}$ of \mathbf{P}_n . Then for a \mathbb{Q} -algebra homomorphism λ from $\mathbf{L}_{n,p}$ to \mathbb{C} , there exists an element $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \dots, \alpha_n(p, \lambda))$ of \mathbf{T}_n satisfying

$$\lambda(\Phi_{n,p}^{-1}(F(X_0, X_1, \dots, X_n))) = F(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \dots, \alpha_n(p, \lambda))$$

for $F \in \mathbf{P}_n^{\mathbf{W}_n}$. The equivalence class of $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \dots, \alpha_n(p, \lambda))$ under the action of $\widetilde{\mathbf{W}}_n$ is uniquely determined by λ . We call this the Satake parameters of $\mathbf{L}_{n,p}$ determined by λ . Now let F be a Hecke eigenform in $M_{\mathbf{k}}(\Gamma^{(n)})$. Then for each prime number p , F defines a \mathbb{Q} -algebra homomorphism $\lambda_{F,p}$ from $\mathbf{L}_{n,p}$ to \mathbb{C} in a usual way, and we denote by $\alpha_0(p), \alpha_1(p), \dots, \alpha_n(p)$ the Satake parameters of $\mathbf{L}_{n,p}$ determined by F . For later purpose, we consider special elements in $\mathbf{L}_{n,p}$; the polynomials $r_n(X_1, \dots, X_n) = \sum_{i=1}^n (X_i + X_i^{-1})$ and $\rho_n(X_0, X_1, \dots, X_n) = X_0^2 X_1 X_2 \cdots X_n r_n(X_1, \dots, X_n)$ are elements of $\mathbf{P}_n^{\mathbf{W}_n}$, and thus we can define elements $\Phi_{n,p}^{-1}(r_n(X_1, \dots, X_n))$ and $\Phi_{n,p}^{-1}(\rho_n(X_0, X_1, \dots, X_n))$ of $\mathbf{L}_{n,p}$, which are denoted by $\mathbf{r}_{n,1}(p)$ and $\tilde{\rho}_{n,1}$, respectively.

PROPOSITION 3.3. (1) *We have*

$$\tilde{\rho}_{n,1}(p) = p^{n(n+1)/2} [p]_n \cdot \mathbf{r}_{n,1}(p),$$

and in particular

$$\lambda_F(\tilde{\rho}_{n,1}(p)) = p^{\sum_{i=1}^n k_i - n(n+1)/2} \sum_{i=1}^n (\alpha_i(p) + \alpha_i(p)^{-1}).$$

(2) Let $\mathbf{k} = (k_1, \dots, k_n)$ be as above. Suppose that $k_n \geq (n + 1)/2$. Then $\tilde{\mathbf{r}}_{n,1}(p) := p^{k_1-1} \mathbf{r}_{n,1}(p)$ belongs to $\mathbf{L}_n^{(\mathbf{k})}$.

PROOF. The assertion (1) can easily be checked remarking that $\Phi_{n,p}(p^{n(n+1)/2}[p]_n) = X_0^2 X_1 \cdots X_n$ (cf. [1, Lemma 3.3.34]). We will prove the assertion (2). Put

$$\mathrm{GSp}_n^+(\mathbb{Q})_\infty = \left\{ \begin{pmatrix} A & B \\ O & D \end{pmatrix} \in \mathrm{GSp}_n^+(\mathbb{Q}) \right\},$$

$\Gamma_\infty^{(n)} = \Gamma^{(n)} \cap \mathrm{GSp}_n^+(\mathbb{Q})_\infty$, and let $\mathbf{L}_{n,\infty} = \mathbf{L}(\Gamma_\infty^{(n)}, \mathrm{GSp}_n^+(\mathbb{Q})_\infty \cap M_{2n}(\mathbb{Z}))$ be the Hecke algebra associated to the Hecke pair $(\Gamma_\infty^{(n)}, \mathrm{GSp}_n^+(\mathbb{Q})_\infty \cap M_{2n}(\mathbb{Z}))$. Then there is a natural injection from \mathbf{L}_n into $\mathbf{L}_{n,\infty}$. For an element $D \in M_n(\mathbb{Z})^{\mathrm{nd}}$ put $D^* = {}^t D^{-1}$ and for non-negative integers a, b such that $a + b \leq n$, put $D_{a,b} = D_{a,b}(p) = 1_{n-a-b} \perp p 1_a \perp p^2 1_b$, $\tilde{U}(D_{a,b}) = (p^2 D_{a,b}^* \perp D_{a,b})$ and $\Pi_{a,b} = \Gamma_\infty^{(n)} \tilde{U}(D_{a,b}) \Gamma_\infty^{(n)}$. For an element $\begin{pmatrix} A & B \\ O & D \end{pmatrix} \in \Gamma_\infty^{(n)}$ with $D \in M_n(\mathbb{Z})^{\mathrm{nd}}$ and $L \in \mathrm{GL}_n(\mathbb{Z}) D \mathrm{GL}_n(\mathbb{Z})$, we define the set $B(L, M)$ as

$$B(L, M) = \left\{ N \in M_n(\mathbb{Z}) \mid \begin{pmatrix} \nu L^* & N \\ O & L \end{pmatrix} \in \Gamma_\infty^{(n)} M \Gamma_\infty^{(n)} \right\},$$

where $\nu = \nu(M)$. Then $\#(B(L, M)/L)$ does not depend on the choice of L , and is uniquely determined by M , which will be denoted by $\alpha(M)$, and in particular put $\beta(D_{a,b}) = \alpha(p^2 D_{a,b}^* \perp D_{a,b})$. Then, by [1, p.160], as an element of $\mathbf{L}_{n,\infty}$, $\tilde{\rho}_{n,1}(p)$ is expressed as

$$\tilde{\rho}_{n,1}(p) = p^{(n-1)n/2} (\beta(D_{n-1,0})^{-1} \Pi_{n-1,0} + p^{n+1} \beta(D_{n-1,1})^{-1} \Pi_{n-1,1}).$$

Let $\mathbf{k}' = (k_1 - k_n, \dots, k_{n-1} - k_n, 0)$ and put $m = \mathrm{depth}(\mathbf{k}')$. Let $F(Z)$ be an element of $M_{\mathbf{k}'}(\Gamma^{(n)})(\mathbb{Z})$. Then we have

$$F(Z) = \sum_{T \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}} a(T, F)(U) \mathbf{e}(\mathrm{tr}(TZ))$$

with $a(T, F)(U) \in \mathbb{Z}[U]_{\mathbf{k}'}$. Then, we have

$$\begin{aligned} F|\tilde{\rho}_{n,1}(p)(Z) &= p^{2(k_1+\dots+k_n)-n(n+1)+(n-1)n/2} \\ &\times \sum_T \left\{ \beta(D_{n-1,0})^{-1} (\det D_{n-1,0})^{-k_n} \right. \\ &\times \sum_{L \in \Lambda_n \setminus \Lambda_n D_{n-1,0} \Lambda_n} a(T, F)(UL^{-1}) \sum_{N \in B(L, \tilde{U}(D_{n-1,0}))/L} \mathbf{e}(\mathrm{tr}(T(p^2 {}^t L^{-1} Z + N)L^{-1})) \\ &+ \beta(D_{n-1,1})^{-1} (\det D_{n-1,1})^{-k_n} \\ &\times \sum_{L \in \Lambda_n \setminus \Lambda_n D_{n-1,1} \Lambda_n} a(T, F)(UL^{-1}) \sum_{N \in B(L, \tilde{U}(D_{n-1,1}))/L} \mathbf{e}(\mathrm{tr}(T(p^2 {}^t L^{-1} Z + N)L^{-1})) \left. \right\}, \end{aligned}$$

where $\Lambda_n = \mathrm{GL}_n(\mathbb{Z})$. For $i = 0, 1$ we have

$$\begin{aligned} & \sum_{N \in B(L, \tilde{U}(D_{n-1,i}))/L} \mathbf{e}\left(\mathrm{tr}(T(p^2 {}^tL^{-1}Z + N)L^{-1})\right) \\ &= \mathbf{e}\left(\mathrm{tr}(p^2 T[L^{-1}]Z)\right) \sum_{N \in B(L, \tilde{U}(D_{n-1,i}))/L} \mathbf{e}\left(\mathrm{tr}(L^{-1}TN)\right). \end{aligned}$$

We have

$$\sum_{N \in B(L, \tilde{U}(D_{n-1,i}))/L} \mathbf{e}\left(\mathrm{tr}(L^{-1}TN)\right) = \begin{cases} \beta(D_{n-1,i}) & \text{if } L^{-1}T \in M_n(\mathbb{Z}), \\ 0 & \text{otherwise.} \end{cases}$$

We note that $L^{-1}T \in M_n(\mathbb{Z})$ if and only if $p^2 T[L^{-1}] \in \mathcal{H}_n(\mathbb{Z})$. Hence we have

$$\begin{aligned} F|\tilde{\rho}_{n,1}(p)(Z) &= p^{2(k_1+\dots+k_n)-n(n+1)+(n-1)n/2} \\ &\times \sum_A \mathbf{e}\left(\mathrm{tr}(AZ)\right) \left\{ p^{-k_n(n-1)} \sum_{L \in \Lambda_n \setminus \Lambda_n D_{n-1,0} \Lambda_n} a(A[L], F)(UL^{-1}) \right. \\ &\quad \left. + p^{-(n+1)k_n+n+1} \sum_{L \in \Lambda_n \setminus \Lambda_n D_{n-1,1} \Lambda_n} a(A[L], F)(UL^{-1}) \right\}, \end{aligned}$$

and therefore

$$\begin{aligned} F|\mathbf{r}_{n,1}(p)(Z) &= p^{k_1+\dots+k_n-n} \sum_A \mathbf{e}\left(\mathrm{tr}(AZ)\right) \left\{ p^{-k_n(n-1)} \sum_{L \in \Lambda_n \setminus \Lambda_n D_{n-1,0} \Lambda_n} a(A[L], F)(UL^{-1}) \right. \\ &\quad \left. + p^{-(n+1)k_n+n+1} \sum_{L \in \Lambda_n \setminus \Lambda_n D_{n-1,1} \Lambda_n} a(A[L], F)(UL^{-1}) \right\}. \end{aligned}$$

We note that $a(T, F)(U)$ is expressed as a \mathbb{Z} -linear combination of polynomials of the following form:

$$P(U) = \prod_{i=1}^m \prod_{j=1}^{l_i - l_{i+1}} U_{J_{ij}}$$

with $l_i = k_i - k_n$, where $(J_{i1}, \dots, J_{i, l_i - l_{i+1}}) \in \mathcal{ST}_{n,i}^{l_i - l_{i+1}}$. Therefore, to prove the assertion (2), it suffices to show that

$$p^{2k_1+k_2+\dots+k_n-n-1} p^{-k_n(n-1+2i)+(n+1)i} P(UL^{-1}) \in \mathbb{Z}[U]_{\mathbf{k}'} \tag{I}$$

for any $L \in \Lambda_n D_{n-1,i} \Lambda_n$ with $i = 0, 1$. We may suppose $L = D_{n-1,i}$ with $i = 0, 1$. First write $D_{n-1,1} = p^{d_1} \perp \dots \perp p^{d_n}$ with $d_1 = \dots = d_{n-1} = 1$ and $d_n = 2$. Then we have

$$P(UD_{n-1,1}^{-1}) = p^{-\sum_{i=1}^m \sum_{j=1}^{l_i - l_{i+1}} \sum_{a=1}^i d_{J_{a+j}}} P(U),$$

where $\{J_{a+j}\}_{1 \leq i \leq m, 1 \leq j \leq l_i - l_{i+1}, 1 \leq a \leq i}$ is a set of integers such that $1 \leq J_{a+j} \leq n$ and

$J_{a+j} \neq J_{a'+j'}$ if $a + j \neq a' + j'$. Then we have $\sum_{a=1}^i d_{J_{a+j}} \leq i + 1$ for any i . Hence we have

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^{l_i-l_{i+1}} \sum_{a=1}^i d_{J_{a+j}} &\leq \sum_{i=1}^m (l_i - l_{i+1})(i + 1) = 2l_1 + \sum_{i=2}^m l_i \\ &= 2k_1 + \sum_{i=2}^m k_i - (m + 1)k_n = 2k_1 + \sum_{i=2}^n k_i - (n + 1)k_n. \end{aligned}$$

Hence (I) holds for any $L \in \Lambda_n D_{n-1,1} \Lambda_n$. Similarly, we have

$$P(UD_{n-1,0}^{-1}) = p^{-\gamma_{k,n}} P(U)$$

with $\gamma_{k,n}$ an integer such that

$$\gamma_{k,n} \leq \sum_{i=1}^{n-1} k_i - (n - 1)k_n.$$

By assumption, we have $k_1 + k_n \geq n + 1$, and hence (I) holds for any $L \in \Lambda_n D_{n-1,0} \Lambda_n$. This proves the assertion (2). □

We write $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$ and write $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$ as usual. Let

$$f(z) = \sum_{m=1}^{\infty} a(m, f) \mathbf{e}(mz)$$

be a primitive form in $S_k(\mathrm{SL}_2(\mathbb{Z}))$, that is let f be a Hecke eigenform whose first coefficient is 1. For a prime number p let $\beta_{1,p}(f)$ and $\beta_{2,p}(f)$ be complex numbers such that $\beta_{1,p}(f) + \beta_{2,p}(f) = a(p, f)$ and $\beta_{1,p}(f)\beta_{2,p}(f) = p^{k-1}$. Then for a Dirichlet character χ we define Hecke's L function $L(s, f)$ twisted by χ as

$$L(s, f, \chi) = \prod_p ((1 - \beta_{1,p}(f)\chi(p)p^{-s})(1 - \beta_{2,p}(f)\chi(p)p^{-s}))^{-1}.$$

We write $L(s, f, \chi) = L(s, f)$ if χ is the principal character.

Let $\{f_1, \dots, f_d\}$ be a basis of $S_k(\Gamma^1)$ consisting of primitive forms. Let K be an algebraic number field containing $\mathbb{Q}(f_1) \cdots \mathbb{Q}(f_d)$, and \mathfrak{D} the ring of integers in K . Let f be a primitive form in $S_k(\mathrm{SL}_2(\mathbb{Z}))$. Then Shimura [54] showed that there exist two complex numbers $c_{\pm}(f)$, uniquely determined up to $\mathbb{Q}(f)^{\times}$ multiple such that the following property holds:

(AL) The value $\frac{\Gamma_{\mathbb{C}}(l)\sqrt{-1}^l L(l, f, \chi)}{\tau(\chi)c_s(f)}$ belongs to $\mathbb{Q}(f)(\chi)$ for any positive integer $l \leq k - 1$ and a Dirichlet character χ , where $\tau(\chi)$ is the Gauss sum of χ , and $s = s(l, \chi) = +$ or $-$ according as $\chi(-1) = (-1)^l$ or $(-1)^{l-1}$.

We note that the above value belongs to $K(\chi)$.

For short, we write

$$\mathbf{L}(l, f, \chi; c_s(f)) = \frac{\Gamma_{\mathbb{C}}(l)\sqrt{-1}^l L(l, f, \chi)}{\tau(\chi)c_s(f)}.$$

We sometimes write $c_{s(l,\chi)}(f) = c_{s(l)}(f)$ and $\mathbf{L}(l, f, \chi; c_{s(l,\chi)}(f)) = \mathbf{L}(l, f; c_{s(l)}(f))$ if χ is the principal character. We note that the value $\mathbf{L}(l, f, \chi; c_s(f))$ depends on the choice of $c_s(f)$, but if $(\chi\eta)(-1) = (-1)^{l+m}$, then $s := s(l, \chi) = s(m, \eta)$ and, the ratio $\frac{\mathbf{L}(l, f, \chi; c_s(f))}{\mathbf{L}(m, f, \eta; c_s(f))}$ does not depend on $c_s(f)$, which will be denoted by $\frac{\mathbf{L}(l, f, \chi)}{\mathbf{L}(m, f, \eta)}$. For two positive integers $l_1, l_2 \leq k - 1$ and Dirichlet characters χ_1, χ_2 such that $\chi_1(-1)\chi_2(-1) = (-1)^{l_1+l_2+1}$, the value

$$\frac{\Gamma_{\mathbb{C}}(l_1)\Gamma_{\mathbb{C}}(l_2)L(l_1, f, \chi_1)L(l_2, f, \chi_2)}{\sqrt{-1}^{l_1+l_2+1}\tau((\chi_1\chi_2)_0)(f, f)}$$

belongs to $\mathbb{Q}(f)(\chi_1, \chi_2)$, where $(\chi_1\chi_2)_0$ is the primitive character associated with $\chi_1\chi_2$ (cf. [53]). We denote this value by

$$\mathbf{L}(l_1, l_2; f; \chi_1, \chi_2).$$

In particular, we put

$$\mathbf{L}(l_1, l_2; f) = \mathbf{L}(l_1, l_2; f; \chi_1, \chi_2)$$

if χ_1 and χ_2 are the principal characters. This value does not depend upon the choice of $c_{\pm}(f)$. Let f be a primitive form in $S_k(\mathrm{SL}_2(\mathbb{Z}))$. Let f_1, \dots, f_d be a basis of $S_k(\mathrm{SL}_2(\mathbb{Z}))$ consisting of primitive forms with $f_1 = f$ and let \mathfrak{D}_f be the ideal of $\mathbb{Q}(f)$ generated by all $\prod_{i=2}^d (\lambda_{f_i}(T(m)) - \lambda_f(T(m)))$'s ($m \in \mathbb{Z}_{>0}$). For a prime ideal \mathfrak{p} of an algebraic number field, let $p_{\mathfrak{p}}$ be the prime number such that $(p_{\mathfrak{p}}) = \mathbb{Z} \cap \mathfrak{p}$. The following proposition is due to [38, Theorem 5.4].

PROPOSITION 3.4. *Let f be a primitive form in $S_k(\mathrm{SL}_2(\mathbb{Z}))$. Let χ_1 and χ_2 be primitive characters with conductors N_1 and N_2 , respectively, and let l_1, l_2 be positive integers such that $k - l_1 + 1 \leq l_2 \leq l_1 - 1 \leq k - 2$. Let \mathfrak{p} be a prime ideal of $\mathbb{Q}(f)(\chi_1, \chi_2)$ with $p_{\mathfrak{p}} > k$. Suppose that \mathfrak{p} divides neither $\mathfrak{D}_f N_1 N_2$ nor $\zeta(1-k)$. Then $\mathbf{L}(l_1, l_2; f; \chi_1, \chi_2)$ is \mathfrak{p} -integral.*

For two primitive forms $f_1 \in S_{k_1}(\mathrm{SL}_2(\mathbb{Z}))$ and $f_2 \in S_{k_2}(\mathrm{SL}_2(\mathbb{Z}))$ we define the Rankin–Selberg L function $L(s, f_1 \otimes f_2)$ as

$$L(s, f_1 \otimes f_2) = \prod_p \left(\prod_{i=1}^2 \prod_{j=1}^2 (1 - \beta_{i,p}(f_1)\beta_{j,p}(f_2)p^{-s}) \right)^{-1}.$$

Let F be a Hecke eigenform in $M_{\mathbf{k}}(\Gamma(n))$, and for a prime number p we take the p -Satake parameters $\alpha_0(p), \alpha_1(p), \dots, \alpha_n(p)$ of F so that

$$\alpha_0(p)^2 \alpha_1(p) \cdots \alpha_n(p) = p^{k_1 + \cdots + k_n - n(n+1)/2}.$$

We define the polynomial $L_p(X, F, \mathrm{Sp})$ by

$$L_p(X, F, \mathrm{Sp}) = (1 - \alpha_0(p)X) \prod_{r=1}^n \prod_{1 \leq i_1 < \cdots < i_r \leq n} (1 - \alpha_0(p)\alpha_{i_1}(p) \cdots \alpha_{i_r}(p)X)$$

and the spinor L function $L(s, F, \text{Sp})$ by

$$L(s, F, \text{Sp}) = \prod_p L_p(p^{-s}, F, \text{Sp})^{-1}.$$

We note that $L(s, f, \text{Sp})$ is Hecke's L function $L(s, f)$ if f is a primitive form. In this case we write $L_p(s, f)$ for $L_p(s, f, \text{Sp})$. We also define the polynomial $L_p(X, F, \text{St})$ by

$$(1 - X) \prod_{i=1}^n (1 - \alpha_i(p)X)(1 - \alpha_i(p)^{-1}X)$$

and the standard L function $L(s, F, \text{St})$ by

$$L(s, F, \text{St}) = \prod_p L_p(p^{-s}, F, \text{St})^{-1}.$$

For a Hecke eigenform $F \in S_k(\Gamma^{(r)})$ put

$$\mathbf{L}(s, F, \text{St}) = \Gamma_{\mathbb{C}}(s) \prod_{i=1}^r \Gamma_{\mathbb{C}}(s + k - i) \frac{L(s, F, \text{St})}{(F, F)}.$$

REMARK 3.5. We note that for a positive integer $m \leq k - r$

$$\mathbf{L}(m, F, \text{St}) = A_{r,k,m} \frac{L(m, F, \text{St})}{\pi^{r(k+m)+m-r(r+1)/2} (F, F)}$$

with an element $A_{r,k,m} \in \mathbb{Z}[2^{-1}]$ such that $\text{ord}_p(A_{r,k,m}) = 0$ for any prime number $p \geq 2k - r - 1$.

PROPOSITION 3.6. *Let F be a Hecke eigenform in $S_k(\Gamma^{(r)})$. We define $n_0 = 3$ if $r \geq 5$ with $r \equiv 1 \pmod{4}$ and $n_0 = 1$ otherwise. Let m be a positive integer $n_0 \leq m \leq k - r$ such that $m \equiv r \pmod{2}$. Then, $a(A, F)\overline{a(B, F)}\mathbf{L}(m, F, \text{St})$ belongs to $\mathbb{Q}(F)$ for any $A, B \in \mathcal{H}_r(\mathbb{Z})_{>0}$.*

PROOF. We note that the value $a(A, F)\overline{a(B, F)}\mathbf{L}(m, F, \text{St})$ remains unchanged if we replace F by γF with any $\gamma \in \mathbb{C}^\times$. By the multiplicity one theorem for Hecke eigenforms (cf. Theorem A.2 (3) and Remark A.2 (2)), we can take some non-zero complex number γ such that $\gamma F \in S_k(\Gamma^{(r)})(\mathbb{Q}(F))$. For this γ , we see $\mathbf{L}(m, \gamma F, \text{St}) \in \mathbb{Q}(F)$ by [50, Appendix A]. This proves the assertion. \square

Let R be a commutative ring, and \mathfrak{a} an ideal of R . For two polynomials $P(X) = \sum_{i=1}^n a_i X^i$ and $Q(X) = \sum_{i=1}^n b_i X^i$, we write

$$P(X) \equiv Q(X) \pmod{\mathfrak{a}}$$

if $a_i \equiv b_i \pmod{\mathfrak{a}}$ for any $1 \leq i \leq m$. Now we will state Harder's conjecture.

CONJECTURE 3.7 ([17]). *Let k and j be non-negative integers such that j is even and $k \geq 3$. Let $f = \sum a(n, f)e(nz) \in S_{2k+j-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a primitive form, and suppose that a “large” prime \mathfrak{p} of $\mathbb{Q}(f)$ divides $\mathbf{L}(k + j, f; c_s(k+j))$. Then, there exists a Hecke eigenform $F \in S_{(k+j,k)}(\Gamma^{(2)})$, and a prime ideal $\mathfrak{p}' \mid \mathfrak{p}$ in (any field containing) $\mathbb{Q}(f)\mathbb{Q}(F)$ such that, for all primes p*

$$L_p(X, F, \mathrm{Sp}) \equiv L_p(X, f)(1 - p^{k-2}X)(1 - p^{j+k-1}X) \pmod{\mathfrak{p}'}$$

In particular,

$$\lambda_F(T(p)) \equiv p^{k-2} + p^{j+k-1} + a(p, f) \pmod{\mathfrak{p}'}$$

REMARK 3.8. (1) The original version of Harder’s conjecture did not mention what “largeness” of \mathfrak{p} means.

- (2) To formulate Harder’s conjecture we must choose the periods $c_s(f)$ in an appropriate way. The original version of Harder’s conjecture did not specify them. After that, Harder suggested assuming another type of divisibility condition instead of the divisibility of $L(k + j, f; c_s(f))$ in his conjecture (cf. [18]). However, it does not seem so easy to confirm such a condition.
- (3) The original version of Harder’s conjecture, which states only the last congruence on $\lambda_F(T(p))$, is naturally included in the above Euler factor version since we have

$$\begin{aligned} L_p(X, F, \mathrm{Sp}) &= 1 - \lambda_F(T(p))X + \lambda_F(\tilde{\rho}_{2,1}(p))X^2 \\ &\quad - \lambda_F(T(p))p^{2k+j-3}X^3 + p^{4k+2j-6}X^4, \end{aligned}$$

and

$$L_p(X, f) = 1 - a(p, f)X + p^{2k+j-3}X^2.$$

- (4) The above congruence is trivial in the case k is even and $j = 0$. Indeed, for the Saito–Kurokawa lift F of f , we have

$$L_p(X, F, \mathrm{Sp}) = L_p(X, f)(1 - p^{k-2}X)(1 - p^{k-1}X),$$

so we have equality, not only congruence.

To avoid the ambiguity in (1) and (2) of Remark 3.8, we propose the following conjecture, which we also call Harder’s conjecture.

CONJECTURE 3.9. *Let k and j be non-negative integers such that j is even and $k \geq 3, j \geq 4$. Let f be as that in Conjecture 3.7. Suppose that a prime ideal \mathfrak{p} of $\mathbb{Q}(f)$ satisfies $p_{\mathfrak{p}} > 2k + j - 2$ and that \mathfrak{p} divides $\frac{\mathbf{L}(k+j,f)}{\mathbf{L}(k_j,f)}$, where $k_j = k + j/2$ or $k + j/2 + 1$ according as $j \equiv 0 \pmod{4}$ or $j \equiv 2 \pmod{4}$. Then the same assertion as Conjecture 3.7 holds.*

REMARK 3.10. There is no ambiguity in the assumptions of Conjecture 3.9. Moreover, since we can compute $\frac{\mathbf{L}(k+j,f)}{\mathbf{L}(k_j,f)}$ rigorously, we can easily check the assumption on \mathfrak{p} .

4. An enhanced version of Harder’s conjecture.

Conjectures 3.7 and 3.9 are not concerning the congruence between the Hecke eigenvalues of two Hecke eigenforms in the same space, and this is one of the reasons that it is not easy to confirm it. To treat the conjecture more accessibly, we reformulate it in the case k is even. (For odd k , see [21], [23].)

To do so, first, we review several results, on the Galois representations attached to automorphic forms, and on liftings. Let R be a locally compact topological ring, and M a free R -module of finite rank. For a profinite group G , let $\rho : G \rightarrow \text{Aut}_R(M)$ be a continuous representation of G . When we fix a basis of M with $\text{rank}_R M = n$, we write $\rho : G \rightarrow \text{GL}_n(R)$. The following result is due to Deligne [13] in the case $n = 1$, and due to Weissauer [59] in the case $n = 2$.

THEOREM 4.1. Let F be a Hecke eigenform in $S_{\mathbf{k}_n}(\Gamma^{(n)})$ with $n \leq 2$, where $\mathbf{k}_n = k$ or $(k + j, k)$ according as $n = 1$ or $n = 2$. Let K be a number field containing $\mathbb{Q}(F)$ and \mathfrak{p} be a prime ideal of K . Then there exists a semi-simple Galois representation $\rho_F = \rho_{F,\mathfrak{p}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_{2n}(K_{\mathfrak{p}})$ such that $\rho_{F,\mathfrak{p}}$ is unramified at any prime number $p \neq p_{\mathfrak{p}}$ and

$$\det(1_{2n} - \rho_{F,\mathfrak{p}}(\text{Frob}_p^{-1})X) = L_p(X, F, \text{Sp}),$$

where Frob_p is the arithmetic Frobenius at p .

THEOREM 4.2. (1) Let $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ with $k_1 \geq \dots \geq k_n > n$, and G be a Hecke eigenform in $S_{\mathbf{k}}(\Gamma^{(n)})$. Let $k \geq 4$ and $j > 0$ and $d > 0$. Assume that

- (a) $k \equiv n \pmod{2}$, $j \equiv 0 \pmod{2}$;
- (b) $k > 2d + 1$ and $j > 2d - 1$;
- (c) $\frac{j}{2} + d < k_i - i < \frac{j}{2} + k - d - 1$ for $i = 1, \dots, n$.

Define $\mathbf{k}' = (k'_1, \dots, k'_{n+4d}) \in \mathbb{Z}^{n+4d}$ so that $k'_1 \geq \dots \geq k'_{n+4d}$ and

$$\begin{aligned} \{k'_1 - 1, k'_2 - 2, \dots, k'_{n+4d} - n - 4d\} &= \{k_1 - 1, k_2 - 2, \dots, k_n - n\} \\ &\cup \left\{ \frac{j}{2} + k + d - 2, \frac{j}{2} + k + d - 3, \dots, \frac{j}{2} + k - d - 1 \right\} \\ &\cup \left\{ \frac{j}{2} + d, \frac{j}{2} + d - 1, \dots, \frac{j}{2} - d + 1 \right\}. \end{aligned}$$

Then, for any Hecke eigenform $F \in S_{(k+j,k)}(\Gamma^{(2)})$ there exists a Hecke eigenform $\mathcal{A}_{n,d,\mathbf{k}}^{(I),\mathbf{k}'}(F, G) \in S_{\mathbf{k}'}(\Gamma^{(n+4d)})$ such that

$$L(s, \mathcal{A}_{n,d,\mathbf{k}}^{(I),\mathbf{k}'}(F, G), \text{St}) = L(s, G, \text{St}) \prod_{i=1}^{2d} L\left(s + d + \frac{j}{2} + k - 1 - i, F, \text{Sp}\right).$$

Here we make the convention that $L(s, G, \text{St}) = \zeta(s)$ if $n = 0$.

- (2) Let k and n be positive even integers such that $k > n > 2$. Let f be a primitive form in $S_{2k-n}(\text{SL}_2(\mathbb{Z}))$ and G be a Hecke eigenform in $S_{(k, k-n+2)}(\Gamma^{(2)})$. Then, there exists a Hecke eigenform $\mathcal{A}_n^{(II)}(f, G) \in S_k(\Gamma^{(n)})$ such that

$$L(s, \mathcal{A}_n^{(II)}(f, G), \text{St}) = L(s, G, \text{St}) \prod_{i=1}^{n-2} L(s + k - 1 - i, f).$$

THEOREM 4.3. Let G be a Hecke eigenform in $S_{\mathbf{k}}(\Gamma^{(n)})$ for a fixed $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ with $k_1 \geq \dots \geq k_n > n$. For positive integers k and d with $k > d$, we assume one of the following conditions:

- (1) $k - d > k_1 - 1$ and $k \equiv d \pmod{2}$.
- (2) $k + d - 1 < k_n - n$, $k > d$ and $k \equiv d + n \pmod{2}$.

Define $\mathbf{k}' = (k'_1, \dots, k'_{n+2d}) \in \mathbb{Z}^{n+2d}$ so that $k'_1 \geq \dots \geq k'_{n+2d}$ and

$$\begin{aligned} & \{k'_1 - 1, k'_2 - 2, \dots, k'_{n+2d} - (n + 2d)\} \\ & = \{k_1 - 1, k_2 - 2, \dots, k_n - n\} \cup \{k + d - 1, k + d - 2, \dots, k - d\}. \end{aligned}$$

Then, for any Hecke eigenform $f \in S_{2k}(\text{SL}_2(\mathbb{Z}))$, there exists a Hecke eigenform $\mathcal{M}_{n,d,\mathbf{k}}^{\mathbf{k}'}(f, G) \in S_{\mathbf{k}'}(\Gamma^{(n+2d)})$ such that

$$L(s, \mathcal{M}_{n,d,\mathbf{k}}^{\mathbf{k}'}(f, G), \text{St}) = L(s, G, \text{St}) \prod_{i=1}^{2d} L(s + k + d - i, f).$$

Here we make the convention that $L(s, G, \text{St}) = \zeta(s)$ if $n = 0$.

Theorem 4.2 (1) for the case $n = 0$, and Theorem 4.2 (2) have been proved in [14, Proposition 5.3] and [14, Proposition 5.2], respectively. (These results were proved under a certain assumption. But such an assumption was proved by Arancibia, Mœglin and Renard [2](cf. Remark A.2), and they are now unconditional results.) A general case of Theorem 4.2 (1) and Theorem 4.3 may be proved similarly. But, for readers' convenience we will give their proofs in Appendix A. Theorem 4.2 was conjectured by Ibukiyama [22] in special cases with numerical examples. We say that the lifts in (1) and (2) are the lifts of types $\mathcal{A}^{(I)}$ and $\mathcal{A}^{(II)}$, respectively. If $n = 0$ and

$$\mathbf{k}' = \left(\overbrace{\left(\frac{j}{2} + k + d - 1, \dots, \frac{j}{2} + k + d - 1 \right)}^{2d}, \overbrace{\left(\frac{j}{2} + 3d + 1, \dots, \frac{j}{2} + 3d + 1 \right)}^{2d} \right),$$

we simply write $\mathcal{A}_{dd}^{(I)}(F)$ instead of $\mathcal{A}_{0,d,\mathbf{k}}^{(I),\mathbf{k}'}(F, G)$ because \mathbf{k} and \mathbf{k}' are determined by F and d . Theorem 4.3 was conjectured by Miyawaki [49] with numerical examples. We also note that $\mathcal{M}_{n,d,\mathbf{k}}^{\mathbf{k}'}(f, G)$ was constructed by Ikeda [33] in the case (2) under the assumption $k_1 = \dots = k_n$ and the non-vanishing condition. In particular in the case $n = 0$, it was

constructed by Ikeda [32], and we write it as $\mathcal{J}_{2d}(f)$. The following proposition is more or less well known.

PROPOSITION 4.4. *Let $\mathbf{k} = (k_1, \dots, k_m, \dots, k_n)$ and $\mathbf{l} = (k_1, \dots, k_m)$ with $k_1 \geq \dots \geq k_m \geq \dots \geq k_n$. Let F be a Hecke eigenform in $S_1(\Gamma^{(m)})$, and suppose that $[F]^{\mathbf{k}} = [F]^{\mathbf{k}}(Z, 0)$ belongs to $M_{\mathbf{k}}(\Gamma^{(n)})$. Then, $[F]^{\mathbf{k}}$ is a Hecke eigenform and*

$$L(s, [F]^{\mathbf{k}}, \text{St}) = L(s, F, \text{St}) \prod_{i=m+1}^n (\zeta(s + k_i - i)\zeta(s - k_i + i)).$$

PROOF. The assertion is well known in the case $k_1 = \dots = k_m = \dots = k_n$ (cf. [1, Exercise 4.3.24]), and a general case can also be proved by the same argument. \square

Let F and G be Hecke eigenforms in $M_{\mathbf{k}}(\Gamma^{(n)})$ and \mathfrak{p} a prime ideal of $\mathbb{Q}(F)$. We say that F is Hecke congruent to G modulo \mathfrak{p} if there is a prime ideal \mathfrak{p}' of $\mathbb{Q}(F) \cdot \mathbb{Q}(G)$ lying above \mathfrak{p} such that

$$\lambda_G(T) \equiv \lambda_F(T) \pmod{\mathfrak{p}} \text{ for any } T \in \mathbf{L}_n^{(\mathbf{k})}.$$

We denote this property by

$$G \equiv_{\text{ev}} F \pmod{\mathfrak{p}}.$$

CONJECTURE 4.5. *Let k, j and n be positive integers. Suppose that*

- (a) $n \equiv k \equiv j \equiv 0 \pmod{2}$ and $j/2 + n/2 \equiv 1 \pmod{2}$.
- (b) $k > n + 1$ and $j > n - 1$.

Put

$$\mathbf{k} = \left(\overbrace{\left(\frac{j}{2} + k + \frac{n}{2} - 1, \dots, \frac{j}{2} + k + \frac{n}{2} - 1 \right)}^n, \overbrace{\left(\frac{j}{2} + \frac{3n}{2} + 1, \dots, \frac{j}{2} + \frac{3n}{2} + 1 \right)}^n \right).$$

Let $f(z) = \sum a(l, f)\mathbf{e}(lz) \in S_{2k+j-2}(\text{SL}_2(\mathbb{Z}))$ be a primitive form. Let \mathfrak{p} be a prime ideal of $\mathbb{Q}(f)$ such that $\mathfrak{p}_{\mathfrak{p}} > 2k + j - 2$ and suppose that \mathfrak{p} divides $\frac{\mathbf{L}(k+j, f)}{\mathbf{L}(j/2+k+n/2-1, f)}$. Then, there exists a Hecke eigenform $F \in S_{(k+j, k)}(\Gamma^{(2)})$ such that

$$\mathcal{A}_{2n}^{(I)}(F) \equiv_{\text{ev}} [\mathcal{J}_n(f)]^{\mathbf{k}} \pmod{\mathfrak{p}}.$$

REMARK 4.6. Since we have $j + 3n/2 + 1 > 3n/2 + 1$, $[\mathcal{J}_n(f)]^{\mathbf{k}}$ belongs to $M_{\mathbf{k}}(\Gamma^{(2n)})$ by Proposition 2.1 (2).

Let \mathfrak{O} be the ring of integers in an algebraic number field K , and \mathfrak{P} a maximal ideal of \mathfrak{O} . Let $\mathcal{A}_{\mathfrak{P}}$ be the Grothendieck group of finite dimensional Galois representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with coefficients $\mathfrak{O}_{\mathfrak{P}}/\mathfrak{P}$ unramified outside \mathfrak{P} . Let \mathcal{S} be the set of isomorphism classes of irreducible representations in $\mathcal{A}_{\mathfrak{P}}$. Write an element H of $\mathcal{A}_{\mathfrak{P}}$ as

$$H = \sum_{S \in \mathfrak{S}} n_S S \text{ with } n_S \in \mathbb{Z}$$

and set

$$\|H\| = \sum_{S \in \mathfrak{S}} |n_S| \dim S.$$

LEMMA 4.7. *Let \mathfrak{P} be as above. Suppose that $p_{\mathfrak{P}}$ is odd. Let H be an element of $\mathcal{A}_{\mathfrak{P}}$. Suppose that $(\bar{\chi}^i + 1)H = 0$ with $i = 1, 2$, where $\bar{\chi}$ is the mod \mathfrak{P} representation of the cyclotomic character $\chi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_1(K_{\mathfrak{P}})$. Then $\|H\|$ is divisible by $(p_{\mathfrak{P}} - 1)/i$.*

PROOF. The assertion for $i = 1$ has been proved in Proposition 10.4.6 of Chenevier and Lannes [12], and the other assertion can also be proved by using the same argument as there. □

THEOREM 4.8. *Let the notation be as in Conjecture 4.5.*

- (1) *Conjecture 3.9 holds for the case $j \equiv 0 \pmod{4}$ if Conjecture 4.5 holds for $n = 2$.*
- (2) *Suppose that $2k + j - 2 \geq 20$. Then Conjecture 3.9 holds for the case $j \equiv 2 \pmod{4}$ if Conjecture 4.5 holds for $n = 4$.*

PROOF. Let f be a primitive form in Conjecture 3.9, and suppose that a prime ideal \mathfrak{p} of $\mathbb{Q}(f)$ satisfies the assumptions in Conjecture 3.9. Then, by Conjecture 4.5, there exists a Hecke eigenform F in $S_{(k+j,k)}(\Gamma^{(2)})$ satisfying the conditions in Conjecture 4.5. Let $K = \mathbb{Q}(f) \cdot \mathbb{Q}(F)$ and \mathfrak{O} the ring of integers in K . Take a prime ideal \mathfrak{P} of \mathfrak{O} lying above \mathfrak{p} . Then it suffices to show that

$$L_p(X, F, \text{Sp}) \equiv L_p(X, f)(1 - p^{k-2}X)(1 - p^{j+k-1}X) \pmod{\mathfrak{P}} \tag{C_p}$$

for any prime number p . By Proposition 4.4, we have

$$L(s, [\mathcal{J}_n(f)]^{\mathbf{k}}, \text{St}) = L(s, \mathcal{J}_n(f), \text{St}) \prod_{i=1}^n \zeta\left(s + \frac{j}{2} + \frac{n}{2} + 1 - i\right) \zeta\left(s - \frac{j}{2} - \frac{n}{2} - 1 + i\right)$$

and

$$L(s, \mathcal{J}_n(f), \text{St}) = \zeta(s) \prod_{i=1}^n L\left(s + \frac{j}{2} + k + \frac{n}{2} - 1 - i, f\right).$$

Hence

$$\begin{aligned} &L(s, [\mathcal{J}_n(f)]^{\mathbf{k}}, \text{St}) \\ &= \zeta(s) \prod_{i=1}^n L\left(s + \frac{j}{2} + k + \frac{n}{2} - 1 - i, f\right) \prod_{i=1}^n \zeta\left(s + \frac{j}{2} + \frac{n}{2} + 1 - i\right) \zeta\left(s - \frac{j}{2} - \frac{n}{2} - 1 + i\right) \\ &= \zeta(s) \prod_{i=1}^n \left(L\left(s + \frac{j}{2} + k + \frac{n}{2} - 1 - i, f\right) \zeta\left(s + \frac{j}{2} + \frac{n}{2} + 1 - i\right) \zeta\left(s - \frac{j}{2} + \frac{n}{2} - i\right) \right). \end{aligned}$$

Then, by (1) of Theorem 4.2, we have

$$L(s, \mathcal{A}_{2n}^{(I)}(F), \text{St}) = \zeta(s) \prod_{i=1}^n L(s + j/2 + k + n/2 - 1 - i, F, \text{Sp}).$$

By [1, (3.3.52), (3.3.53), Theorem 3.3.30, Lemma 3.3.34], for any prime number p , the i -th coefficient of $L_p(X, [J_n(f)]^{\mathbf{k}}, \text{St})$ and $L_p(X, \mathcal{A}_{2n}^{(I)}(F), \text{St})$ are of the form $p^{n_i} \lambda_{[J_n(f)]^{\mathbf{k}}}(T_i)$ and $p^{n_i} \lambda_{\mathcal{A}_{2n}^{(I)}(F)}(T_i)$ with $n_i \in \mathbb{Z}_{\leq 0}$ and $T_i \in \mathbf{L}_n^{\mathbf{k}}$, respectively. Therefore, for any prime number $p \neq p_{\mathfrak{P}}$, they belong to $\mathfrak{D}_{\mathfrak{P}}$, and by the assumption, we have

$$L_p(X, \mathcal{A}_{2n}^{(I)}(F), \text{St}) \equiv L_p(X, [J_n(f)]^{\mathbf{k}}, \text{St}) \pmod{\mathfrak{P}}.$$

Hence we have

$$\prod_{i=1}^n L_p(p^{i-1}X, F, \text{Sp}) \equiv \prod_{i=1}^n L_p(p^{i-1}X, f)(1 - p^{i-1}p^{k-2}X)(1 - p^{i-1}p^{j+k-1}X) \pmod{\mathfrak{P}} \tag{D_p}$$

for any prime number $p \neq p_{\mathfrak{P}}$. Let $\rho_F : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_4(K_{\mathfrak{P}})$ and $\rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(K_{\mathfrak{P}})$ be the Galois representation attached to the spin L functions of F and f , respectively. For $\rho = \rho_F, \rho_f$ let $\bar{\rho}$ be the mod \mathfrak{P} representation of ρ . Then, by (D_p), in the Grothendieck ring $\mathcal{A}_{\mathfrak{P}}$,

$$(1 + \bar{\chi}^{-1})\bar{\rho}_F = (1 + \bar{\chi}^{-1})(\bar{\rho}_f + \bar{\chi}^{2-k} + \bar{\chi}^{-j-k+1})$$

or

$$(1 + \bar{\chi}^{-1})(1 + \bar{\chi}^{-2})\bar{\rho}_F = (1 + \bar{\chi}^{-1})(1 + \bar{\chi}^{-2})(\bar{\rho}_f + \bar{\chi}^{2-k} + \bar{\chi}^{-j-k+1})$$

according as $n = 2$ or 4 . Define an element H of $\mathcal{A}_{\mathfrak{P}}$ as

$$H = \bar{\rho}_F - (\bar{\rho}_f + \bar{\chi}^{2-k} + \bar{\chi}^{-j-k+1}).$$

Then we have $\|H\| \leq 8$. Let $n = 2$. Then, we have

$$(1 + \bar{\chi}^{-1})H = 0.$$

Since we have $2k + j - 2 \equiv 2 \pmod{4}$, by assumption we have $p_{\mathfrak{P}} = p_{\mathfrak{p}} \geq 2k + j - 2 \geq 18$. Hence, by Lemma 4.7, we have $H = 0$. Let $n = 4$. Then,

$$(1 + \bar{\chi}^{-1})(1 + \bar{\chi}^{-2})H = 0.$$

Since we have $p_{\mathfrak{P}} = p_{\mathfrak{p}} > 2k + j - 2 \geq 20$, using Lemma 4.7 repeatedly, we also have $H = 0$. Hence, in $\mathcal{A}_{\mathfrak{P}}$ we have

$$\bar{\rho}_F = \bar{\rho}_f + \bar{\chi}^{2-k} + \bar{\chi}^{-j-k+1}.$$

This implies that the congruence relation (C_p) holds for any prime number $p \neq p_{\mathfrak{P}}$.

Let $p = p_{\mathfrak{p}}$. Then we have

$$\lambda_{\mathcal{A}_{2n}^{(I)}(F)}(\mathbf{r}_{2n,1}(p)) = p^{-j/2-k-n/2+1} \lambda_F(T(p)) \sum_{i=1}^n p^i,$$

and

$$\lambda_{[\mathcal{J}_n(f)]^{\mathbf{k}}}(\mathbf{r}_{2n,1}(p)) = p^{-j/2-k-n/2+1} a(p, f) \sum_{i=1}^n p^i + p^{-j/2-n/2-1} \sum_{i=1}^n p^i + p^{j+n/2+1} \sum_{i=1}^n p^{-i}.$$

Since the Hecke operator $\tilde{\mathbf{r}}_{2n,1}(p) = p^{k+j/2+n/2-2} \mathbf{r}_{2n,1}(p)$ belongs to $\mathbf{L}_{2n}^{\mathbf{k}}$ by Proposition 3.3, we have

$$\lambda_F(T(p)) \equiv \lambda_{\mathcal{A}_{2n}^{(I)}(F)}(\tilde{\mathbf{r}}_{2n,1}(p)) \equiv \lambda_{[\mathcal{J}_n(f)]^{\mathbf{k}}}(\tilde{\mathbf{r}}_{2n,1}(p)) \equiv a(p, f) \pmod{\mathfrak{P}}.$$

Moreover, all the coefficients of X^m with $m \geq 2$ of the both polynomials $L_p(X, F, \text{Sp})$ and $L(X, f)(1 - p^{k-2}X)(1 - p^{j+k-1}X)$ are congruent to 0 modulo \mathfrak{P} . This proves the assertion. \square

REMARK 4.9. (1) Our conjecture is stronger than Conjecture 3.7 in the case k is even.

(2) The above conjecture tells nothing about the case k is odd. However, we can propose a similar conjecture in the case k is odd.

5. Pullback formula.

5.1. Differential operators with automorphic property.

In this section, we explain some explicit differential operators that are used in the pullback formula.

5.1.1. Setting. Now for an integer $n \geq 2$, fix a partition (n_1, n_2) with $n = n_1 + n_2$ with $n_i \geq 1$. Let λ be a dominant integral weight with $\text{depth}(\lambda) \leq \min(n_1, n_2)$. For $i = 1, 2$, let $(\rho_{n_i, \lambda}, V_{n_i, \lambda})$ be the representation of $\text{GL}_{n_i}(\mathbb{C})$ defined in Section 1. Put $V_{\lambda, n_1, n_2} = V_{n_1, \lambda} \otimes V_{n_2, \lambda}$. We regard $\mathbb{H}_{n_1} \times \mathbb{H}_{n_2}$ as a subset of \mathbb{H}_n by the diagonal embedding. We consider V_{λ, n_1, n_2} valued differential operators \mathbb{D} on scalar valued functions of \mathbb{H}_n , satisfying Condition 1 below on automorphy: We fix λ, n_1, n_2 as above. For variables $Z_i \in \mathbb{H}_{n_i}$, irreducible representations (ρ_i, V_i) of $\text{GL}_{n_i}(\mathbb{C})$ for $i = 1, 2$, a $V_1 \otimes V_2$ valued function $f(Z_1, Z_2)$ on $\mathbb{H}_{n_1} \times \mathbb{H}_{n_2}$, and $g_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \text{Sp}_{n_i}(\mathbb{R})$, we write

$$(f|_{\rho_1, \rho_2}[g_1, g_2])(Z_1, Z_2) = \rho_1(C_1 Z_1 + D_1)^{-1} \otimes \rho_2(C_2 Z_2 + D_2)^{-1} f(g_1 Z_1, g_2 Z_2).$$

We regard $\text{Sp}_{n_1}(\mathbb{R}) \times \text{Sp}_{n_2}(\mathbb{R})$ as a subgroup of $\text{Sp}_n(\mathbb{R})$ by

$$\iota(g_1, g_2) = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix} \quad (g_i \in \text{Sp}_{n_i}(\mathbb{R}) \text{ for } i = 1, 2).$$

For $Z = (z_{ij}) \in \mathbb{H}_n$, we denote by ∂_Z the following $n \times n$ symmetric matrix of partial derivations

$$\partial_Z = \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial z_{ij}} \right)_{1 \leq i, j \leq n}.$$

For a V_{λ, n_1, n_2} valued polynomial $P(T)$ in components of $n \times n$ symmetric matrix T , we put $\mathbb{D}_P = P(\partial_Z)$.

CONDITION 1. We fix k and λ . Let $\mathbb{D} = P(\partial_Z)$ as above. For any holomorphic function F on \mathbb{H}_n and any $(g_1, g_2) \in \text{Sp}_{n_1}(\mathbb{R}) \times \text{Sp}_{n_2}(\mathbb{R})$, the operator \mathbb{D} satisfies

$$\text{Res}(\mathbb{D}(F|_k[\iota(g_1, g_2)])) = (\text{Res } \mathbb{D}(F))|_{\det^k \rho_{n_1, \lambda}, \det^k \rho_{n_2, \lambda}}[g_1, g_2],$$

where Res means the restriction of a function on \mathbb{H}_n to $\mathbb{H}_{n_1} \times \mathbb{H}_{n_2}$.

For $Z = \begin{pmatrix} Z_1 & Z_{12} \\ {}_t Z_{12} & Z_2 \end{pmatrix} \in \mathbb{H}_n$ with $Z_1 \in \mathbb{H}_{n_1}, Z_2 \in \mathbb{H}_{n_2}, Z_{12} \in M_{n_1, n_2}(\mathbb{C})$, we sometimes write $\mathbb{D}(F) \begin{pmatrix} Z_1 & O \\ O & Z_2 \end{pmatrix}$ instead of $\text{Res } \mathbb{D}(F(Z))$. This condition on \mathbb{D} is, roughly speaking, the condition that, if F is a Siegel modular form of degree n of weight k , then $\text{Res}(\mathbb{D}(F))$ is a Siegel modular form of weight $\det^k \rho_{n_i, \lambda}$ for each variable Z_i for $i = 1, 2$. Here, if $2k \geq n$, the condition that ρ_1 and ρ_2 correspond to the same λ is a necessary and sufficient condition for the existence of \mathbb{D} ([20]). A characterization for P is given in [20]. We review it since we need it later. For an $m \times 2k$ matrix $X = (x_{i\nu})$ of variables and for any (i, j) with $1 \leq i, j \leq m$, we put

$$\Delta_{ij}(X) = \sum_{\nu=1}^{2k} \frac{\partial^2}{\partial x_{i\nu} \partial x_{j\nu}}.$$

We say that a polynomial $\tilde{P}(X)$ in $x_{i\nu}$ is pluri-harmonic if

$$\Delta_{ij}(X)\tilde{P}(X) = 0$$

for any i, j with $1 \leq i, j \leq m$.

THEOREM 5.1 ([20]). We assume that $2k \geq n$. Notation and assumptions being as above, the operator $\mathbb{D} = P(\partial_Z)$ satisfies Condition 1 if and only if the V_{λ, n_1, n_2} valued polynomial P satisfies the following two conditions.

(1) For $i = 1, 2$, let X_i be an $n_i \times 2k$ matrix of variables. Then the polynomial

$$\tilde{P}(X_1, X_2) := P \begin{pmatrix} X_1 {}^t X_1 & X_1 {}^t X_2 \\ X_2 {}^t X_1 & X_2 {}^t X_2 \end{pmatrix}$$

is pluri-harmonic for each X_1, X_2 , that is, $\Delta_{ij}(X_1)\tilde{P} = \Delta_{ij}(X_2)\tilde{P} = 0$, regarding that the variables in X_1 and in X_2 are independent.

(2) For any $A_1 \in \text{GL}_{n_1}(\mathbb{C})$ and $A_2 \in \text{GL}_{n_2}(\mathbb{C})$, we put

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

Then we have

$$P(AT^tA) = \rho_{n_1,\lambda}(A_1) \otimes \rho_{n_2,\lambda}(A_2)P(T).$$

Besides, for any fixed k with $2k \geq n = n_1 + n_2$ and λ , the polynomial $P(T)$ satisfying (1) and (2) exists and is unique up to constant.

There are a lot of results concerning explicit description of P , notably in [25], [27]. But still we need more explicit formula for our purpose and we will explain it in the next subsection.

5.1.2. Explicit formula. In this section, we consider some special type of λ . We assume that $\lambda = (l, \dots, l, 0, \dots, 0)$ with depth m . Put $\lambda_0 = (l, \dots, l)$. Then first we explain some general way to construct V_{λ, n_1, n_2} polynomial $P(T)$ satisfying Condition 1 from a scalar valued polynomial $P_0(S)$ satisfying Condition 1 for $\rho_{m, \lambda_0} \otimes \rho_{m, \lambda_0}$. Here T is an $n \times n$ symmetric matrix and S is an $2m \times 2m$ symmetric matrix. Then for the case $m \leq 2$ and any l , we give a completely explicit description of $P(T)$. (The case $m = 1$ has been already given in [20] and a new point is the case $m = 2$.) Here we note that $\rho_{m, \lambda_0} = \det^l$ and $\det^k \rho_{m, \lambda_0} = \det^{k+l}$.

For any positive integers k, l , we consider Condition 1 for $n = 2m, (n_1, n_2) = (m, m)$ and from weight k to weight $\det^{k+l} \otimes \det^{k+l}$. If we denote by $P_{k, k+l}(S)$ a non-zero polynomial satisfying Condition 1 for this case, this is a scalar valued polynomial in components of an $2m \times 2m$ symmetric matrix S . We assume that we know $P_{k, k+l}(S)$, and then we consider how to give more general case starting from this $P_{k, k+l}$.

First we review realization of representations of $GL_{n_1}(\mathbb{C}) \times GL_{n_2}(\mathbb{C})$ by bideterminants. Let U, V be $m \times n_1$ and $m \times n_2$ matrices of independent variables respectively. Let $\lambda = (l, \dots, l, 0, \dots, 0)$ such that $\text{depth}(\lambda) = m$. For a integers n_1 and n_2 such that $n_1, n_2 \geq m$, put $\mathbf{k}_1' = (\overbrace{l, \dots, l}^m, \overbrace{0, \dots, 0}^{n_1-m})$ and $\mathbf{k}_2' = (\overbrace{l, \dots, l}^m, \overbrace{0, \dots, 0}^{n_2-m})$, and let $\mathbb{C}[U, V]_{\mathbf{k}_1', \mathbf{k}_2'}$ be the vector space defined in Section 2. Then, we can take $\mathbb{C}[U, V]_{\mathbf{k}_1', \mathbf{k}_2'}$ as a representation space of $\rho_{n_1, \lambda} \otimes \rho_{n_2, \lambda}$ as explained in Section 2. We denote by \mathbb{U} the following $2m \times n$ matrix, where $n = n_1 + n_2$:

$$\mathbb{U} = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

PROPOSITION 5.2. *Notation being as above, consider $\lambda = (l, \dots, l, 0, \dots, 0)$ such that $\text{depth}(\lambda) = m$. For a partition $n = n_1 + n_2$, we assume that $m \leq \min(n_1, n_2)$. Let T be an $n \times n$ symmetric matrix. Then for $Q(T) = P_{k, k+l}(\mathbb{U}T^t\mathbb{U})$, the differential operator $\mathbb{D}_{\lambda, n_1, n_2} = Q(\partial_Z)$ for $Z \in \mathbb{H}_n$ satisfies Condition 1 for k and $\det^k \rho_{n_1, \lambda}, \det^k \rho_{n_2, \lambda}$.*

PROOF. For $A_1 \in GL_{n_1}(\mathbb{C})$ and $A_2 \in GL_{n_2}(\mathbb{C})$, we put

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

The fact that $Q(T)$ is in the representation space of $\rho_{n_1, \lambda} \otimes \rho_{n_2, \lambda}$ for the action $U \rightarrow UA_1$ and $V \rightarrow VA_2$ is concretely proved by using a structure theorem on the shape of $P_{k, k+l}(S)$

in [29, Proposition 3.1], but we will later give a more abstract proof in the lemma below. So here we prove the rest. We write

$$T = \begin{pmatrix} T_{11} & T_{12} \\ {}^tT_{12} & T_{22} \end{pmatrix}, \tag{1}$$

where T_{11} is an $n_1 \times n_1$, T_{12} is an $n_1 \times n_2$, and T_{22} is an $n_2 \times n_2$ matrix. Then

$$Q(AT^tA) = P_{k,k+l} \begin{pmatrix} UA_1T_{11}^t(UA_1) & UA_1T_{12}^t(VA_2) \\ (VA_2)^tT_{12}^t(UA_1) & VA_2T_{22}^t(VA_2) \end{pmatrix}.$$

So surely the action of A to T gives the action of A on U, V given by UA_1 and VA_2 . This means that

$$Q(AT^tA) = \rho_{n_1,\lambda}(A_1) \otimes \rho_{n_2,\lambda}(A_2)Q(T).$$

Finally we see the pluri-harmonicity. Let X and Y be $n_1 \times 2k$ and $n_2 \times 2k$ matrices, respectively. We put

$$\tilde{Q}(X, Y) = Q \begin{pmatrix} X^tX & X^tY \\ Y^tX & Y^tY \end{pmatrix}$$

and we must show that \tilde{Q} is pluri-harmonic for each X and Y . As before, for $m \times 2k$ matrices X_1 and X_2 , we write

$$\tilde{P}_{k,k+l}(X_1, X_2) = P_{k,k+l} \begin{pmatrix} X_1^tX_1 & X_1^tX_2 \\ X_2^tX_1 & X_2^tX_2 \end{pmatrix}.$$

Then we have $\tilde{Q}(X, Y) = \tilde{P}_{k,k+l}(UX, VY)$. We write $U = (u_{ij})$, $V = (v_{ij})$ and put

$$(\xi_{i\mu})_{1 \leq i \leq m, 1 \leq \mu \leq 2k} = UX, \quad (\eta_{i\nu})_{1 \leq i \leq m, 1 \leq \nu \leq 2k} = VY.$$

Then we have

$$\xi_{i\mu} = \sum_{l=1}^{2k} u_{il}x_{l\mu}.$$

So we have

$$\begin{aligned} \frac{\partial \tilde{Q}(X, Y)}{\partial x_{l\mu}} &= \sum_{i=1}^m u_{il} \frac{\partial \tilde{P}_{k,k+l}}{\partial \xi_{i\mu}}(UX, VY), \\ \frac{\partial^2 \tilde{Q}(X, Y)}{\partial x_{l\mu} \partial x_{t\mu}} &= \sum_{i,j=1}^m u_{il}u_{jt} \frac{\partial^2 \tilde{P}_{k,k+l}}{\partial \xi_{i\mu} \partial \xi_{j\mu}}(UX, VY). \end{aligned}$$

So for any l, t with $1 \leq t, l \leq n_1$, we have

$$\sum_{\mu=1}^{2k} \frac{\partial^2 \tilde{Q}(X, Y)}{\partial x_{l\mu} \partial x_{t\mu}} = \sum_{i,j=1}^m u_{il}u_{jt} \sum_{\mu=1}^{2k} \frac{\partial^2 \tilde{P}_{k,k+l}}{\partial \xi_{i\mu} \partial \xi_{j\mu}}(UX, VY).$$

The last expression is 0 by the pluri-harmonicity of $P_{k,k+l}$. In the same way, we can show that $\tilde{Q}(X, Y)$ is pluri-harmonic also for Y . □

LEMMA 5.3. *Let n_1, m be integers such that $1 \leq m \leq n_1$. Let U be an $m \times n_1$ matrix of variables. Let $Q(U)$ be a (scalar valued) polynomial in the components of U such that $Q(BU) = \det(B)^l Q(U)$ for any $B \in \text{GL}_m(\mathbb{C})$. Then $Q(U)$ is a linear combination of products $\prod_{i=1}^l U_{I_i}$, where $I_i \subset \{1, \dots, n_1\}$ with $|I_i| = m$ and U_{I_i} is the $m \times m$ minor consisting of p_ν -th columns for $p_\nu \in I_i$.*

PROOF. We regard $B = (b_{ij})_{1 \leq i, j \leq m}$ as a matrix of variables and define a matrix of operators by

$$\frac{\partial}{\partial B} = \left(\frac{\partial}{\partial b_{ij}} \right)_{1 \leq i, j \leq m}.$$

We consider

$$\det \left(\frac{\partial}{\partial B} \right) = \sum_{\sigma \in S_m} \text{sgn}(\sigma) \frac{\partial}{\partial b_{1\sigma(1)}} \cdots \frac{\partial}{\partial b_{m\sigma(m)}},$$

where S_m is the permutation group on m letters. By Cayley type identity ([11]), we have

$$\det \left(\frac{\partial}{\partial B} \right) \det(B)^l = (l)_m \det(B)^{l-1},$$

where $(x)_m = x(x+1) \cdots (x+m-1)$ is the ascending Pochhammer symbol, so by the assumption $Q(BU) = \det(B)^l Q(U)$, we have

$$\det \left(\frac{\partial}{\partial B} \right) Q(BU) = (l)_m \det(B)^{l-1} Q(U).$$

Repeating this process, we have

$$\det \left(\frac{\partial}{\partial B} \right)^l Q(BU) = (l)_m (l-1)_m \cdots (1)_m Q(U).$$

On the other hand, writing $U = (u_{i\nu})$ and $BU = (v_{i\nu})$, we have $v_{i\nu} = \sum_{p=1}^m b_{ip} u_{p\nu}$ and

$$\frac{\partial}{\partial b_{i\sigma(i)}} (Q(BU)) = \sum_{j=1}^m \sum_{\nu=1}^{n_1} \frac{\partial v_{j\nu}}{\partial b_{i\sigma(i)}} \frac{\partial Q}{\partial v_{j\nu}} (BU) = \sum_{\nu=1}^{n_1} u_{\sigma(i)\nu} \frac{\partial Q}{\partial v_{i\nu}} (BU).$$

So we have

$$\det \left(\frac{\partial}{\partial B} \right) Q(BU) = \sum_{\nu_1, \dots, \nu_m=1}^{n_1} \left(\sum_{\sigma \in S_m} \text{sgn}(\sigma) u_{\sigma(1)\nu_1} \cdots u_{\sigma(m)\nu_m} \right) \frac{\partial^m Q}{\partial v_{1\nu_1} \cdots \partial v_{m\nu_m}} (BU).$$

Here if we fix ν_1, \dots, ν_m , then we have

$$\sum_{\sigma \in \mathcal{S}_m} \operatorname{sgn}(\sigma) u_{\sigma(1)\nu_1} \cdots u_{\sigma(m)\nu_m} = \begin{vmatrix} u_{1\nu_1} & \cdots & u_{1\nu_m} \\ \vdots & \ddots & \vdots \\ u_{m\nu_1} & \cdots & u_{m\nu_m} \end{vmatrix}.$$

If $\nu_i = \nu_j$ for some $i \neq j$, then of course this is 0 and if the cardinality $|I|$ of $I = \{\nu_1, \dots, \nu_m\}$ is m , then this is U_I up to sign. By taking B to be scalar, we see that $Q(U)$ is a homogeneous polynomial of the total degree ml , so the ml -th derivative of $Q(U)$ is a constant. So we see that

$$\det \left(\frac{\partial}{\partial B} \right)^l Q(BU)$$

is a linear combination of l products of $m \times m$ minors of U . Since this is equal to $(l)_m(l-1)_{m-1} \cdots (1)_m Q(U)$, we see that $Q(U)$ is a linear combination of l products of minors of degree m of U . \square

REMARK 5.4. By Proposition 5.2, the operator $\operatorname{Res} \mathbb{D}_{\lambda, n_1, n_2}$ sends $M_k(\Gamma^{(n_1+n_2)})$ to $M_{\det^k \rho_{n_1, \lambda}}(\Gamma^{(n_1)}) \otimes M_{\det^k \rho_{n_2, \lambda}}(\Gamma^{(n_2)})$. Moreover, if $l > 0$ and $n_1 = m$, by the property of $P_{k, k+l}(\partial Z)$, for $A = \begin{pmatrix} A_1 & R/2 \\ {}^t R/2 & A_2 \end{pmatrix} \in \mathcal{H}_{m+n_2}(\mathbb{Z})_{\geq 0}$ with $A_1 \in \mathcal{H}_m(\mathbb{Z}), A_2 \in \mathcal{H}_{n_2}(\mathbb{Z})$ and $R \in M_{m, n_2}(\mathbb{Z})$, we have $\mathbb{D}_{\lambda, m, n_2} \mathbf{e}(\operatorname{tr}(AZ)) = 0$ unless $A_1 > 0$. Hence we have $(\operatorname{Res} \mathbb{D}_{\lambda, m, n_2})(M_k(\Gamma^{(m+n_2)})) \subset S_{\det^k \rho_{m, \lambda}}(\Gamma^{(m)}) \otimes M_{\det^k \rho_{n_2, \lambda}}(\Gamma^{(n_2)})$.

If we write

$$P_{k, k+l}(S) = P_{k, k+l} \begin{pmatrix} S_{11} & S_{12} \\ {}^t S_{12} & S_{22} \end{pmatrix}$$

for $m \times m$ matrices S_{ij} , then by definition, for $B_i \in \operatorname{GL}_m(\mathbb{C})$, we have

$$P_{k, k+l} \begin{pmatrix} B_1 S_{11} {}^t B_1 & B_1 S_{12} {}^t B_2 \\ B_2 {}^t S_{12} B_1 & B_2 S_{22} {}^t B_2 \end{pmatrix} = \det(B_1 B_2)^l P_{k, k+l}(S).$$

So

$$\begin{aligned} & P_{k, k+l} \begin{pmatrix} (B_1 U) T_{11} {}^t (B_1 U) & (B_1 U) T_{12} {}^t (B_2 V) \\ (B_2 V) {}^t T_{12} {}^t (B_1 U) & (B_2 V) T_{22} {}^t (B_2 V) \end{pmatrix} \\ &= \det(B_1 B_2)^l P_{k, k+l} \begin{pmatrix} U T_{11} {}^t U & U T_{12} {}^t V \\ V {}^t T_{12} {}^t U & V T_{22} {}^t V \end{pmatrix}. \end{aligned}$$

So applying Lemma 5.3, we see that $Q(T)$ is in the representation space of $\rho_{n_1, \lambda} \otimes \rho_{n_2, \lambda}$.

Now we apply this for concrete cases. For $m = 1$, the polynomial $P_{k, k+l}$ is essentially the (homogeneous) Gegenbauer polynomial of degree l . Based on these facts and Lemma 5.2, this case gives differential operators for $n = n_1 + n_2$ and λ with depth 1, that is, the case $\lambda = (l, 0, \dots, 0)$. The corresponding representation is the symmetric tensor representation $Sym(l)$ of degree l , so \mathbb{D} is from weight k to $\det^k Sym(l) \otimes \det^k Sym(l)$. An explicit generating function of such operators for general $n = n_1 + n_2$ has been already given in [20, pp.113–114].

Here we give the depth 2 case with $\lambda = (l, l, 0, \dots, 0)$. This means $m = 2$ and an explicit generating function of $P_{k,k+l}(S)$ for $l \geq 0$ is given in [20, p.114] explicitly, where S is a 4×4 symmetric matrix. Polynomials for general $n = n_1 + n_2$ for $\lambda = (l, l, 0, \dots, 0)$ based on Proposition 5.2 are given as follows. For a 4×4 symmetric matrix $S = \begin{pmatrix} S_{11} & S_{12} \\ {}^t S_{12} & S_{22} \end{pmatrix}$ with 2×2 matrices S_{ij} of variables for $i, j = 1, 2$, we put

$$\begin{aligned} f_1(S) &= \det(S_{12}), \\ f_2(S) &= \det(S_{11}) \det(S_{22}), \\ f_3(S) &= \det(S). \end{aligned}$$

For an indeterminate t , we put

$$\begin{aligned} \Delta_0(S, t) &= 1 - 2f_1(S)t + f_2(S)t^2, \\ R(S, t) &= (\Delta_0(S, t) + \sqrt{\Delta_0(S, t) - 4f_3(S)t^2})/2. \end{aligned}$$

Then for each l , we define a polynomial $Q_l(T, U, V) = Q_{l,n_1,n_2}(T, U, V)$ by the following generating function.

$$\frac{1}{R(\mathbb{U}T {}^t\mathbb{U}, t)^{k-5/2} \sqrt{\Delta_0(\mathbb{U}T {}^t\mathbb{U}, t) - 4f_3(\mathbb{U}T {}^t\mathbb{U})t^2}} = \sum_{l=0}^{\infty} Q_l(T, U, V)t^l.$$

Here Q_l is a non-zero polynomial. For $Z \in \mathbb{H}_n$, we put

$$\mathbb{D}_l = \mathbb{D}_{l,n_1,n_2} = Q_{l,n_1,n_2}(\partial_Z, U, V). \tag{2}$$

Then \mathbb{D}_l is a differential operator satisfying Condition 1 for k and $\lambda = (l, l, 0, \dots, 0)$, where the representation space is realized by bideterminants as we explained. When $2k \geq n$, such differential operator \mathbb{D}_l is unique up to constant.

Actually, the generating series is easily expanded by a well-known formula, and more explicitly we have the following formula,

LEMMA 5.5. *The polynomials f_i being the same as above, we put*

$$F_i(T, U, V) = f_i(\mathbb{U}T {}^t\mathbb{U})$$

for $i = 1, 2, 3$. Then we have

$$\begin{aligned} Q_l(T, U, V) &= \sum_{\substack{0 \leq a,b,c \\ a+2b+2c=l}} \frac{(-1)^b 2^a}{a!b!c!} \left(k + c - \frac{3}{2}\right)_{a+b+c} F_1(T, U, V)^a F_2(T, U, V)^b F_3(T, U, V)^c. \end{aligned}$$

5.2. Weak pullback formula.

Let n_1, n_2 be positive integers such that $n_1 \leq n_2$. Let λ be a dominant integral weight such that $\text{depth}(\lambda) \leq n_1$. We consider a differential operator $\mathbb{D}_\lambda = \mathbb{D}_{\lambda,n_1,n_2}$ on $\mathbb{H}_{n_1+n_2}$ satisfying Condition 1 for k and $\det^k \rho_{n_1,\lambda} \otimes \det^k \rho_{n_2,\lambda}$. For an integer r such that $\text{depth}(\lambda) \leq r$, we put $\rho_r = \det^k \rho_{r,\lambda}$. For a Hecke eigenform $f \in S_{\rho_r}(\Gamma^{(r)})$ we define

$D(s, f)$ as

$$D(s, f) = \zeta(s)^{-1} \prod_{i=1}^r \zeta(2s - 2i)^{-1} L(s - r, f, \text{St}).$$

For any polynomial $Q(U)$ with complex coefficients, we denote by $\overline{Q}(U) = \overline{Q(U)}$ the polynomial obtained by changing the coefficients of $Q(U)$ by the complex conjugates. For any function $f(Z)$, we write $(\theta f)(Z) = \overline{f(-\overline{Z})}$. This means that if $f(z)$ is a Fourier series of the following form

$$f(z) = \sum_T a(T) \mathbf{e}(\text{tr}(TZ))$$

with $a(T) = a(T)(U)$ a polynomial in U , then we have

$$(\theta f)(Z) = \sum_T \overline{a(\overline{T})} \mathbf{e}(\text{tr}(TZ)).$$

So if we take $a(T)$ to be real, (which is possible), we just have $\theta f = f$.

The next theorem is (a pullback formula) essentially due to Kozima [46].

THEOREM 5.6. *Let $\lambda = (l, \dots, l, 0, \dots, 0), n_1, n_2, k$ and $\mathbb{D}_{\lambda, n_1, n_2}$ be those in Proposition 5.2. Besides we assume that k is even and $n_2 \geq n_1$. Let $s \in \mathbb{C}$ such that $2\text{Re}(s) + k > n_1 + n_2 + 1$. Then for any Hecke eigenform $f \in S_{\rho_{n_1}}(\Gamma^{(n_1)})$ we have*

$$\left(f, \mathbb{D}_{\lambda, n_1, n_2} E_{n_1 + n_2, k} \left(\begin{pmatrix} * & O \\ O & -\overline{W} \end{pmatrix}, \overline{s} \right) \right) = c(s, \rho_{n_1}) D(2s + k, f) [f]_{\rho_{n_1}}^{\rho_{n_2}}(W, s),$$

where $c(s, \rho_{n_1})$ is a function of s depending on ρ_{n_1} but not on n_2 .

REMARK 5.7. This type of formula has been proved in the case $k > n_1 + n_2 + 1$ and $s = 0$ in [45] in more general setting, and it can also be generalized in the case $s \neq 0$ using the same method as in [45] (cf. [46]). Kozima [46] gave an abstract pullback formula for general λ assuming that P in Condition 1 is realized in his special way. The existence of P satisfying Condition 1 itself has been known in [20]. For further development on realization of P and exact pullback formula, see [28].

Now we prove Proposition 2.1 (2), that is, we prove the following statement:

Let $\mathbf{k} = (\overbrace{k+l, \dots, k+l}^m, \overbrace{k, \dots, k}^{n-m})$ such that $l \geq 0$ and $k > 3m/2 + 1$ and let f be a Hecke eigenform in $S_{k+l}(\Gamma^{(m)})$. Then $[f]^{\mathbf{k}}(Z, s)$ can be continued meromorphically to the whole s -plane as a function of s , and holomorphic at $s = 0$. Moreover suppose that $k > (n + m + 3)/2$. Then $[f]^{\mathbf{k}}(Z)$ belongs to $M_{\mathbf{k}}(\Gamma^{(n)})$.

PROOF. Suppose that $l > 0$. Let $\lambda = (\overbrace{l, \dots, l}^m, 0, \dots, 0)$. Then for any $n_2 \geq m$ we have

$$\left(f, \mathbb{D}_{\lambda, m, n_2} E_{m+n_2, k} \left(\begin{pmatrix} * & O \\ O & -\overline{W} \end{pmatrix}, \bar{s} \right) \right) = c(s, \rho_m) D(2s + k, f) [f]_{\rho_m}^{\rho_{n_2}}(W, s).$$

In particular,

$$\left(f, \mathbb{D}_{\lambda, m, m} E_{2m, k} \left(\begin{pmatrix} * & O \\ O & -\overline{W} \end{pmatrix}, \bar{s} \right) \right) = c(s, \rho_m) D(2s + k, f) f(W).$$

We claim that $c(s, \rho_m)$ is a meromorphic function of s , and holomorphic and non-zero at $s = 0$. We note that $\mathbb{D}_{\lambda, m, m}$ coincides with the differential operator $\mathfrak{D}_{m, k}^l$ in [10, (1.14)] up to constant multiple not depending on s . Moreover, by [10, Theorem 3.1], we have

$$\left(f, \mathfrak{D}_{m, k}^l E_{2m, k} \left(\begin{pmatrix} * & O \\ O & -\overline{W}_* \end{pmatrix}, \bar{s} \right) \right) = \Omega_{k+l, l}(s) D(k + 2s, f) f(W),$$

where

$$\begin{aligned} &\Omega_{k+l, l}(s) \\ &= (-1)^{\frac{m(k+l)}{2}} 2^{-m(k+l) + \frac{m(m+3)}{2} - 2ms} \pi^{\frac{m(m+1)}{2}} \frac{\Gamma_m(k+l+s-\frac{m}{2}) \Gamma_m(k+l-\frac{m+1}{2})}{\Gamma_m(k+s) \Gamma_m(k+s-\frac{m}{2})}. \end{aligned}$$

(There is a minor misprint in [10]. On page 1393, line 9, “ $2^{1+n(n+1)/2-2ns}$ ” should be “ $2^{1-nl+n(n+3)/2-2ns}$ ”.) Therefore $c(s, \rho_m)$ coincides with $\Omega_{k+l, l}(s)$ up to constant multiple. Hence, $c(s, \rho_m)$ is a meromorphic function of s , and holomorphic and non-zero at $s = 0$.

We have

$$\left(f, \mathbb{D}_{\lambda, m, n} E_{m+n, k} \left(\begin{pmatrix} * & O \\ O & -\overline{W} \end{pmatrix}, \bar{s} \right) \right) = c(s, \rho_m) D(2s + k, f) [f]^{\mathbf{k}}(W, s).$$

As stated before, $E_{m+n, k}(\left(\frac{Z}{O} \overline{O}\right), s)$ can be continued meromorphically to the whole s -plane, and holomorphic at $s = 0$, and therefore so is $\overline{\mathbb{D}_{\lambda, m, n} E_{m+n, k}(\left(\begin{pmatrix} * & O \\ O & -\overline{W} \end{pmatrix}, \bar{s})}$. Moreover, $D(2s + k, f)$ can be continued meromorphically to the whole s -plane. Moreover, since we have $k > 3m/2 + 1$, by [55, Theorem 21.3], it is holomorphic and non-zero at $s = 0$. This proves the first part of the assertion. Moreover, if $k \geq (n + m + 1)/2$, then by Proposition 2.1 (1), $\overline{\mathbb{D}_{\lambda, m, n} E_{m+n, k}(\left(\begin{pmatrix} * & O \\ O & -\overline{W} \end{pmatrix}, \bar{s})}$ belongs to $M_{\mathbf{k}}(\Gamma^{(n)})$ as a function of W except in the cases $k = (n + m + 2) \equiv 2 \pmod{4}$ and $k = (n + m + 3)/2 \equiv 2 \pmod{4}$. Since $c(0, \rho_m) D(k, f) \neq 0$, this proves the second part of the assertion. \square

We note that the constant c_r in the pullback formula depends on two things. One is a definition of the differential operator, and the other is a definition of the Petersson inner product (P) in Section 2. First we fix a definition of the inner product. For a while, we fix a dominant integral weight $\lambda = (l_1, \dots, l_m, 0, \dots, 0)$ of depth m .

For an integer r such that $m \leq r$, let U be an $m \times r$ matrix of variables and $\mathbf{k}'_r = (l_1, \dots, l_m, \overbrace{0, \dots, 0}^{r-m})$, and we take $V_{r, \lambda} = \mathbb{C}[U]_{\mathbf{k}'_r}$ as the representation space of $\rho_{r, \lambda}$

as stated before. Here we make the convention that $\mathbb{C}[U]_{\mathbf{k}'_r} = \mathbb{C}$ if $m = 0$. We fix an inner product $\langle v, w \rangle$ of $V_{r,\lambda}$ such that

$$\langle \rho_{r,\lambda}(g)v, w \rangle = \langle v, \rho_{r,\lambda}({}^t\bar{g})w \rangle$$

as in (H) in Section 2. This relation is valid also for the representation $\rho_r = \det^k \rho_{r,\lambda}$, so we often use the same inner product for these. Now we must fix an inner product $\langle *, * \rangle$ of $V_{r,\lambda}$ explicitly. Since we have $V_{r,\lambda} = \mathbb{C}[U]_{\mathbf{k}'_r}$, an element of $V_{r,\lambda}$ is a polynomial in the components of an $m \times r$ matrix U where the action of $\rho_{r,\lambda}$ is induced by $U \rightarrow UA$ for any $A \in \text{GL}_r(\mathbb{C})$: $(\rho_{r,\lambda}(A)Q)(U) = Q(UA)$. For any $B \in \text{GL}_r(\mathbb{C})$, we obviously have

$$\overline{Q(\overline{UB})} = \overline{Q}(UB).$$

Here the right-hand side means to substitute the argument U in $\overline{Q}(U)$ by UB and the left-hand side means to replace coefficients of $Q(\overline{UB})$ by complex conjugates. We put

$$\frac{\partial}{\partial U} = \left(\frac{\partial}{\partial u_{ij}} \right)_{1 \leq i \leq m, 1 \leq j \leq r}.$$

For two homogeneous polynomials $P(U)$ and $Q(U)$ of the same degree, we define

$$\langle P, Q \rangle_0 = P \left(\frac{\partial}{\partial U} \right) \overline{Q}(U).$$

Then we have

$$\langle \rho_{r,\lambda}(A)P, Q \rangle_0 = \langle P, \rho_{r,\lambda}({}^t\bar{A})Q \rangle_0. \tag{3}$$

Indeed, if we put $V = UB$, then by the chain rule we have

$$\frac{\partial}{\partial U} = \frac{\partial}{\partial V} {}^tB,$$

so if we put $A = {}^tB^{-1}$, then we have

$$P \left(\frac{\partial}{\partial U} A \right) = P \left(\frac{\partial}{\partial V} \right),$$

and

$$\begin{aligned} P \left(\frac{\partial}{\partial U} A \right) \overline{Q}(U) &= P \left(\frac{\partial}{\partial V} \right) \overline{Q}(V {}^tA) \\ &= P \left(\frac{\partial}{\partial V} \right) \overline{Q(V {}^t\bar{A})}. \end{aligned}$$

So (3) is proved. Of course such an inner product is determined only up to constant, and there is no canonical choice, but we must fix something. When $\rho_{r,\lambda}$ is scalar valued representation \det^l , if we define an inner product by $\langle P, Q \rangle_0 / (l)_r (l-1)_r \cdots (1)_r$ then by the Cayley type identity [11], this just means to take a product of scalars, so

$$(f, g) = \int_{\Gamma^{(r)} \backslash \mathbb{H}_r} f(Z) \overline{g(Z)} \det(\text{Im}(Z))^{l-r-1} dZ.$$

Then we have a weak type of the pullback formula. Let k and l be non-negative integers. For the dominant integral $\lambda = (l, l, 0, \dots, 0)$ of depth m and integers n_1, n_2 such that $2 \leq n_1 \leq n_2$, let $\rho_{n_1} = \det^k \rho_{n_1, \lambda}$ and $\rho_{n_2} = \det^k \rho_{n_2, \lambda}$ be the representations of $\text{GL}_{n_1}(\mathbb{C})$ and $\text{GL}_{n_2}(\mathbb{C})$, respectively, as above. We note that $m = 0$ or 2 according as $l = 0$ or $l > 0$. Moreover, let \mathbb{D}_{l, n_1, n_2} be the differential operator corresponding to the polynomial Q_{l, n_1, n_2} in Lemma 5.5.

THEOREM 5.8. *Let the notation be as above. We define a subspace $\widetilde{M}_{\rho_{n_1}}(\Gamma^{(n_1)})$ of $M_{\rho_{n_1}}(\Gamma^{(n_1)})$ as*

$$\widetilde{M}_{\rho_{n_1}}(\Gamma^{(n_1)}) = \{F \in M_{\rho_{n_1}}(\Gamma^{(n_1)}) \mid \Phi_2^{n_1}(F) \in S_{\rho_2}(\Gamma^{(2)})\}$$

or $M_{\rho_{n_1}}(\Gamma^{(n_1)})$ according as $l > 0$ or $l = 0$. Let $\{f_{2,j}\}_{1 \leq j \leq d(2)}$ be a basis of $S_{\rho_2}(\Gamma^{(2)})$ consisting of Hecke eigenforms, and take Hecke eigenforms $\{F_j\}_{d(2)+1 \leq j \leq d}$ so that $\{[f_{2,j}]_{\rho_2}^{\rho_{n_1}} \ (1 \leq j \leq d(2)), F_j \ (d(2)+1 \leq j \leq d)\}$ forms a basis of $\widetilde{M}_{\rho_{n_1}}(\Gamma^{(n_1)})$. Suppose that $k \geq \max((n_1 + n_2 + 1)/2, 6)$ and that neither $k = (n_1 + n_2 + 2)/2 \equiv 2 \pmod{4}$ nor $k = (n_1 + n_2 + 3) \equiv 2 \pmod{4}$. Then

$$\begin{aligned} \mathbb{D}_{l, n_1, n_2} E_{n_1+n_2, k} \begin{pmatrix} Z & O \\ O & W \end{pmatrix} &= c(0, \rho_2) \sum_{j=1}^{d(2)} \frac{D(k, f_{2,j})}{(f_{2,j}, f_{2,j})} [f_{2,j}]_{\rho_2}^{\rho_{n_1}}(Z)(U) [\theta f_{2,j}]_{\rho_2}^{\rho_{n_2}}(W)(V) \\ &\quad + \sum_{j=d(2)+1}^d F_j(Z)(U) G_j(W)(V) \quad (Z \in \mathbb{H}_{n_1}, W \in \mathbb{H}_{n_2}), \end{aligned}$$

where G_j is a certain element of $M_{\rho_{n_2}}(\Gamma^{(n_2)})$. Here, U and V are $m \times n_1$ and $m \times n_2$ matrices of variables, respectively, and we regard $[f_{2,j}]_{\rho_2}^{\rho_{n_1}}$ and F_j (resp. $[\theta f_{2,j}]_{\rho_2}^{\rho_{n_2}}$ and G_j) as elements of $\text{Hol}(\mathbb{H}_{n_1})[U]_{\mathbf{k}'_{n_1}}$ (resp. $\text{Hol}(\mathbb{H}_{n_2})[V]_{\mathbf{k}'_{n_2}}$). Moreover we have

$$c(0, \rho_2) = \frac{2^{9-2(k+2l)} (-1)^{k+l} \pi^3 (2k-3)_l (2k-1)_{2l-3}}{l!}.$$

PROOF. First suppose that $l > 0$. Let $d_0 = \dim M_{\rho_{n_1}}(\Gamma^{(n_1)})$ and $\{F_j\}_{d+1 \leq j \leq d_0}$ be a basis of the orthogonal complement of $\widetilde{M}_{\rho_{n_1}}(\Gamma^{(n_1)})$ in $M_{\rho_{n_1}}(\Gamma^{(n_1)})$ with respect to the Petersson inner product. Then we have

$$\begin{aligned} \mathbb{D}_{l, n_1, n_2} E_{n_1+n_2, k} \begin{pmatrix} Z & O \\ O & W \end{pmatrix} &= \sum_{j=1}^{d(2)} [f_{2,j}]_{\rho_2}^{\rho_{n_1}}(Z)(U) G_j(W)(V) \\ &\quad + \sum_{j=d(2)+1}^{d_0} F_j(Z)(U) G_j(W)(V). \end{aligned}$$

For $m_1 \geq m_2$ and $l_1 \geq l_2$ and $H(Z, W)(U, V) = \sum_j A_j(Z)(U) B_j(W)(V) \in M_{\rho_{m_1}}(\Gamma^{(m_1)}) \otimes M_{\rho_{l_1}}(\Gamma^{(l_1)})$, we define $\Phi_{m_2}^{m_1} \otimes \Phi_{l_2}^{l_1}(H(Z, W)(U, V))$ as

$$\Phi_{m_2}^{m_1} \otimes \Phi_{l_2}^{l_1}(H(Z, W)(U, V)) = \sum_j \Phi_{m_2}^{m_1}(A_j(Z)(U))\Phi_{l_2}^{l_1}(B_j(W)(V)).$$

We note that we have $\Phi_2^{n_1}(F_j) = 0$ for $d(2) + 1 \leq j \leq d$. Hence we have

$$\begin{aligned} & (\Phi_2^{n_1} \otimes \Phi_{n_2}^{n_2}) \left(\mathbb{D}_{l, n_1, n_2} E_{n_1+n_2, k} \begin{pmatrix} Z & O \\ O & W \end{pmatrix} \right) \\ &= \sum_{j=1}^{d(2)} f_{2,j}(Z^{(2)})(U^{(2)})G_j(W)(V) + \sum_{j=d+1}^{d_0} \Phi_2^{n_1}(F_j(Z)(U))\Phi_{n_2}^{n_2}(G_j(W)(V)), \end{aligned}$$

where $Z^{(2)} = \text{pr}_2^{n_1}(Z)$ for $Z \in \mathbb{H}_{n_1}$, and $U^{(2)} = (u_{ij})_{1 \leq i, j \leq 2}$ for $U = (u_{ij})_{1 \leq i \leq 2, 1 \leq j \leq n_1}$. On the other hand, let $A = \begin{pmatrix} A_1 & O & R_1/2 \\ O & O & O \\ {}^t R_1/2 & O & D_1 \end{pmatrix} \in \mathcal{H}_{n_1+n_2}(\mathbb{Z})_{\geq 0}$ with $A_1 \in \mathcal{H}_2(\mathbb{Z})$, $D_1 \in \mathcal{H}_{n_2}(\mathbb{Z})$ and $R_1 \in M_{2, n_2}(\mathbb{Z})$. Then we have

$$\begin{aligned} & (\Phi_2^{n_1} \otimes \Phi_{n_2}^{n_2}) \left(\text{Res } \mathbb{D}_{l, n_1, n_2} \left(\mathbf{e} \left(\text{tr} \left(A \begin{pmatrix} Z & Z_{12} \\ {}^t Z_{12} & W \end{pmatrix} \right) \right) \right) \right) \\ &= \text{Res } \mathbb{D}_{l, 2, n_2} \left(\mathbf{e} \left(\text{tr} \left(\begin{pmatrix} A_1 & R_1/2 \\ {}^t R_1/2 & D_1 \end{pmatrix} \begin{pmatrix} Z^{(2)} & Z_{12}^{(2)} \\ {}^t Z_{12}^{(2)} & W \end{pmatrix} \right) \right) \right) \end{aligned}$$

for $\begin{pmatrix} Z & Z_{12} \\ {}^t Z_{12} & W \end{pmatrix} \in \mathbb{H}_{n_1+n_2}$ with $Z \in \mathbb{H}_{n_1}$, $W \in \mathbb{H}_{n_2}$ and $Z_{12} \in M_{n_1, n_2}(\mathbb{C})$. Here $Z_{12}^{(2)}$ is the upper-left $2 \times n_2$ block of Z_{12} . Hence we have

$$(\Phi_2^{n_1} \otimes \Phi_{n_2}^{n_2}) \left(\mathbb{D}_{l, n_1, n_2} E_{n_1+n_2, k} \begin{pmatrix} Z & O \\ O & W \end{pmatrix} \right) = \mathbb{D}_{l, 2, n_2} E_{2+n_2, k} \begin{pmatrix} Z^{(2)} & O \\ O & W \end{pmatrix},$$

and therefore, by Remark 5.4, $(\Phi_2^{n_1} \otimes \Phi_{n_2}^{n_2})(\mathbb{D}_{l, n_1, n_2} E_{n_1+n_2, k} \begin{pmatrix} Z & O \\ O & W \end{pmatrix})$ belongs to $S_{\rho_2}(\Gamma^{(2)}) \otimes M_{\rho_{n_2}}(\Gamma^{(n_2)})$. We note that $\Phi_{\rho_2}^{\rho_{n_1}}(F_j) \notin S_{\rho_2}(\Gamma^{(2)})$ for $d + 1 \leq j \leq d_0$. Hence we have $G_j(W)(V) = 0$ for $d + 1 \leq j \leq d_0$ and

$$\mathbb{D}_{l, 2, n_2} E_{2+n_2, k} \begin{pmatrix} Z^{(2)} & O \\ O & W \end{pmatrix} = \sum_{j=1}^{d(2)} f_{2,j}(Z^{(2)})(U^{(2)})G_j(W)(V).$$

By Theorem 5.6, we see that

$$(f_{2,j}, f_{2,j})\theta G_j(W)(V) = c(0, \rho_2)D(k, f_{2,j})[f_{2,j}]_{\rho_2}^{\rho_{n_2}}(W)(V).$$

We note that $D(k, f_{2,j})$ is real number, and hence

$$G_j(W)(V) = c(0, \rho_2) \frac{D(k, f_{2,j})}{(f_{2,j}, f_{2,j})} [\theta f_{2,j}]_{\rho_2}^{\rho_{n_2}}(W)(V).$$

This proves the first part of the assertion. By using the same argument as above, we have

$$\mathbb{D}_{l,2,2} E_{4,k} \begin{pmatrix} Z^{(2)} & O \\ O & W^{(2)} \end{pmatrix} = \sum_{j=1}^{d(2)} f_{2,j}(Z^{(2)})(U^{(2)}) \theta f_{2,j}(W^{(2)})(V^{(2)}),$$

where $W^{(2)} = \text{pr}_2^{n_1}(Z)$ for $W \in \mathbb{H}_{n_2}$, and $V^{(2)} = (v_{ij})_{1 \leq i,j \leq 2}$ for $V = (v_{ij})_{1 \leq i \leq 2, 1 \leq j \leq n_1}$.
 Let

$$\iota : S_{k+l}(\Gamma^{(2)}) \longrightarrow S_{\rho_2}(\Gamma^{(2)})$$

be the isomorphism stated before. Take an element $g_{2,j} \in S_{k+l}(\Gamma^{(2)})$ such that $f_{2,j} = \iota(g_{2,j})$. Then $\{g_{2,j}\}_j$ forms a basis of $S_{k+l}(\Gamma^{(2)})$ and

$$D(k, f_{2,j}) = D(k, g_{2,j})$$

and

$$(f_{2,j}, f_{2,j}) = (g_{2,j}, g_{2,j}).$$

We also note that

$$\mathbb{D}_{l,2,2} = (\det U^{(2)} \det V^{(2)})^l \mathbb{D}_{l,k+l},$$

where $\mathbb{D}_{k,k+l} = P_{k,k+l}(\partial_{\tilde{Z}})$ with $\tilde{Z} \in \mathbb{H}_4$. Hence we have

$$\mathbb{D}_{k,k+l} E_{4,k} \begin{pmatrix} Z^{(2)} & O \\ O & W^{(2)} \end{pmatrix} = c(0, \rho_2) \sum_{j=1}^{d(2)} \frac{D(k, g_{2,j})}{(g_{2,j}, g_{2,j})} g_{2,j}(Z^{(2)}) \otimes \theta g_{2,j}(W^{(2)}).$$

Hence we have

$$\left(g_{2,j}, \mathbb{D}_{k,k+l} E_{4,k} \begin{pmatrix} * & O \\ O & -W^{(2)} \end{pmatrix} \right) = c(0, \rho_2) D(k, g_{2,j}) g_{2,j}(W^{(2)}).$$

On the other hand, let $\mathfrak{D}_{2,k}^l$ be the differential operator in [10, (1.14)]. Then by [10, Theorem 3.1] we have

$$\left(g_{2,j}, \mathfrak{D}_{2,k}^l E_{4,k} \begin{pmatrix} * & O \\ O & -W^{(2)} \end{pmatrix} \right) = \tilde{c}_2 D(k, g_{2,j}) g_{2,j}(W^{(2)})$$

with

$$\tilde{c}_2 = (-1)^{k+l} 2^{6-2(k+l)} \pi^3 \frac{\Gamma(k+l-1)\Gamma(k+l-3/2)^2\Gamma(k+l-2)}{\Gamma(k)\Gamma(k-1/2)\Gamma(k-1)\Gamma(k-3/2)}.$$

By page 71 in [37], we have

$$\mathbb{D}_\lambda = d_{k,l} \mathfrak{D}_{2,k}^l$$

with

$$d_{k,l} = \frac{\binom{2k+2l-5}{l}}{\prod_{i=1}^l (k+l-2-i/2)(k+l-3/2-i/2)}.$$

Hence we have $c(0, \rho_2) = d_{k,l} \tilde{c}_2$, and by a simple computation we prove the assertion. Next suppose that $l = 0$. Then the assertion can be proved using the same argument as above. □

REMARK 5.9. (1) The second part of the assertion can be also proved by the fact that the differential operator is realized uniformly in Lemma 5.5 and its operation on the automorphy factor is essentially the same as the case when $U = V = 1_2$.

(2) If $k > n_1 + n_2 + 1$, then $[f]_{\rho_r}^{\rho_{n_1}}(Z, U)$ and $[f]_{\rho_r}^{\rho_{n_2}}(W, V)$ are holomorphic modular forms for any Hecke eigenform f in $S_{\rho_r}(\Gamma^{(r)})$, and we can get an explicit pullback formula (cf. Theorems B.1 and B.13). However, if $k \leq n_1 + n_2 + 1$, it does not necessarily hold. This is why we say that the formula in the above theorem is a weak type of pullback formula. We note that it is sufficient for proving our main results in Section 8.

6. Congruence for Klingen–Eisenstein lifts.

To explain why Conjecture 4.5 is reasonable, we consider congruence for Klingen–Eisenstein series, which is a generalization of [39]. For $\lambda = (k - l, k - l, 0, 0)$ with $k \geq l$ and $2 \leq m \leq 4$, let $(\rho_{m,\lambda}, V_{m,\lambda})$ be the representation of $\text{GL}_m(\mathbb{C})$ defined in the previous section, and put $\rho_m = \det^l \otimes \rho_{m,\lambda}$ and $\mathbf{k}'_m = (k - l, k - l, \overbrace{0, \dots, 0}^{m-2})$ and $\mathbf{k}_m = (k, k, \overbrace{l, \dots, l}^{m-2})$. Let U and V be $2 \times n_1$ and $2 \times n_2$ matrices of variables, respectively. Then we recall that $V_{n_1,\lambda} = \mathbb{C}[U]_{\mathbf{k}'_{n_1}}, V_{n_2,\lambda} = \mathbb{C}[V]_{\mathbf{k}'_{n_2}}$ and that every element F of $M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes M_{\rho_{n_2}}(\Gamma^{(n_2)})$ is expressed as

$$F(Z_1, Z_2) = \sum_{A_1 \in \mathcal{H}_{n_1}(\mathbb{Z})_{\geq 0}, A_2 \in \mathcal{H}_{n_2}(\mathbb{Z})_{\geq 0}} c(A_1, A_2; F)(U, V) \mathbf{e}(\text{tr}(A_1 Z_1 + A_2 Z_2))$$

with $c(A_1, A_2; F)(U, V) \in \mathbb{C}[U, V]_{\mathbf{k}'_{n_1}, \mathbf{k}'_{n_2}}$. For a subring R of \mathbb{C} , we denote by $(M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes M_{\rho_{n_2}}(\Gamma^{(n_2)}))(R)$ the submodule of $M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes M_{\rho_{n_2}}(\Gamma^{(n_2)})$ consisting of all F 's such that $c(A_1, A_2; F)(U, V) \in R[U, V]_{\mathbf{k}'_{n_1}, \mathbf{k}'_{n_2}}$ for all $A_1 \in \mathcal{H}_{n_1}(\mathbb{Z})_{\geq 0}, A_2 \in \mathcal{H}_{n_2}(\mathbb{Z})_{\geq 0}$. We also note that every element F of $M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes V_{n_2,\lambda}$ is expressed as

$$F(Z_1) = \sum_{A_1 \in \mathcal{H}_{n_1}(\mathbb{Z})_{\geq 0}} c(A_1; F)(U, V) \mathbf{e}(\text{tr}(A_1 Z_1))$$

with $c(A_1; F)(U, V) \in \mathbb{C}[U, V]_{\mathbf{k}'_{n_1}, \mathbf{k}'_{n_2}}$. We then define a submodule $(M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes V_{n_2,\lambda})(R)$ of $M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes V_{n_2,\lambda}$ consisting of all F 's such that $c(A_1; F)(U, V) \in R[U, V]_{\mathbf{k}'_{n_1}, \mathbf{k}'_{n_2}}$ for all $A_1 \in \mathcal{H}_{n_1}(\mathbb{Z})_{\geq 0}$.

For positive integers n and l , put

$$Z(n, l) = \zeta(1 - l) \prod_{j=1}^{[n/2]} \zeta(1 + 2j - 2l).$$

We define $\tilde{E}_{n,l}$ as

$$\tilde{E}_{n,l}(Z) = Z(n, l)E_{n,l}(Z)$$

and we set

$$\mathcal{E}(Z_1, Z_2) = \mathcal{E}_{k,l,n_1,n_2}(Z_1, Z_2) = (k - l)!(2\pi\sqrt{-1})^{-2(k-l)}\mathbb{D}_{k-l,n_1,n_2}\tilde{E}_{n_1+n_2,l} \begin{pmatrix} Z_1 & O \\ O & Z_2 \end{pmatrix}.$$

Moreover, for positive integers m, l and a Hecke eigenform $F \in S_k(\Gamma^{(2)})$ put

$$\mathcal{C}_{m,l}(F) = \frac{Z(m, l)}{Z(4, l)}\mathbf{L}(l - 2, F, \text{St}).$$

We also use the same symbol $\mathcal{C}_{m,l}(f)$ to denote the value $\frac{Z(m, l)}{Z(4, l)}\mathbf{L}(l - 2, f, \text{St})$ for a Hecke eigenform $f \in S_{\rho_2}(\Gamma^{(2)})$. As stated before, we have the following isomorphism:

$$\iota : S_k(\Gamma^{(2)}) \ni F \mapsto \tilde{F} := F(\det U)^{k-l} \in S_{\rho_2}(\Gamma^{(2)}), \tag{IS}$$

where U is 2×2 matrix of variables. Then we note that $\mathcal{C}_{m,l}(\tilde{F}) = \mathcal{C}_{m,l}(F)$ for a Hecke eigenform $F \in S_k(\Gamma^{(2)})$.

Now, for our later purpose, we rewrite a special case of Theorem 5.8 as follows.

PROPOSITION 6.1. *Let n_1, n_2 be integers such that $2 \leq n_1 \leq n_2 \leq 4$ and let k, l be even positive integers such that $k \geq l$. Then we have*

$$\begin{aligned} \mathcal{E}_{k,l,n_1,n_2}(Z_1, Z_2) &= \gamma_2 \sum_{j=1}^{d(2)} \mathcal{C}_{n_1+n_2,l}(f_{2,j})[f_{2,j}]_{\rho_2}^{\rho_{n_1}}(Z_1)(U)[\theta f_{2,j}]_{\rho_2}^{\rho_{n_2}}(Z_2)(V) \\ &+ \sum_{j=d(2)+1}^d F_j(Z_1)(U)\tilde{G}_j(Z_2)(V), \end{aligned}$$

where γ_2 is a certain rational number which is p -unit for any prime number $p > 2k$, and $\tilde{G}_j(Z_2)(V)$ is an element of $M_{\rho_{n_2}}(\Gamma^{(n_2)})$.

We write $\mathcal{E}(Z_1, Z_2)$ as

$$\mathcal{E}(Z_1, Z_2) = \sum_{N \in \mathcal{H}_{n_2}} g_{(k,l,n_1,n_2),N}^{(n_1)}(Z_1)\mathbf{e}(\text{tr}(NZ_2)). \tag{*}$$

Then $g_N^{(n_1)} = g_{(k,l,n_1,n_2),N}^{(n_1)}$ belongs to $M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes V_{n_2,\lambda}$. To consider congruence between Klingen–Eisenstein lift and another modular form of the same weight, we rewrite the above proposition as follows:

COROLLARY 6.2. *Under the same notation and the assumption as above, let $N \in \mathcal{H}_{n_2}(\mathbb{Z})_{>0}$. Then,*

$$g_N^{(n_1)}(Z_1) = \gamma_2 \sum_{j=1}^{d(2)} C_{n_1+n_2,l}(f_{2,j}) [f_{2,j}]_{\rho_2}^{\rho_{n_1}}(Z_1) \overline{a(N, [f_{2,j}]_{\rho_2}^{\rho_{n_2}})(V)} + \sum_{j=d(2)+1}^d F_j(Z_1)(U) a(N, \tilde{G}_j)(V).$$

Observe that the first term in the right-hand side of the above is invariant if we multiply $f_{2,j}$ by an element of \mathbb{C}^\times .

To see the Fourier expansion of $\tilde{E}_{n,k}(Z)$, we review the polynomial $F_p(B, X)$ attached to the local Siegel series $b_p(B, s)$ for an element B of $\mathcal{H}_n(\mathbb{Z}_p)$ (cf. [35]). We define $\chi_p(a)$ for $a \in \mathbb{Q}_p^\times$ as follows:

$$\chi_p(a) := \begin{cases} +1 & \text{if } \mathbb{Q}_p(\sqrt{a}) = \mathbb{Q}_p, \\ -1 & \text{if } \mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p \text{ is quadratic unramified,} \\ 0 & \text{if } \mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p \text{ is quadratic ramified.} \end{cases}$$

For an element $B \in \mathcal{H}_n(\mathbb{Z}_p)^{\text{nd}}$ with n even, we define $\xi_p(B)$ by

$$\xi_p(B) := \chi_p((-1)^{n/2} \det B).$$

For a non-degenerate half-integral matrix B of size n over \mathbb{Z}_p define a polynomial $\gamma_p(B, X)$ in X by

$$\gamma_p(B, X) := \begin{cases} (1 - X) \prod_{i=1}^{n/2} (1 - p^{2i} X^2) (1 - p^{n/2} \xi_p(B) X)^{-1} & \text{if } n \text{ is even,} \\ (1 - X) \prod_{i=1}^{(n-1)/2} (1 - p^{2i} X^2) & \text{if } n \text{ is odd.} \end{cases}$$

Then it is well known that there exists a unique polynomial $F_p(B, X)$ in X over \mathbb{Z} with constant term 1 such that

$$b_p(B, s) = \gamma_p(B, p^{-s}) F_p(B, p^{-s})$$

(e.g. [35]). For $B \in \mathcal{H}_n(\mathbb{Z})_{>0}$ with n even, let \mathfrak{d}_B be the discriminant of $\mathbb{Q}(\sqrt{(-1)^{n/2} \det B})/\mathbb{Q}$, and $\chi_B = (\frac{\mathfrak{d}_B}{*})$ the Kronecker character corresponding to $\mathbb{Q}(\sqrt{(-1)^{n/2} \det B})/\mathbb{Q}$. We note that we have $\chi_B(p) = \xi_p(B)$ for any prime p .

We define a polynomial $F_p^*(T, X)$ for any $T \in \mathcal{H}_n(\mathbb{Z}_p)$ which is not-necessarily non-degenerate as follows: For an element $T \in \mathcal{H}_n(\mathbb{Z}_p)$ of rank $m \geq 1$, there exists an element $\tilde{T} \in \mathcal{H}_m(\mathbb{Z}_p)^{\text{nd}}$ such that $T \sim_{\mathbb{Z}_p} \tilde{T} \perp O_{n-m}$. We note that $F_p(\tilde{T}, X)$ does not depend on the choice of \tilde{T} . Then we put $F_p^*(T, X) = F_p(\tilde{T}, X)$. For an element $T \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}$ of rank $m \geq 1$, there exists an element $\tilde{T} \in \mathcal{H}_m(\mathbb{Z})_{>0}$ such that $T \sim_{\mathbb{Z}} \tilde{T} \perp O_{n-m}$. Then $\chi_{\tilde{T}}$ does not depend on the choice of \tilde{T} . We write $\chi_T^* = \chi_{\tilde{T}}$ if m is even.

PROPOSITION 6.3. *Let $k \in 2\mathbb{Z}$. Assume that $k \geq (n + 1)/2$ and that neither $k = (n + 2)/2 \equiv 2 \pmod{4}$ nor $k = (n + 3)/2 \equiv 2 \pmod{4}$. Then for $T \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}$ of rank m , we have*

$$a(T, \tilde{E}_{n,k}) = 2^{\lfloor (m+1)/2 \rfloor} \prod_{p|\det(2\tilde{T})} F_p^*(T, p^{k-m-1}) \\ \times \begin{cases} \prod_{i=m/2+1}^{\lfloor n/2 \rfloor} \zeta(1+2i-2k)L(1+m/2-k, \chi_T^*) & \text{if } m \text{ is even,} \\ \prod_{i=(m+1)/2}^{\lfloor n/2 \rfloor} \zeta(1+2i-2k) & \text{if } m \text{ is odd.} \end{cases}$$

Here we make the convention $F_p^*(T, p^{k-m-1}) = 1$ and $L(1+m/2-k, \chi_T^*) = \zeta(1-k)$ if $m = 0$.

To consider the integrality of $a(T, \tilde{E}_{n,k})$, we provide the following lemma.

LEMMA 6.4. *Let $T \in \mathcal{H}_m(\mathbb{Z}_p)^{\text{nd}}$. Then, we have $F_p(p^{-\lfloor (m+1)/2 \rfloor}T, X) \in \mathbb{Z}[X]$.*

PROOF. The assertion has been proved in [32, Lemma 15] in the case m is even, and the assertion for odd case can also be proved in the same manner. \square

PROPOSITION 6.5. *Let the notation and the assumption be as in Proposition 6.3. Then, we have $\tilde{E}_{n,k}$ belongs to $M_k(\Gamma^{(n)})(\mathbb{Q})$. In particular, for any prime number $p > 2k$, $\tilde{E}_{n,k}$ belongs to $M_k(\Gamma^{(n)})(\mathbb{Z}_{(p)})$.*

PROOF. The first assertion is well known. We prove the second assertion. Let $T \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}$ of rank m . Since we have $k \geq (n+1)/2$, by Lemma 6.4, the product $\prod_{p|\det(2\tilde{T})} F_p^*(T, p^{k-m-1})$ is an integer. Moreover, since we have $p > 2k$, by the theorem of von Staudt–Clausen, the value $\zeta(1-k)$ and $\zeta(1+2i-2k)$ for a positive integer $i \leq \lfloor n/2 \rfloor$ belong to $\mathbb{Z}_{(p)}$, and by [6, (5.1), (5.2)], the value $L(1+m/2-k, \chi_T^*)$ belongs to $\mathbb{Z}_{(p)}$ if $m \geq 2$ is even. Thus the assertion follows from Proposition 6.3. \square

PROPOSITION 6.6. *Let the notation and the assumption be as in Proposition 6.1. Then*

$$\mathcal{E}_{k,l,n_1,n_2}(Z_1, Z_2) \in (M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes M_{\rho_{n_2}}(\Gamma^{(n_2)}))(\mathbb{Q}),$$

and more precisely

$$\mathcal{E}_{k,l,n_1,n_2}(Z_1, Z_2) \in (M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes M_{\rho_{n_2}}(\Gamma^{(n_2)}))(\mathbb{Z}_{(p)})$$

for any prime number $p > 2k$.

PROOF. For $T_1 \in \mathcal{H}_{n_1}$ and $T_2 \in \mathcal{H}_{n_2}$, put

$$\begin{aligned} \epsilon(T_1, T_2)(U, V) &= \epsilon_{k,l,n_1,n_2}(T_1, T_2)(U, V) \\ &= \sum_{R \in M_{n_1, n_2}(\mathbb{Z})} a \left(\begin{pmatrix} T_1 & R/2 \\ tR/2 & T_2 \end{pmatrix}, \tilde{E}_{n_1+n_2, l} \right) \\ &\quad \times Q_{k-l, n_1, n_2} \left(\begin{pmatrix} T_1 & R/2 \\ tR/2 & T_2 \end{pmatrix}, U, V \right), \end{aligned} \tag{E}$$

where Q_{k-l, n_1, n_2} is the polynomial in Section 5.1.2. Then we have

$$\mathcal{E}_{k,l,n_1,n_2}(Z_1, Z_2) = \sum_{T_1 \in \mathcal{H}_{n_1}(\mathbb{Z})_{\geq 0}, T_2 \in \mathcal{H}_{n_2}(\mathbb{Z})_{\geq 0}} \epsilon(T_1, T_2)(U, V) \mathbf{e}(\text{tr}(T_1 Z_1 + T_2 Z_2)).$$

Hence the assertion follows from Proposition 6.5. □

COROLLARY 6.7. *For each $N \in \mathcal{H}_{n_2}(\mathbb{Z})_{>0}$ let $g_N^{(n_1)}$ be that defined above. Then*

$$g_{(k,l,n_1,n_2),N}^{(n_1)}(Z_1) \in (M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes V_{n_2,\lambda})(\mathbb{Q})$$

and moreover

$$g_{(k,l,n_1,n_2),N}^{(n_1)}(Z_1) \in (M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes V_{n_2,\lambda})(\mathbb{Z}_{(p)})$$

for any prime number $p > 2k$.

PROOF. $g_{(k,l,n_1,n_2),N}^{(n_1)}(Z_1)$ is expressed as

$$g_{(k,l,n_1,n_2),N}^{(n_1)}(Z_1) = \sum_{T \in \mathcal{H}_{n_1}(\mathbb{Z})} \epsilon_{k,l,n_1,n_2}(T, N)(U, V) \mathbf{e}(\text{tr}(TZ_1)).$$

Hence the assertion directly follows from the above proposition. □

PROPOSITION 6.8. *Let the notation and the assumptions be as in Theorem 5.8, and let $2 \leq m \leq 4$. Then for any Hecke eigenform f in $S_{\rho_2}(\Gamma^{(2)})(\mathbb{Q}(f))$, $[f]_{\rho_2}^{\rho_m} \in M_{\rho_m}(\Gamma^{(m)})(\mathbb{Q}(f))$.*

PROOF. The assertion in the case $k = l$ has been proved by Mizumoto [50], and the other case can also be proved by using the same method. □

PROPOSITION 6.9. *Let the notation and the assumption be as in Proposition 6.1. Let f be a Hecke eigenform in $S_k(\Gamma^{(2)})$. Then, for any $N \in \mathcal{H}_{n_2}(\mathbb{Z})_{>0}$ and $N_1 \in \mathcal{H}_2(\mathbb{Z})_{>0}$, the value $C_{n_2+2,l}(f)a(N, [\tilde{f}]_{\rho_2}^{\rho_{n_2}})(V)a(N_1, f)$ belongs to $\mathbb{Q}(f)[V]_{\mathbf{k}'_2}$, where \tilde{f} is that in (IS).*

PROOF. The value in question remains unchanged if we replace f by cf with $c \in \mathbb{C}^\times$. Moreover we can take $c \in \mathbb{C}^\times$ so that $cf \in S_{\mathbf{k}_2}(\Gamma^{(2)})(\mathbb{Q}(f))$. Thus the assertion follows from Proposition 6.8 remarking that $a(N_1, [\tilde{f}]_{\rho_2}^{\rho_2}) = a(N_1, f)(\det U)^{k-l}$. □

The following lemma can be proved by a careful analysis of the proof of [36, Lemma 5.1].

LEMMA 6.10. *Let F_1, \dots, F_d be Hecke eigenforms in $M_{\rho_{n_1}}(\Gamma^{(n_1)})$ linearly independent over \mathbb{C} . Let K be the composite field $\mathbb{Q}(F_1) \cdots \mathbb{Q}(F_d)$, \mathfrak{D} the ring of integers in K and \mathfrak{p} a prime ideal of K . Let $G(Z, U, V) \in (M_{\rho_{n_1}}(\Gamma^{(n_1)}) \otimes V_{n_1,\lambda})(\mathfrak{D}_{(\mathfrak{p})})$ and assume the following conditions*

(1) G is expressed as

$$G(Z, U, V) = \sum_{i=1}^d c_i(V) F_i(Z)(U)$$

with $c_i(V) \in \widetilde{V}_{n_1, \lambda}$.

- (2) $c_1(V)a(A_1, F_1)(U) \in (V_{n_1, \lambda} \otimes V_{n_1, \lambda})(K)$ and $\text{ord}_{\mathfrak{p}}(c_1(V)a(A_1, F_1)(U)) < 0$ for some $A_1 \in \mathcal{H}_{n_1}(\mathbb{Z})$.

Then there exists $i \neq 1$ such that

$$F_i \equiv_{\text{ev}} F_1 \pmod{\mathfrak{p}}.$$

THEOREM 6.11. *Let k and l be positive even integers such that $k \geq l \geq 6$ and put $\mathbf{k} = (k, k, l, l)$ and $\widetilde{M}_{\mathbf{k}}(\Gamma^{(4)}) = \widetilde{M}_{\rho_4}(\Gamma^{(4)})$. Let $F \in S_k(\Gamma^{(2)})$ be a Hecke eigenform, and \mathfrak{p} a prime ideal of $\mathbb{Q}(F)$. Suppose that \mathfrak{p} divides $|a(A_1, F)|^2 \mathbf{L}(l - 2, F, \text{St})$ and does not divide*

$$\mathcal{C}_{8,l}(F)a(A_1, F)\overline{a(A, [F]^{\mathbf{k}})}$$

for some $A_1 \in \mathcal{H}_2(\mathbb{Z})_{>0}$ and $A \in \mathcal{H}_4(\mathbb{Z})_{>0}$, where $[F]^{\mathbf{k}} = [\widetilde{F}]_{\rho_2}^{\rho_4}$ as stated in Section 1. Then there exists a Hecke eigenform $G \in \widetilde{M}_{\mathbf{k}}(\Gamma^{(4)})$ such that G is not a constant multiple of $[F]^{\mathbf{k}}$ and

$$G \equiv_{\text{ev}} [F]^{\mathbf{k}} \pmod{\mathfrak{p}}.$$

PROOF. The assertion in the case $k = l$ has been proved in [39] in more general setting and the other case can also be proved using the same argument as in its proof. But for the sake of convenience, we here give an outline of the proof. Suppose that $k > l$. Take a basis $\{F_j\}_{1 \leq j \leq d}$ of $\widetilde{M}_{\mathbf{k}}(\Gamma^{(4)})$ such that $F_1 = [F]^{\mathbf{k}}$. Then, by Corollary 6.2, for any $A \in \mathcal{H}_4(\mathbb{Z})_{>0}$ we have

$$g_{(k,l,4,4),A}^{(4)}(Z_1) = \sum_{j=1}^d c_j(A, V)F_j(Z_1)(U),$$

where $c_1(A, V) = \gamma_2 \mathcal{C}_{8,l}(F)\overline{a(A, [F]^{\mathbf{k}})(V)}$ and $c_j(A, V) = \overline{a(A, \widetilde{G}_j)(V)}$ for some $\widetilde{G}_j \in M_{\mathbf{k}}(\Gamma^{(4)})$ for $2 \leq j \leq d$. We have

$$\begin{aligned} & \frac{Z(8, l)}{Z(4, l)} \mathcal{C}_{8,l}(F)a(A, [F]^{\mathbf{k}})(U)\overline{a(A, [F]^{\mathbf{k}})(V)} \\ &= \left(\frac{Z(8, l)}{Z(4, l)}\right)^2 \frac{\mathbf{L}(l - 2, F, \text{St})a(A_1, F)a(A, [F]^{\mathbf{k}})(U)\mathbf{L}(l - 2, F, \text{St})\overline{a(A_1, F)a(A, [F]^{\mathbf{k}})(V)}}{|a(A_1, F)|^2 \mathbf{L}(l - 2, F, \text{St})}. \end{aligned}$$

We note that the reduced denominator of $\frac{Z(8, l)}{Z(4, l)}$ is not divisible by \mathfrak{p} by the theorem of von Staudt–Clausen. Hence we have

$$\text{ord}_{\mathfrak{p}}(\gamma_2 \mathcal{C}_{8,l}(F)a(N, [F]^{\mathbf{k}})(U)\overline{a(N, [F]^{\mathbf{k}})(V)}) < 0.$$

Hence the assertion follows from Lemma 6.10. □

PROPOSITION 6.12. *Let k and l be even integers such that $6 \leq l \leq k$. Let f be a primitive form in $S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$. Suppose that a prime ideal \mathfrak{p} in $\mathbb{Q}(f)$ satisfies the following conditions (1), (2), (3):*

- (1) $p_{\mathfrak{p}} \geq 2k - 2$.
- (2) \mathfrak{p} divides $\mathbf{L}(k + l - 4, f)/\mathbf{L}(k, f)$.
- (3) \mathfrak{p} divides neither \mathfrak{D}_f nor $\zeta(3 - 2k)$, where \mathfrak{D}_f is the ideal of $\mathbb{Q}(f)$ defined in Proposition 3.4.

Then for any $N \in \mathcal{H}_2(\mathbb{Z})_{>0}$ such that $\mathfrak{p} \nmid \mathfrak{d}_N$, we have

$$\mathrm{ord}_{\mathfrak{p}}(|a(N, \mathcal{J}_2(f))|^2 \mathbf{L}(l - 2, \mathcal{J}_2(f), \mathrm{St})) > 0.$$

Here, \mathfrak{d}_N is the discriminant of $\mathbb{Q}(\sqrt{-\det N})/\mathbb{Q}$ as defined before.

PROOF. Let g be a Hecke eigenform in the Kohnen plus space $S_{k-1/2}^+(\Gamma_0(4))$ corresponding to f under the Shimura correspondence (cf. [41]). For any $N \in \mathcal{H}_2(\mathbb{Z})_{>0}$ we have $a(N, \mathcal{J}_2(f)) = ba(|\mathfrak{d}_N|, g)$ with $b \in \mathbb{Z}$. Hence

$$\mathbf{L}(l - 2, \mathcal{J}_2(f), \mathrm{St})|a(N, \mathcal{J}_2(f))|^2 = b^2 A_{2,k,l-2} \frac{L(l - 2, \mathcal{J}_2(f), \mathrm{St})}{\pi^{2k+3l-9} \langle \mathcal{J}_2(f), \mathcal{J}_2(f) \rangle} |a(|\mathfrak{d}_N|, g)|^2$$

with $A_{2,k,l-2} \in \mathbb{Z}_{(p_{\mathfrak{p}})}$ (cf. Remark 3.5). By definition, we have

$$L(l - 2, \mathcal{J}_2(f), \mathrm{St}) = \zeta(l - 2)L(k + l - 3, f)L(k + l - 4, f).$$

Moreover, by [42], we have

$$\langle \mathcal{J}_2(f), \mathcal{J}_2(f) \rangle = 2^{k-2} \langle g, g \rangle \Gamma_{\mathbb{C}}(2)\zeta(2)\Gamma_{\mathbb{C}}(k)L(k, f),$$

and by [43] we have

$$\frac{|a(|\mathfrak{d}_N|, g)|^2}{\langle g, g \rangle} = \frac{2^{k-2} |\mathfrak{d}_N|^{k-3/2} \Gamma_{\mathbb{C}}(k - 1)L(k - 1, f, (\frac{\mathfrak{d}_N}{\mathfrak{p}}))}{\langle f, f \rangle}.$$

We note that $\tau((\frac{\mathfrak{d}_N}{\mathfrak{p}})) = \sqrt{-1} |\mathfrak{d}_N|^{1/2}$, and $\pi^{2-l}\zeta(l - 2)$ and $\pi^{-2}\zeta(2)$ belong to $\mathbb{Z}_{(p_{\mathfrak{p}})}$. Hence, by a simple computation, we have

$$\mathbf{L}(l - 2, \mathcal{J}_2(f), \mathrm{St})|a(N, \mathcal{J}_2(f))|^2 = \epsilon_{k,N} \frac{\mathbf{L}(k + l - 4, f)}{\mathbf{L}(k, f)} \mathbf{L}\left(k + l - 3, k - 1; f; \mathbf{1}, \left(\frac{\mathfrak{d}_N}{\mathfrak{p}}\right)\right)$$

where $\epsilon_{k,N}$ is a \mathfrak{p} -integral rational number. Since \mathfrak{p} divides neither $\mathfrak{D}_f \mathfrak{d}_N$ nor $\zeta(3 - 2k)$, by Proposition 3.4, the value $\mathbf{L}(k + l - 3, k - 1; f; \mathbf{1}, (\frac{\mathfrak{d}_N}{\mathfrak{p}}))$ is \mathfrak{p} -integral. Thus the assertion holds. □

The next theorem clarifies what we need to look at to try to prove Conjecture 4.5.

THEOREM 6.13. *Let k and l positive even integers such that $6 \leq l \leq k$, and put $\mathbf{k} = (k, k, l, l)$. Let f be a primitive form in $S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ and \mathfrak{p} a prime ideal of $\mathbb{Q}(f)$ such that*

- (1) $p_{\mathfrak{p}} \geq 2k - 2$.
- (2) \mathfrak{p} divides $\mathbf{L}(k + l - 4, f)/\mathbf{L}(k, f)$.
- (3) \mathfrak{p} divides neither \mathfrak{D}_f nor $\zeta(3 - 2k)$.
- (4) \mathfrak{p} divides neither $\mathcal{C}_{8,l}(\mathcal{J}_2(f))a(A_1, \mathcal{J}_2(f))\overline{a(A, [\mathcal{J}_2(f)]^{\mathbf{k}})}$ nor \mathfrak{d}_{A_1} for some $A_1 \in \mathcal{H}_2(\mathbb{Z})_{>0}$ and $A \in \mathcal{H}_4(\mathbb{Z})_{>0}$.

Then there exists a Hecke eigenform G in $\widetilde{M}_{\mathbf{k}}(\Gamma^{(4)})$ such that G is not a constant multiple of $[\mathcal{J}_2(f)]^{\mathbf{k}}$ and

$$G \equiv_{\mathrm{ev}} [\mathcal{J}_2(f)]^{\mathbf{k}} \pmod{\mathfrak{p}}.$$

PROOF. The assertion follows from Theorem 6.11 and Proposition 6.12. □

7. Fourier coefficients of Klingen–Eisenstein lift.

Let $\mathbf{k} = (k, k, l, l)$ with k, l positive even integers such that $k \geq l$. To confirm the condition (4) in Theorem 6.13, we give a formula for computing $\mathbf{L}(l - 2, F, \mathrm{St})a(T, F)a(N, [F]^{\mathbf{k}})$ for a Hecke eigenform F in $S_k(\Gamma^{(2)})$, $T \in \mathcal{H}_2(\mathbb{Z})_{>0}$ and $N \in \mathcal{H}_4(\mathbb{Z})_{>0}$. For $T \in \mathcal{H}_2(\mathbb{Z})_{>0}$ and $N \in \mathcal{H}_4(\mathbb{Z})_{>0}$, let $\epsilon_{k,l,2,4}(T, N)(U, V)$ be as in (E) and put $g_N = g_{(k,l,2,4),N}^{(2)}$. Recall that U and V are 2×2 and 2×4 matrices, respectively, of variables. We note that $\epsilon_{k,l,2,4}(T, N)(U, V)$ can be expressed as

$$\epsilon_{k,l,2,4}(T, N)(U, V) = (\det U)^{k-l} \epsilon_{k,\mathbf{k}}(T, N)(V)$$

with $\epsilon_{k,\mathbf{k}}(T, N) = \epsilon_{k,\mathbf{k}}(T, N)(V) \in \mathbb{C}[V]_{(k-l, k-l, 0, 0)}$. Then g_N is expressed as

$$g_N(W) = \sum_{T \in \mathcal{H}_2(\mathbb{Z})} (\det U)^{k-l} \epsilon_{k,\mathbf{k}}(T, N) \mathbf{e}(\mathrm{tr}(TW)).$$

Now, for a positive integer m , let $T(m)$ be the element of \mathbf{L}_2 defined in Section 3. For a positive integer $m = p_1 \cdots p_r$ with p_i a prime number, we define the Hecke operator $T^{(m)} = T(p_1) \cdots T(p_r)$. We make the convention that $T^{(1)} = T(1)$. We note that $T^{(m)} = T(m)$ if p_1, \dots, p_r are distinct, but in general it is not. For each $m \in \mathbb{Z}_{>0}$ and $N \in \mathcal{H}_4(\mathbb{Z})_{>0}$, write $g_N|T^{(m)}(W)$ as

$$g_N|T^{(m)}(W) = \sum_{T \in \mathcal{H}_2(\mathbb{Z})_{>0}} (\det U)^{k-l} \epsilon_{k,\mathbf{k}}(m, T, N) \mathbf{e}(\mathrm{tr}(TW))$$

with $\epsilon_{k,\mathbf{k}}(m, T, N) \in \mathbb{C}[V]_{(k-l, k-l, 0, 0)}$.

Let $\mathcal{M}_{k,l} = M_k(\Gamma^{(2)})$ or $S_k(\Gamma^{(2)})$ according as $k = l$ or not, and let $\{F_j\}_{j=1}^d$ be a basis of $\mathcal{M}_{k,l}$ consisting of Hecke eigenforms. Furthermore write

$$F_j|T^{(m)}(z) = \lambda_{j,m}F_j(z).$$

Then the following proposition is a consequence of applying $T(m)$ to the formula in Corollary 6.2 with $n_1 = 2$ and $n_2 = 4$.

PROPOSITION 7.1. *Notation being as above, we have*

$$\epsilon_{k,\mathbf{k}}(m, T, N) = \sum_{j=1}^d \lambda_{j,m} a(T, F_j) B(F_j)$$

for any $N \in \mathcal{H}_4(\mathbb{Z})_{>0}$, $T \in \mathcal{H}_2(\mathbb{Z})_{>0}$ and $m \in \mathbb{Z}_{>0}$, where $B(F_j)$ is a certain element of $\mathbb{C}[V]_{(k-l, k-l, 0, 0)}$, and in particular we have

$$B(F_j) = \gamma_2 \mathcal{C}_{6,l}(F_j) \overline{a(N, [F_j]^{\mathbf{k}})}$$

if $F_j \in S_k(\Gamma^{(2)})$. Here, γ_2 is the rational number in Proposition 6.1.

We note that $\mathcal{C}_{8,l}(F) = \zeta(9-2l)\mathcal{C}_{6,l}(F)$ for a Hecke eigenform F in $S_k(\Gamma^{(2)})$. Hence by the above proposition, we have the following formula:

PROPOSITION 7.2. *For $N_1 \in \mathcal{H}_2(\mathbb{Z})_{>0}$, $N \in \mathcal{H}_4(\mathbb{Z})_{>0}$ let $e_m = \epsilon_{k,\mathbf{k}}(m, N_1, N)$. Let F be a Hecke eigenform in $S_k(\Gamma^{(2)})$ and $\{F_j\}_{j=1}^d$ a basis of $\mathcal{M}_{k,l}$ consisting of Hecke eigenforms such that $F_1 = F$. For positive integers m_1, \dots, m_d put $\Delta = \Delta(m_1, \dots, m_d) = \det(\lambda_{j,m_i})_{1 \leq i, j \leq d}$. Then,*

$$\Delta \gamma_2 \mathcal{C}_{8,l}(F) a(N_1, F) \overline{a(N, [F]^{\mathbf{k}})} = \zeta(9-2l) \begin{vmatrix} e_1 & \lambda_{1,2} & \dots & \lambda_{1,d} \\ \vdots & \vdots & \vdots & \vdots \\ e_d & \lambda_{d,2} & \dots & \lambda_{d,d} \end{vmatrix}.$$

COROLLARY 7.3. *Let the notation and the assumption as above. Let \mathfrak{p} be a prime ideal of $\mathbb{Q}(F)$ such that $p_{\mathfrak{p}} > 2k$. Suppose that \mathfrak{p} divides neither $\zeta(9-2l)$ nor $\begin{vmatrix} e_1 & \lambda_{1,2} & \dots & \lambda_{1,d} \\ \vdots & \vdots & \vdots & \vdots \\ e_d & \lambda_{d,2} & \dots & \lambda_{d,d} \end{vmatrix}$. Then, \mathfrak{p} does not divide $\mathcal{C}_{8,l}(F) a(N_1, F) \overline{a(N, [F]^{\mathbf{k}})}$.*

PROOF. By Proposition 3.2, Δ is an algebraic integer, and by the assumption, γ_2 is a \mathfrak{p} -unit. Thus the assertion holds. \square

The following lemma will be used in the next section.

LEMMA 7.4. *Let $N \in \mathcal{H}_4(\mathbb{Z})_{>0}$. Then for any $T \in \mathcal{H}_2(\mathbb{Z})_{>0}$ and a prime number p , we have the following recursion formula for $\epsilon_{k,\mathbf{k}}(m, T, N)$:*

$$\epsilon_{k,\mathbf{k}}(1, T, N) = \epsilon_{k,\mathbf{k}}(T, N),$$

and for $m > 1$,

$$\epsilon_{k,\mathbf{k}}(m, T, N) = \epsilon_{k,\mathbf{k}}(mp^{-1}, pT, N) + p^{2k-3} \epsilon_{k,\mathbf{k}}(mp^{-1}, T/p, N)$$

$$+ p^{k-2} \sum_{D \in \text{GL}_2(\mathbb{Z})U_p\text{GL}_2(\mathbb{Z})/\text{GL}_2(\mathbb{Z})} \epsilon_{k,\mathbf{k}}(mp^{-1}, T[D]/p, N),$$

where p is a prime factor of m and $U_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$.

PROOF. The assertion follows from [1, Exercise 4.2.10]. □

Let U and V be the matrices of variables stated above.

THEOREM 7.5. For $A_0 \in \mathcal{H}_2(\mathbb{Z})_{>0}$, $A_1 \in \mathcal{H}_4(\mathbb{Z})_{>0}$ and $R \in M_{2,4}(\mathbb{Z})$, put $r(R) = r(A_0, A_1, R) = \text{rank} \begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix}$ and

$$\mathcal{Z}(A_0, A_1, R, l) = \begin{cases} L(4-l, \chi \begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix}) & \text{if } r(A_0, A_1, R) = 6, \\ \zeta(7-2l) & \text{if } r(A_0, A_1, R) = 5, \\ \zeta(7-2l)L(3-l, \chi_{A_1}) & \text{if } r(A_0, A_1, R) = 4. \end{cases}$$

Moreover, put

$$P \left(\begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix} \right) (V) = (k-l)! \sum_{\substack{a,b,c \geq 0 \\ a+2b+2c=k-l}} \frac{(-1)^b 2^a}{a!b!c!} \left(l + c - \frac{3}{2} \right)_{a+b+c} \\ \times |R {}^tV|^a (|V A_1 {}^tV| |A_0|)^b \left| \begin{matrix} A_0 & R {}^tV/2 \\ V {}^tR/2 & V A_1 {}^tV \end{matrix} \right|^c.$$

Then

$$\epsilon_{k,\mathbf{k}}(A_0, A_1)(V) = \sum_{\substack{R \in M_{2,4}(\mathbb{Z}) \\ \begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix} \geq 0}} 2^{[r(R)+1]/2} \mathcal{Z}(A_0, A_1, R, l) P \left(\begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix} \right) (V) \\ \times \prod_p F_p^* \left(\begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix}, p^{l-r(R)-1} \right).$$

PROOF. Let $Q_{k-l,2,4}$ be the polynomial in Section 5.1.2. Then, by Lemma 5.5, we have

$$Q_{k-l,2,4} \left(\begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix}, U, V \right) = (\det U)^{k-l} P \left(\begin{pmatrix} A_0 & R/2 \\ {}^tR/2 & A_1 \end{pmatrix} \right) (V).$$

Thus by (E) in the proof of Proposition 6.6, Lemma 5.5, and Proposition 6.3, we prove the assertion. □

We have an explicit formula for $F_p(T, X)$ for any nondegenerate half-integral matrix T over \mathbb{Z}_p (cf. [35]), and an algorithm for computing it (cf. Lee [48]). Therefore, by using Proposition 7.2 and Theorem 7.5 we can compute the Fourier coefficients of the Klingen–Eisenstein series in question.

8. Main results.

For $l = 12, 16, 18, 22, 26$ let ϕ_l be a unique primitive form in $S_l(\mathrm{SL}_2(\mathbb{Z}))$, and for $l = 24, 28, 30, 32, 34, 38$ let ϕ_l^\pm be a unique primitive form in $S_l(\mathrm{SL}_2(\mathbb{Z}))$ such that $a(2, \phi^\pm) = a \pm b\sqrt{D}$ with $a \in \mathbb{Q}, b \in \mathbb{Q}_{>0}$, where D is the discriminant of $\mathbb{Q}(\phi^\pm)$. For each $(k, j) = (10, 4), (14, 2)$, or $(4, 24)$ let $G_{k,j}$ be a Hecke eigenform in $S_{(k+j,k)}(\Gamma^{(2)})$ uniquely determined up to constant multiple, and $\{G_{14,4}^+, G_{14,4}^-\}$ is a basis of $S_{(18,14)}(\Gamma^{(2)})$ consisting of Hecke eigenforms.

(1) Let $f = \phi_{22}$. Put $\mathbf{k} = (12, 12, 6, 6)$. Then, by Taibi [56] and the numerical table in [57], we have

$$\begin{aligned} S_{\mathbf{k}}(\Gamma^{(4)}) &= \langle \mathcal{A}_4^{(I)}(G_{10,4}) \rangle_{\mathbb{C}}, \\ S_{(12,12,6)}(\Gamma^{(3)}) &= \{0\}, \\ S_{12}(\Gamma^{(2)}) &= \langle \mathcal{J}_2(\phi_{22}) \rangle_{\mathbb{C}}. \end{aligned}$$

Hence we have $\widetilde{M}_{\mathbf{k}}(\Gamma^{(4)}) = \langle [\mathcal{J}_2(\phi_{22})]^{\mathbf{k}}, \mathcal{A}_4^{(I)}(G_{10,4}) \rangle_{\mathbb{C}}$. Then 41 is the only prime number which satisfies the assumptions in Conjecture 3.9, and it divides $\mathbf{L}(14, f)/\mathbf{L}(12, f)$.

Let $N = \begin{pmatrix} 1 & 1/2 & 0 & 1/2 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \\ 1/2 & 0 & 1/2 & 1 \end{pmatrix}$ and $N_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then, substituting V for $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 3 \end{pmatrix}$ in Theorem 7.5, by a computation with Mathematica [60]

$$\epsilon_{12,\mathbf{k}}(N_1, N) \left(\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 3 \end{pmatrix} \right) = \frac{-20874555}{28} \equiv 11 \pmod{41}.$$

Hence, applying Corollary 7.3 to $d = 1$ and $\lambda_{1,1} = 1$, we see that \mathfrak{p} does not divide $\mathcal{C}_{8,6}(\mathcal{J}_2(\phi_{22}))\overline{a(N, [\mathcal{J}_2(\phi_{22})]^{\mathbf{k}})a(N_1, \mathcal{J}_2(\phi_{22}))}$, and by Theorem 6.13 we prove the following theorem.

THEOREM 8.1. *There exists a Hecke eigenform G in $S_{(14,10)}(\Gamma^{(2)})$ such that*

$$\mathcal{A}_4^{(I)}(G) \equiv_{\mathrm{ev}} [\mathcal{J}_2(f)]^{\mathbf{k}} \pmod{41}.$$

COROLLARY 8.2. *Conjecture 3.9 holds for $(k, j) = (10, 4)$.*

We note that Harder’s conjecture for $(k, j) = (10, 4)$ has been already proved by Chenevier and Lannes [12].

(2) Let $f = \phi_{30}^+$ and $f' = \phi_{30}^-$ and put

$$\alpha = 4320 + 96\sqrt{51349} \quad \text{and} \quad \alpha' = 4320 - 96\sqrt{51349}.$$

Then

$$a(2, f) = \alpha, \quad a(3, f) = -552\alpha - 99180$$

and

$$a(2, f') = \alpha', \quad a(3, f') = -552\alpha' - 99180.$$

We also have

$$\lambda_{\mathcal{J}_2(f)}(T(2)) = \alpha + 49152, \quad \lambda_{\mathcal{J}_2(f')}(T(2)) = \alpha' + 49152,$$

and

$$\lambda_{\mathcal{J}_2(f)}(T(3)) = -552\alpha + 19032696, \quad \lambda_{\mathcal{J}_2(f')}(T(3)) = -552\alpha' + 19032696.$$

(2.1) Let $(k, j) = (14, 4)$. Then the prime number 4289 divides $N_{\mathbb{Q}(f)/\mathbb{Q}}(\mathbf{L}(18, f)/\mathbf{L}(16, f))$ and it splits in $\mathbb{Q}(f)$. Hence there exists prime ideals $\mathfrak{q}, \mathfrak{q}'$ of $\mathbb{Q}(f)$ such that $(4289) = \mathfrak{q}\mathfrak{q}'$ and \mathfrak{q} divides $\mathbf{L}(18, f)/\mathbf{L}(16, f)$. The prime ideal \mathfrak{q} is the only prime ideal which satisfies the assumptions in Conjecture 3.9. Put $\mathbf{k} = (16, 16, 6, 6)$. Let N and N_1 be as those in (1). For $V = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 3 \end{pmatrix}$, put $\epsilon_{16,\mathbf{k}}(N_1, N) = \epsilon_{16,\mathbf{k}}(N_1, N)(V)$, $\epsilon_{16,\mathbf{k}}(2N_1, N) = \epsilon_{16,\mathbf{k}}(2N_1, N)(V)$, and $e_i = \epsilon_{16,\mathbf{k}}(i, N_1, N)$ ($i = 1, 2$). Then

$$e_1 = \epsilon_{16,\mathbf{k}}(N_1, N), \quad e_2 = \epsilon_{16,\mathbf{k}}(2N_1, N) + 2^{14}\epsilon_{16,\mathbf{k}}(N_1, N)$$

and

$$\alpha_{16,\mathbf{k}}(N_1, N) = \begin{vmatrix} e_1 & 1 \\ e_2 & \lambda_{\mathcal{J}_2(f')}(T(2)) \end{vmatrix}.$$

By a computation with Mathematica [60], we have

$$\epsilon_{16,\mathbf{k}}(N_1, N) = 1744286277555/28672, \quad \epsilon_{16,\mathbf{k}}(2N_1, N) = 309108562779375/112,$$

and hence

$$\alpha_{16,\mathbf{k}}(N_1, N) = 405(-1114174584071 + 12920639093\sqrt{51349})/896.$$

Using Theorem 7.5, by a computation with Mathematica [60] we have

$$N_{\mathbb{Q}(f)/\mathbb{Q}}(\alpha_{16,\mathbf{k}}(N_1, N)) \equiv 2206 \pmod{4289}.$$

Hence, by Corollary 7.3, \mathfrak{p} does not divide $\mathcal{C}_{8,6}(\mathcal{J}_2(f))\overline{a(N, [\mathcal{J}_2(f)]^{\mathbf{k}})}a(N_1, \mathcal{J}_2(f))$. Moreover, by Taibi [56] and the numerical table in [57], we have

$$\begin{aligned} S_{\mathbf{k}}(\Gamma^{(4)}) &= \langle \mathcal{A}_4^{(I)}(G_{14,4}^+), \mathcal{A}_4^{(I)}(G_{14,4}^-) \rangle_{\mathbb{C}}, \\ S_{(16,16,6)}(\Gamma^{(3)}) &= \{0\}, \\ S_{16}(\Gamma^{(2)}) &= \langle \mathcal{J}_2(\phi_{30}^+), \mathcal{J}_2(\phi_{30}^-) \rangle_{\mathbb{C}}. \end{aligned}$$

The prime ideal \mathfrak{q} does not divide \mathfrak{D}_f and $[\mathcal{J}_2(\phi_{30}^+)]^{\mathbf{k}} \not\equiv [\mathcal{J}_2(\phi_{30}^-)]^{\mathbf{k}} \pmod{\mathfrak{q}}$. Hence, by Theorem 6.13 we prove the following theorem.

THEOREM 8.3. *There exists a Hecke eigenform G in $S_{(18,14)}(\Gamma^{(2)})$ such that*

$$\mathcal{A}_4^{(I)}(G) \equiv_{\text{ev}} [\mathcal{J}_2(f)]^{\mathbf{k}} \pmod{\mathfrak{q}}.$$

COROLLARY 8.4. *Conjecture 3.9 holds for $(k, j) = (14, 4)$.*

(2.2) Let $(k, j) = (4, 24)$ and put $\mathbf{k} = (16, 16, 16, 16)$. Then the prime number 97 divides $N_{\mathbb{Q}(f)/\mathbb{Q}}(\mathbf{L}(28, f)/\mathbf{L}(16, f))$ and it splits in $\mathbb{Q}(f)$. Hence there exist prime ideals $\mathfrak{q}, \mathfrak{q}'$ of $\mathbb{Q}(f)$ such that $(97) = \mathfrak{q}\mathfrak{q}'$ and \mathfrak{q} divides $\mathbf{L}(28, f)/\mathbf{L}(16, f)$. The prime ideal \mathfrak{q} is the only prime ideal which satisfies the assumptions in Conjecture 3.9. We have

$$M_{16}(T^{(2)}) = \langle E_{2,16}, [\phi_{16}]^{(16,16)}, \mathcal{J}_2(f), \mathcal{J}_2(f') \rangle_{\mathbb{C}}.$$

Let N be as that in (1), $N_1 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, and $N_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. Put $e_i = \epsilon_{16,\mathbf{k}}(i, N_1, N)$ ($i = 1, 2, 3, 4$). Then

$$\begin{aligned} e_1 &= \epsilon_{16,\mathbf{k}}(N_1, N), \\ e_2 &= \epsilon_{16,\mathbf{k}}(2N_1, N), \\ e_3 &= \epsilon_{16,\mathbf{k}}(3N_1, N) + 3^{14}\epsilon_{16,\mathbf{k}}(N_1, N), \\ e_4 &= \epsilon_{16,\mathbf{k}}(4N_1, N) + 2^{29}\epsilon_{16,\mathbf{k}}(N_1, N) + 3 \cdot 2^{14}\epsilon_{16,\mathbf{k}}(N_2, N) \end{aligned}$$

and

$$\alpha_{16,\mathbf{k}}(N_1, N) = \begin{vmatrix} e_1 & 1 & 1 & 1 \\ e_2 & \lambda_{\mathcal{J}_2(f')}(T(2)) & \lambda_{[\phi_{16}]^{(16,16)}}(T(2)) & \lambda_{E_{2,16}}(T(2)) \\ e_3 & \lambda_{\mathcal{J}_2(f')}(T(3)) & \lambda_{[\phi_{16}]^{(16,16)}}(T(3)) & \lambda_{E_{2,16}}(T(3)) \\ e_4 & (\lambda_{\mathcal{J}_2(f')}(T(2)))^2 & (\lambda_{[\phi_{16}]^{(16,16)}}(T(2)))^2 & (\lambda_{E_{2,16}}(T(2)))^2 \end{vmatrix}.$$

Then, by Lemma 7.4, we have $e_i = \epsilon(i, N_1, N)$ for $i = 1, 2, 3, 4$. Using Theorem 7.5, by a computation with Mathematica [60], we have

$$\begin{aligned} \epsilon_{16,\mathbf{k}}(N_1, N) &= 38740804007974226508744800778240/6232699579062017, \\ \epsilon_{16,\mathbf{k}}(2N_1, N) &= 8035873503466715618094093067152998400/6232699579062017, \\ \epsilon_{16,\mathbf{k}}(3N_1, N) &= -29430266109700665036971047394543222568960/6232699579062017, \\ \epsilon_{16,\mathbf{k}}(4N_1, N) &= 7060754204175435666580204417230810153615360/6232699579062017 \end{aligned}$$

and

$$\epsilon_{16,\mathbf{k}}(N_2, N) = 337608542664093039037162829831689850880/6232699579062017.$$

We also have

$$\begin{aligned} \lambda_{[\phi_{16}]^{(16,16)}}(T(2)) &= a(2, \phi_{16})(1 + 2^{14}) = 216(1 + 2^{14}), \\ \lambda_{[\phi_{16}]^{(16,16)}}(T(3)) &= a(3, \phi_{16})(1 + 3^{14}) = -3348(1 + 3^{14}), \\ \lambda_{E_{2,16}}(T(q)) &= (1 + q^{14})(1 + q^{15}) \quad \text{for } q = 2, 3. \end{aligned}$$

Hence by a simple computation we have

$$N_{\mathbb{Q}(f)/\mathbb{Q}}(\alpha_{k,\mathbf{k}}(N_1, N)) \not\equiv 0 \pmod{97}.$$

Hence by Corollary 7.3, \mathfrak{p} does not divide $\mathcal{C}_{8,16}(\mathcal{J}_2(f))a(N_1, \mathcal{J}_2(f))\overline{a(N, [\mathcal{J}_2(f)]^{\mathbf{k}})}$. The

prime ideal \mathfrak{q} does not divide \mathfrak{D}_f . Hence, by Theorem 6.13, there exists a Hecke eigenform F in $M_{16}(\Gamma^{(4)})$ such that

$$F \equiv_{\text{ev}} [\mathcal{J}_2(f)]^{\mathbf{k}} \pmod{\mathfrak{q}}.$$

To show that F is a lift of type $\mathcal{A}^{(I)}$, we classify the Hecke eigenforms in $M_{16}(\Gamma^{(4)})$ following [52] and [30].

PROPOSITION 8.5. *We have $\dim M_{16}(\Gamma^{(4)}) = 14$ and $\dim S_{16}(\Gamma^{(4)}) = 7$, and we have the following:*

- (1) *We can take a basis $\{h_i\}_{i=1}^7$ of $S_{16}(\Gamma^{(4)})$ such that*

$$h_i = \begin{cases} \mathcal{J}_4(\phi_{28}^+) & i = 1, \\ \mathcal{J}_4(\phi_{28}^-) & i = 2, \\ \mathcal{A}_4^{(II)}(\phi_{30}^+, G_{14,2}) & i = 3, \\ \mathcal{A}_4^{(II)}(\phi_{30}^-, G_{14,2}) & i = 4, \\ \mathcal{M}^{(I)}(\phi_{26}, \mathcal{J}_2(\phi_{30}^+)) & i = 5, \\ \mathcal{M}^{(I)}(\phi_{26}, \mathcal{J}_2(\phi_{30}^-)) & i = 6, \\ \mathcal{A}_4^{(I)}(G_{4,24}) & i = 7. \end{cases}$$

Moreover we have

$$\lambda_{h_i}(T(2)) = \begin{cases} 12960(67989 + 443\sqrt{18209}) & i = 1, \\ 12960(67989 - 443\sqrt{18209}) & i = 2, \\ -230400(1703 + 9\sqrt{18209}) & i = 3, \\ -230400(1703 - 9\sqrt{18209}) & i = 4, \\ 1175040(557 + \sqrt{51349}) & i = 5, \\ 1175040(557 - \sqrt{51349}) & i = 6, \\ 230400000 & i = 7. \end{cases}$$

- (2) *We can take a basis $\{h_i\}_{i=8}^{14}$ of $S_{16}(\Gamma^{(4)})^\perp$ such that*

$$h_i = \begin{cases} E_{4,16} & i = 8, \\ [\phi_{16}]^{\mathbf{k}} & i = 9, \\ [\mathcal{M}^{(I)}(\phi_{16}, \phi_{28}^+)]^{\mathbf{k}} & i = 10, \\ [\mathcal{M}^{(I)}(\phi_{16}, \phi_{28}^-)]^{\mathbf{k}} & i = 11, \\ [H_{16}^{(3)}]^{\mathbf{k}} & i = 12, \\ [\mathcal{J}_2(\phi_{30}^-)]^{\mathbf{k}} & i = 13, \\ [\mathcal{J}_2(\phi_{30}^+)]^{\mathbf{k}} & i = 14, \end{cases}$$

where $H_{16}^{(3)}$ is a unique tempered Hecke eigenform, up to constant multiple, in $S_{16}(\Gamma^{(3)})$. For the definition of tempered forms, see Appendix A.

Moreover, we have

$$\lambda_{h_i}(T(2)) = \begin{cases} 18022646021156865 & i = 8, \\ 118797996294360 & i = 9, \\ 4097(4414176 + 23328\sqrt{18209}) & i = 10, \\ 4097(4414176 - 23328\sqrt{18209}) & i = 11, \\ -471974400 & i = 12, \\ 33566721(53472 - \sqrt{51349}) & i = 13, \\ 33566721(53472 + \sqrt{51349}) & i = 14. \end{cases}$$

REMARK 8.6. We have $\mathcal{M}^{(I)}(\phi_{26}, J_2(\phi_{30}^\pm)) = \mathcal{M}^{(II)}(\phi_{30}^\pm, J_2(\phi_{26}))$.

By Proposition 8.5, we have

$$\lambda_{h_i}(T(2)) \not\equiv \lambda_{[J_2(f)]^k}(T(2)) \pmod{\mathfrak{q}}$$

for any $1 \leq i \leq 13$ such that $i \neq 7$. Hence F coincides with h_7 up to constant multiple. Hence we have the following theorem.

THEOREM 8.7. *There exists a Hecke eigenform G in $S_{(28,4)}(\Gamma^{(2)})$ such that*

$$\mathcal{A}_4^{(I)}(G)^k \equiv_{\text{ev}} [J_2(f)]^k \pmod{\mathfrak{q}}.$$

COROLLARY 8.8. *Conjecture 3.9 holds for $(k, j) = (4, 24)$.*

Appendix A. Proofs of Theorems 4.2 and 4.3.

In this appendix, we will give proofs of Theorems 4.2 and 4.3. These theorems are a simple application of Arthur’s endoscopic classification [3] to Siegel modular forms as in the book of Chenevier–Lannes [12, Section 8.5.1].

First we recall the explicit multiplicity formula for $\text{Sp}_n(\mathbb{Z})$. The following theorem is just a reformulation of [12, Theorem 8.5.2, Corollary 8.5.4].

THEOREM A.1 (explicit multiplicity formula for $S_{\mathbf{k}}(\text{Sp}_n(\mathbb{Z}))$). *Let $\mathbf{k} = (k_1, \dots, k_n)$ be a sequence of positive integers such that $k_1 \geq \dots \geq k_n \geq n + 1$.*

- (1) *We can associate each Hecke eigenform G in $S_{\mathbf{k}}(\text{Sp}_n(\mathbb{Z}))$ with its A -parameter ψ_G which is a formal sum*

$$\psi_G = \boxplus_{i=1}^t \pi_i[d_i],$$

where π and d_i satisfy the following conditions (a) to (f):

- (a) $\pi_i = \otimes_v \pi_{i,v}$ is an irreducible unitary cuspidal automorphic self-dual representation of $\text{PGL}_{n_i}(\mathbb{A}_{\mathbb{Q}})$;
- (b) d_i is a positive integer such that $\sum_{i=1}^t n_i d_i = 2n + 1$;

- (c) $\pi_{i,p}$ is unramified for any prime $p < \infty$;
- (d) if we denote the infinitesimal character of $\pi_{i,\infty}$ by $c_{i,\infty}$ which is a semisimple conjugacy class of $\mathfrak{sl}_{n_i}(\mathbb{R})$, and if we set

$$e_d = \text{diag} \left(\frac{d-1}{2}, \frac{d-3}{2}, \dots, -\frac{d-1}{2} \right) \in \mathfrak{sl}_d(\mathbb{R}),$$

then the eigenvalues of the semisimple conjugacy class

$$\bigoplus_{i=1}^t c_{i,\infty} \otimes e_{d_i}$$

in $\mathfrak{sl}_{2n+1}(\mathbb{R})$ are the distinct integers

$$k_1 - 1 > \dots > k_n - n > 0 > -(k_n - n) > \dots > -(k_1 - 1);$$

- (e) there exists $1 \leq i_0 \leq t$ such that $d_{i_0} = 1$, $n_{i_0} \equiv 1 \pmod{2}$, and $n_i d_i \equiv 0 \pmod{4}$ for any $i \neq i_0$;
- (f) the sign condition

$$\prod_{\substack{1 \leq j \leq t \\ j \neq i}} \epsilon(\pi_i \times \pi_j)^{\min\{d_i, d_j\}} = \begin{cases} (-1)^{\frac{n_i d_i}{4}} & \text{if } d_i \equiv 0 \pmod{2}, \\ (-1)^{|K_i|} & \text{if } d_i \equiv 1 \pmod{2} \end{cases}$$

holds for any $i \neq i_0$, where K_i is the set of odd indices $1 \leq j \leq n$ such that $k_j - j$ is an eigenvalue of $c_{i,\infty}$.

The A -parameter ψ_G is characterized by

$$L(s, G, \text{St}) = \prod_{i=1}^t \prod_{d=1}^{d_i} L^\infty \left(s + \frac{d_i + 1}{2} - d, \pi_i \right),$$

where the right-hand side is a product of the finite parts of the Godement–Jacquet L -functions.

- (2) Conversely, for any formal sum $\psi = \boxplus_{i=1}^k \pi_i[d_i]$ satisfying (a)–(f) above, there exists a Hecke eigenform G such that $\psi = \psi_G$.
- (3) For two Hecke eigenforms G_1, G_2 in $S_{\mathbf{k}}(\text{Sp}_n(\mathbb{Z}))$, the following are equivalent:
 - G_1 and G_2 are constant multiples of one another;
 - $L(s, G_1, \text{St}) = L(s, G_2, \text{St})$;
 - $\psi_{G_1} = \psi_{G_2}$.

Here, a formal sum means that it is an equivalence class defined so that $\boxplus_{i=1}^t \tau_i[d_i] \sim \boxplus_{i=1}^{t'} \tau'_i[d'_i]$ if $t = t'$ and if there exists a permutation $\sigma \in \mathfrak{S}_t$ such that $\pi'_i \cong \pi_{\sigma(i)}$ and $d'_i = d_{\sigma(i)}$ for any $1 \leq i \leq t$.

REMARK A.2. (1) In [12, Theorem 8.5.2], Chenevier and Lannes assumed [12, Conjecture 8.4.22]. As is written in the postface in that book, this conjecture has been proved by Arancibia, Mœglin and Renard [2].

- (2) Theorem A.1 (3) is a multiplicity one theorem. In [12, Corollary 8.5.4], not to use [12, Conjecture 8.4.22], the stronger assumption $k_1 > \dots > k_n > n + 1$ is assumed. Using [2], the same proof is available even when $k_1 \geq \dots \geq k_n \geq n + 1$.
- (3) By condition (d), π_i is an irreducible regular algebraic cuspidal self-dual automorphic representation of $\mathrm{PGL}_{n_i}(\mathbb{A}_{\mathbb{Q}})$ ([12, Definition 8.2.7]). As explained in [12, Theorem 8.2.17], thanks to numerous mathematicians, one can prove that such a representation satisfies the Ramanujan conjecture. Namely, for any p , all the eigenvalues of the Satake parameter of $\pi_{i,p}$ have absolute value 1. In particular, a Hecke eigenform G in $S_k(\mathrm{Sp}_n(\mathbb{Z}))$ satisfies the Ramanujan conjecture if and only if its A -parameter is of the form $\psi_G = \boxplus_{i=1}^t \pi_i[1]$. In this case we call G tempered.
- (4) By the purity lemma of Clozel [12, Proposition 8.2.13], the Langlands parameter of $\pi_{i,\infty}$ is completely determined by the eigenvalues of the infinitesimal character $c_{i,\infty}$. In particular, one can compute the Rankin–Selberg root number $\epsilon(\pi_i \times \pi_j)$ explicitly in terms of the eigenvalues of $c_{i,\infty}$ and $c_{j,\infty}$ by

$$\epsilon(\pi_i \times \pi_j) = \prod_{w_i > 0} \prod_{w_j > 0} (-1)^{1 + \max\{2w_i, 2w_j\}} \times \begin{cases} 1 & \text{if } i, j \neq i_0, \\ \prod_{w_i > 0} (-1)^{\frac{1+2w_i}{2}} & \text{if } i \neq j = i_0, \end{cases}$$

where w_i (resp. w_j) runs over the positive eigenvalues of $c_{i,\infty}$ (resp. $c_{j,\infty}$). Note that w_i and w_j are in $(1/2)\mathbb{Z}$ and that $2w_i \equiv d_i - 1 \pmod{2}$ for any (positive) eigenvalue w_i of $c_{i,\infty}$.

- (5) By [3, Theorem 1.5.3], we know that $\epsilon(\pi_i \times \pi_j) = 1$ if $d_i \equiv d_j \pmod{2}$. This is easily checked when $i, j \neq i_0$.
- (6) Theorem A.1 is an existence theorem. To construct a modular form G from a parameter ψ is a different problem.
- (7) If we were not to assume that $k_n > n$, the statement of theorem would be much more difficult. At least when the scalar weight case, i.e., when $k_1 = \dots = k_n = k$ with $k \leq n$, a similar theorem, in particular a multiplicity one theorem, would follow from a result of Mœglin–Renard [51].

To obtain several lifting theorems, we need the following proposition which comes from accidental isomorphisms.

PROPOSITION A.3. (1) *Suppose that $k > 0$ is even. For any primitive form f in $S_k(\mathrm{SL}_2(\mathbb{Z}))$, there exists an irreducible unitary cuspidal automorphic self-dual representation π_f of $\mathrm{PGL}_2(\mathbb{A}_{\mathbb{Q}})$ such that*

$$L\left(s + \frac{k-1}{2}, f\right) = L^\infty(s, \pi_f)$$

and such that the eigenvalues of the infinitesimal character of $\pi_{f,\infty}$ are $\pm(k-1)/2$.

- (2) Suppose that $j > 0$ is even and that $k \geq 4$. For any Hecke eigenform F in $S_{(k+j,k)}(\mathrm{Sp}_2(\mathbb{Z}))$, there exists an irreducible unitary cuspidal automorphic self-dual representation π_F of $\mathrm{PGL}_4(\mathbb{A}_{\mathbb{Q}})$ such that

$$L\left(s + \frac{2k + j - 3}{2}, F, \mathrm{Sp}\right) = L^\infty(s, \pi_F)$$

and such that the eigenvalues of the infinitesimal character of $\pi_{F,\infty}$ are $\pm(j+2k-3)/2, \pm(j+1)/2$.

- (3) Suppose that $k_1 \geq k_2 > 0$ are even. For any primitive forms $f_1 \in S_{k_1}(\mathrm{SL}_2(\mathbb{Z}))$ and $f_2 \in S_{k_2}(\mathrm{SL}_2(\mathbb{Z}))$, there exists an irreducible unitary cuspidal automorphic self-dual representation π_{f_1,f_2} of $\mathrm{PGL}_4(\mathbb{A}_{\mathbb{Q}})$ such that

$$L\left(s + \frac{k_1 + k_2 - 2}{2}, f_1 \otimes f_2\right) = L^\infty(s, \pi_{f_1,f_2})$$

and such that the eigenvalues of the infinitesimal character of $\pi_{f_1,f_2,\infty}$ are $\pm(k_1 + k_2 - 2)/2$ and $\pm(k_1 - k_2)/2$.

PROOF. (1) is well-known (see, e.g., [12, Proposition 9.1.5]). (2) is [12, Proposition 9.1.4]. (3) can be proved in a way similar to [12, Proposition 9.1.4]. \square

Now we explain Theorems 4.2 and 4.3.

PROOF OF THEOREM 4.2 (1). Let G be a Hecke eigenform in $S_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z}))$, and $\psi_G = \boxplus_{i=1}^t \pi_i[d_i]$ be its A -parameter. Here we make the convention that $\psi_G = \mathbf{1}_{\mathrm{GL}_1(A_{\mathbb{Q}})}[1]$ if $n = 0$. Let $F \in S_{(k+j,k)}(\mathrm{Sp}_2(\mathbb{Z}))$ be a Hecke eigenform with $k \geq 4$ and $j > 0$ even, and π_F be the irreducible unitary cuspidal automorphic self-dual representation of $\mathrm{PGL}_4(\mathbb{A}_{\mathbb{Q}})$ associated by F by Proposition A.3 (2). It suffices to show that under the assumptions of Theorem 4.2 (1), the parameter

$$\psi' = \psi_G \boxplus \pi_F[2d]$$

satisfies the conditions (a)–(f) of Theorem A.1 (1) with respect to $\mathbf{k}' = (k'_1, \dots, k'_{n+4d}) \in \mathbb{Z}^{n+4d}$ defined in Theorem 4.2 (1). The conditions (a), (b), (c) and (e) are obvious. The condition (d) follows from the definition of \mathbf{k}' . To check the sign condition (f), we will compute $\epsilon(\pi_i \times \pi_F)$. By Remark A.2 (5), we have $\epsilon(\pi_i \times \pi_F) = 1$ if d_i is even. When d_i is odd, any positive eigenvalue w_i of $c_{i,\infty}$ belongs to $\{k_1 - 1, \dots, k_n - n\}$ so that $j+1 < 2w_i < j+2k-3$. Hence when d_i is odd and $i \neq i_0$ so that $n_i \equiv 0 \pmod{4}$, we have

$$\begin{aligned} \epsilon(\pi_i \times \pi_F)^{\min\{d_i, 2d\}} &= \left(\prod_{w_i > 0} (-1)^{\max\{2w_i, j+2k-3\} + \max\{2w_i, j+1\}} \right)^{\min\{d_i, 2d\}} \\ &= \left(\prod_{w_i > 0} (-1)^{j+2k-3+2w_i} \right)^{\min\{d_i, 2d\}} = (-1)^{\frac{n_i}{2}(j+2k-3)\min\{d_i, 2d\}} = 1. \end{aligned}$$

Since the cardinality of the set K_i for ψ' is the same as the one for ψ_G , we obtain the sign condition for π_i . Also,

$$\begin{aligned} \prod_{j=1}^t \epsilon(\pi_F \times \pi_j)^{\min\{2d, d_j\}} &= \epsilon(\pi_F \times \pi_{i_0})^{\min\{2d, d_{i_0}\}} \\ &= (-1)^{\frac{n_{i_0}-1}{2}(j+2k-3) + \frac{1+(j+2k-3)}{2} + \frac{1+(j+1)}{2}} = (-1)^{n+k} \end{aligned}$$

since $n_{i_0} \equiv 2n + 1 \pmod 4$ and $j \equiv 0 \pmod 2$. Hence the sign condition for π_F is equivalent to $k \equiv n \pmod 2$. \square

The proofs of Theorem 4.2 (2) and Theorem 4.3 are similar. Let G and f be as in the statements, $\psi_G = \boxplus_{i=1}^t \pi_i[d_i]$ be the A -parameter of G . and π_f be the irreducible unitary cuspidal automorphic self-dual representation of $\mathrm{PGL}_2(\mathbb{A}_{\mathbb{Q}})$ associated to f . We have to check the conditions (a)–(f) in Theorem A.1 for

$$\psi' = \psi_G \oplus \pi_f[2d]$$

with $2d = n - 2$ in the proof of Theorem 4.2 (2). Only the condition (f) is non-trivial.

When G is in Theorem 4.2 (2), we claim that ψ_G is of the form $\pi_G[1]$ for some irreducible unitary cuspidal automorphic self-dual representation π_G of $\mathrm{PGL}_5(\mathbb{A}_{\mathbb{Q}})$. If not, by the condition (e) and by $n > 2$, we would have $\psi_G = \pi_1[1] \boxplus \mathbf{1}_{\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})}[1]$. In this case, the sign condition fails since $|K_1| = 1$. Since

$$\begin{aligned} \epsilon(\pi_f \times \pi_G) &= (-1)^{(1+\max\{2k-n-1, 2(k-1)\}) + (1+\max\{2k-n-1, 2(k-n)\}) + \frac{1+(2k-n-1)}{2}} \\ &= (-1)^{2(k-1) + (2k-n-1) + k - \frac{n}{2}} = (-1)^{k - \frac{n-2}{2}} = (-1)^{k + \frac{2d}{2}}, \end{aligned}$$

the sign condition for ψ' holds if and only if k is even.

When G is in Theorem 4.3, since

$$\epsilon(\pi_f \times \pi_j)^{\min\{2d, d_j\}} = \begin{cases} 1 & \text{if } j \neq i_0, \\ (-1)^k & \text{if } j = i_0 \text{ in case (1),} \\ (-1)^{k+n} & \text{if } j = i_0 \text{ in case (2)} \end{cases}$$

and the right-hand side of (f) for $\pi_f[2d]$ is $(-1)^d$, we can check the sign conditions. This completes the proofs of Theorems 4.2 and 4.3.

Appendix B. An explicit pullback formula.

Suppose that $k > n_1 + n_2 + 1$. Then for any $r \leq \min(n_1, n_2)$ and a Hecke eigenform $f \in S_{\rho_r}(\Gamma^{(r)})$, the Klingen–Eisenstein series $[f]_{\rho_r}^{\rho_{n_1}}(Z, U)$ and $[f]_{\rho_r}^{\rho_{n_2}}(W, V)$ become holomorphic modular forms, and we obtain more explicit results. The proof of the following theorem is independent of Böcherer’s argument. The proof here is a brute force but we still believe this way of calculation would be useful in some cases. For more conceptual description of a complete general exact pullback formula, see [28].

THEOREM B.1. *Notation being as in Theorem 5.6, suppose that $l > 0$. Then we have*

$$\begin{aligned} & (\mathbb{D}_{\lambda, n_1, n_2} E_{n_1+n_2, k}) \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix} \\ &= \sum_{r=2}^{\min(n_1, n_2)} c_r \sum_{j=1}^{d(r)} \frac{D(k, f_{r,j})}{(f_{r,j}, f_{r,j})} [f_{r,j}]_{\rho_r}^{\rho_{n_1}}(Z)(U) [\theta f_{r,j}]_{\rho_r}^{\rho_{n_2}}(W)(V), \end{aligned}$$

where c_r is a certain constant depending on $\mathbb{D}_{\lambda, n_1, n_2}$.

This has been already proved by Kozima [45] in more general setting. However he does not give explicit values of c_r in general. He gave in [45] one strategy how to calculate the constant c_r in the formula. Actually this calculation is difficult to execute in general, but in the next section, we explain how to do this when $\mathbb{D}_\lambda = Q_l(\partial_Z, U, V)$, where Q_l is defined by (2) before Lemma 5.5. Though this calculation is not necessary for proving our main results, it is interesting in its own right, and may have an application for computing exact standard L -values for $f \in S_{\rho_r}(\Gamma^{(r)})$.

B.0.1. Calculation of the constant. To calculate the constant c_r , we follow Kozima’s formulation in [45]. We fix $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_{n_1+n_2}(\mathbb{R})$ and put $\delta = \det(CZ + D)$. To obtain c_r for our differential operator \mathbb{D} we need, it is a key to calculate $\mathbb{D}(\delta^{-k})$ explicitly. For $Z \in \mathbb{H}_{n_1+n_2}$, we write

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ {}^t Z_{12} & Z_{22} \end{pmatrix} \quad \text{with } Z_{11} \in \mathbb{H}_{n_1}, Z_{22} \in \mathbb{H}_{n_2}, Z_{12} \in M_{n_1, n_2}(\mathbb{C}).$$

For an $m \times n_1$ matrix U and $m \times n_2$ matrix V of variables, define \mathbb{U} as before and we write

$$\partial_{U, Z} = (\partial_{ij}^U) = \mathbb{U}Z^t\mathbb{U} = \begin{pmatrix} UZ_{11} {}^tU & UZ_{12} {}^tV \\ V {}^tZ_{12} {}^tU & VZ_{22} {}^tV \end{pmatrix}.$$

We put $\Delta = (CZ + D)^{-1}C$ and

$$\Delta^{\mathbb{U}} = (\Delta_{ij}) = \mathbb{U}(CZ + D)^{-1}C {}^t\mathbb{U}.$$

It is well-known and easy to see that this is a symmetric matrix. We write blocks of Δ by

$$\Delta^{\mathbb{U}} = \begin{pmatrix} \Delta^{(11)} & \Delta^{(12)} \\ {}^t\Delta^{(12)} & \Delta^{(22)} \end{pmatrix},$$

where $\Delta^{(ij)}$ are $m \times m$ matrices. We consider a differential operator $\mathbb{D} = P(\partial_Z)$ which satisfies Condition 1. The following proposition is the same as Proposition 4.1 in page 241 in [45] except for the point that the realization of the representation is slightly different.

PROPOSITION B.2. *There exists a polynomial $Q(X)$ in the components of $m \times m$ matrix X such that*

$$\mathbb{D}(\delta^{-k}) = \delta^{-k}Q(\Delta^{(12)}).$$

The key point is that the polynomial does not contain components of $\Delta^{(11)}$ and $\Delta^{(22)}$.

Since the realization of our polynomials and Kozima's are different, we explain the relation. We fix $\lambda = (l, \dots, l, 0, \dots, 0)$ with $\text{depth}(\lambda) = m$. We denote by u_i and v_j the i -th row vector of U and the j -th row vector of V respectively and write $U = (u_i)$, $V = (v_j)$. Our polynomial is a polynomial in components of T , U and V . We write this as $P(T, U, V)$ to emphasize its dependence on U and V . We define polarization of P for each rows u_i, v_j in the usual way as follows. We prepare ml row vectors $\xi_{i\nu}$ ($1 \leq i \leq m, 1 \leq \nu \leq l$) and other ml vectors $\eta_{j\mu}$ ($1 \leq j \leq m, 1 \leq \mu \leq l$) of variables. We write $\xi = (\xi_{i\nu})$ and $\eta = (\eta_{j\mu})$ for short. For P we define P^* by

$$P^*(T, \xi, \eta) = l^{-2ml} \sum_{i,j=1}^l \frac{\partial^l}{\partial c_{i1} \cdots \partial c_{il}} \frac{\partial^l}{\partial d_{j1} \cdots \partial d_{jl}} P \left(T, \left(\sum_{\nu=1}^l c_{i\nu} \xi_{i\nu} \right), \left(\sum_{\mu=1}^l d_{j\mu} \eta_{j\mu} \right) \right) \Big|_{c_{i\nu}=d_{j\mu}=0}.$$

In other words, replacing u_i and v_j by $u_i = c_{i1}\xi_{i1} + \cdots + c_{il}\xi_{il}$ and $v_j = d_{j1}\eta_{j1} + \cdots + d_{jl}\eta_{jl}$ respectively in $P(T, U, V)$ and take coefficients of

$$\prod_{i,j=1}^m \prod_{\nu,\mu=1}^l c_{i\nu} d_{j\mu}.$$

Here by definition, the polynomial P^* is a multilinear polynomial in $\xi_{i\nu} T_{11}{}^t \xi_{j\mu}$, $\xi_{i\nu} T_{12}{}^t \eta_{j\mu}$, $\eta_{i\nu} T_{22}{}^t \eta_{j\mu}$ and it is homogeneous in the sense of Kozima. Since the polarization $P \rightarrow P^*$ commutes with $\Delta_{ij}(X)$ and $\Delta_{ij}(Y)$, the polynomial P^* is also pluriharmonic. The action of $A_1 \in \text{GL}(n_1, \mathbb{C})$ and $A_2 \in \text{GL}(n_2, \mathbb{C})$ is the same as P since we have $u_i A_1 = \sum_{\nu=1}^l c_{i\nu} x_{i\nu} A_1$ and $v_j A_2 = \sum_{\mu=1}^l d_{j\mu} \eta_{j\mu} A_2$. So if we use P^* instead of P , our formulation becomes Kozima's formulation. So we can use Kozima's Proposition 4.1 in [45], and the interpretation in our case is given in Proposition B.2 above.

Our next problem is to obtain Q in Proposition B.2.

For any row vector x, y of length n , we write

$$\partial[x, y] = x \partial_Z{}^t y = \sum_{1 \leq i, j \leq n} \frac{1 + \delta_{ij}}{2} x_i y_j \frac{\partial}{\partial z_{ij}}.$$

The following formulas are given in Kozima [45]. (See also [24] for a precise proof.) For any row vectors u_1, u_2, u_3, u_4 of length n and any functions f, g of Z , we have

- (D1) $\partial[u_1, u_2](fg) = (\partial[u_1, u_2]f)g + f(\partial[u_1, u_2]g),$
- (D2) $\partial[u_1, u_2](\delta^{-k}) = -k\delta^{-k}u_1\Delta{}^t u_2,$
- (D3) $\partial[u_1, u_2](u_3\Delta{}^t u_4) = -\frac{1}{2}((u_1\Delta{}^t u_3)(u_2\Delta{}^t u_4) + (u_1\Delta{}^t u_4)(u_2\Delta{}^t u_3)).$

By iterate use of these formulas, we can calculate the action of $P(U\partial_Z{}^t U)$ for any polynomial P . But actual calculation is a bit complicated. For our case, we have a following

result. The rest of this section is devoted to prove this theorem.

THEOREM B.3. *We assume that $m = 2$ and we define differential operator $\mathbb{D}_l = Q_l(\partial_Z, U, V)$ from \det^k to $\det^k \rho_{n_1, \lambda} \otimes \det^k \rho_{n_2, \lambda}$ for $\lambda = (l, l, 0, \dots, 0)$ by (2) before Lemma 5.5. We write $\mathbb{U}(CZ + D)^{-1}C^t\mathbb{U} = (\Delta_{ij})_{1 \leq i, j \leq 4}$. Then we have*

$$\mathbb{D}_l(\delta^{-k}) = \frac{1}{2^{2l}l!} (2k - 3)_l (2k - 1)_{2l} (\Delta_{13}\Delta_{24} - \Delta_{14}\Delta_{23})^{2l}.$$

In the notation before, we have $\det(\Delta^{(12)}) = \Delta_{13}\Delta_{24} - \Delta_{14}\Delta_{23}$.

A simple theoretical proof of Theorem B.3 is given in [28], but here we give an original proof prepared for the present paper. This proof consists of complicated combinatorial brute force calculations, and we believe such alternative proof is not useless.

To make it readable, we first explain rough idea of calculation, and then we give actual calculation. Define $F_i(T, U, V)$ as in Lemma 5.5. We also define

$$F_4(T, U, V) = \det(UT_{11} {}^tU), \quad F_5(T, U, V) = \det(VT_{22} {}^tV),$$

where T_{ij} are defined by (1) in Section 5. Then of course we have $F_2 = F_4F_5$. We write

$$\mathcal{F}_i = F_1(\partial_Z, U, V), \quad i = 1, 2, 3, 4, 5.$$

Here $\mathcal{F}_1, \mathcal{F}_4, \mathcal{F}_5$ are differential operators of order 2 and \mathcal{F}_3 of order 4. We put

$$\begin{aligned} \Delta^{\mathbb{U}} &= \mathbb{U}(CZ + D)^{-1}C^t\mathbb{U} = (\Delta_{ij}), & C_1 &= \Delta_{13}\Delta_{24} - \Delta_{14}\Delta_{23}, \\ C_4 &= \Delta_{11}\Delta_{22} - \Delta_{12}^2, & C_5 &= \Delta_{33}\Delta_{44} - \Delta_{34}^2, & C_2 &= C_4C_5, & C_3 &= \det(\Delta^{\mathbb{U}}). \end{aligned}$$

Now our strategy of calculation is as follows.

- (1) For any a, b, c , we see easily that $\mathcal{F}_1^a \mathcal{F}_2^b \mathcal{F}_3^c (\delta^{-k})$ is written as a product of δ^{-k} and a polynomial in Δ_{ij} by virtue of the formulas (D1), (D2), (D3). But in fact, more strongly, we can show that it is a product of δ^{-k} and a polynomial in C_1, C_2, C_3 that is a weighted homogeneous polynomial of total degree $a + 2b + 2c$ if we put $\deg(C_1) = 1, \deg(C_2) = 2, \deg(C_3) = 2$.
- (2) Here we can show that C_1, C_2 and C_3 are algebraically independent for generic g and $\Delta^{\mathbb{U}}$, so by virtue of Lemma B.2, we need only the coefficient of $C_1^{a+2b+2c}$ in $\mathcal{F}_1^a \mathcal{F}_2^b \mathcal{F}_3^c (\delta^{-k})$ to describe $\mathbb{D}_l(\delta^{-k})$. So we calculate these coefficients for all (a, b, c) and sum them up according to the explicit definition of \mathbb{D}_l .

Now we execute these calculations.

LEMMA B.4. *For any non-negative integer, we have*

$$\mathcal{F}_3(\delta^{-k}C_3^r) = \frac{(k + r - 1)(k + r)(2k + 2r - 3)(2k + 2r - 1)}{4} \delta^{-k}C_3^{r+1}.$$

PROOF. The operator \mathcal{F}_3 is a differential operator of order 4. There are many ways to prove Lemma B.4. One way is to use computer directly. Actually, by (D1), (D2), (D3), it is clear that

$$\mathcal{F}_3(\delta^{-k}C_3^r) = \delta^{-k}C_3^{r-4} \times P(r, \Delta_{ij}),$$

where $P(r, \Delta_{ij})$ is a polynomial in r of degree 4 whose coefficients are polynomials in Δ_{ij} that do not depend on r . So the calculation for $r = 0, \dots, 4$ is enough and executing these we obtain the above formula. An alternative way is to specialize Δ to the case $n_1 = n_2 = 2$ and $U = 1_2, V = 1_2, C = 1_4, D = 0_4$. Then we have $\Delta^U = Z^{-1}$ for $Z \in \mathbb{H}_4$. Then under this specialization on Δ^U and \mathcal{F}_3 , we have

$$\begin{aligned} \mathcal{F}_3(\delta^{-k}C_3^r) &= \det(\partial_Z)(\det(Z)^{-k-r}) \\ &= (k+r) \left(k+r+\frac{1}{2}\right) (k+r+1) \left(k+r+\frac{3}{2}\right) \det(Z)^{-k-r-1} \\ &= \frac{(k+r)(2k+2r+1)(k+r+1)(2k+2r+3)}{4} \delta^{-k}C_3^{r+1}. \end{aligned}$$

The second equality is nothing but the Cayley type identity for symmetric matrices ([11]). Since the calculation is formally the same, we get Lemma B.4. \square

Next we consider \mathcal{F}_1 and $\mathcal{F}_2 = \mathcal{F}_4\mathcal{F}_5$. Since $\mathcal{F}_1, \mathcal{F}_4, \mathcal{F}_5$ are differential operators of order 2, the operation of these on products of functions can be calculated if we have several fundamental operations on factors. To explain this, we assume that \mathcal{F} is a differential operator of homogeneous order 2 and define a bracket $\{A, B\}_{\mathcal{F}}$ by

$$\mathcal{F}(AB) = \mathcal{F}[A]B + A\mathcal{F}[B] + \{A, B\}_{\mathcal{F}}.$$

We have $\{B, A\}_{\mathcal{F}} = \{A, B\}_{\mathcal{F}}$. For the operator $\partial_1\partial_2$ where ∂_1 and ∂_2 are differential operators of the first order, we have

$$\{A, B\}_{\partial_1\partial_2} = (\partial_1A)(\partial_2B) + (\partial_2A)(\partial_1B),$$

so for general \mathcal{F} of order 2 and functions A, B, C , we have

$$\{A, BC\}_{\mathcal{F}} = \{A, B\}_{\mathcal{F}}C + \{A, C\}_{\mathcal{F}}B.$$

So for example, the operation of \mathcal{F} on a product $A_1 \cdots A_\nu$ of functions A_i can be calculated if we have $\mathcal{F}(A_i)$ and $\{A_i, A_j\}_{\mathcal{F}}$. More generally, for δ^{-k} , any functions A, B, C , and non-negative integers p, q, r and for a differential operator \mathcal{F} of second order, we can give the following general formula by repeating the above consideration.

LEMMA B.5.

$$\begin{aligned} F(\delta^{-k}A^pB^qC^r) &= F(\delta^{-k})A^pB^qC^r + pA^{p-1}B^qC^r\{\delta^{-k}, A\}_{\mathcal{F}} + qA^pB^{q-1}C^r\{\delta^{-k}, B\}_{\mathcal{F}} \\ &+ rA^pB^qC^{r-1}\{\delta^{-k}, C\}_{\mathcal{F}} + \delta^{-k} \left(pA^{p-1}B^qC^rF(A) + qA^pB^{q-1}C^rF(B) \right. \\ &+ rA^pB^qC^{r-1}F(C) + pqA^{p-1}B^{q-1}C^r\{A, B\}_{\mathcal{F}} + qrA^pB^{q-1}C^{r-1}\{B, C\}_{\mathcal{F}} \\ &+ pqA^{p-1}B^qC^{r-1}\{A, C\}_{\mathcal{F}} + \frac{p(p-1)}{2}A^{p-2}B^qC^r\{A, A\}_{\mathcal{F}} + \frac{q(q-1)}{2}A^pB^{q-2}C^r\{B, B\}_{\mathcal{F}} \\ &\left. + \frac{r(r-1)}{2}A^pB^qC^{r-2}\{C, C\}_{\mathcal{F}} \right). \end{aligned}$$

So for δ^{-k} , C_1, C_2, C_3 (and C_4, C_5 when necessary), and $\mathcal{F} = \mathcal{F}_1, \mathcal{F}_4, \mathcal{F}_5$, we list up all these fundamental data below. For $i = 1, 4, 5$, we write $\{*, *\}_{\mathcal{F}_i} = \{*, *\}_i$.

Next we consider \mathcal{F}_1 and \mathcal{F}_2 . We give *fundamental formulas* below.

$$\begin{aligned} \mathcal{F}_1(\delta^{-k}) &= \frac{k(2k-1)}{2}C_1\delta^{-k}, & \mathcal{F}_1(C_1) &= \frac{1}{2}C_2, & \mathcal{F}_1(C_2) &= 3C_1C_2, & \mathcal{F}_1(C_3) &= \frac{1}{2}C_1C_3, \\ \{\delta^{-k}, C_1\}_1 &= \frac{k}{2}(3C_1^2 + C_2 - C_3)\delta^{-k}, & \{\delta^{-k}, C_2\}_1 &= 4kC_1C_2\delta^{-k}, \\ \{\delta^{-k}, C_3\}_1 &= 2kC_1C_3\delta^{-k}, & \{C_1, C_1\}_1 &= \frac{1}{2}(2C_1^2 + 2C_2 - C_3), \\ \{C_1, C_2\}_1 &= C_2(3C_1^2 + C_2 - C_3), & \{C_1, C_3\}_1 &= \frac{1}{2}C_3(3C_1^2 + C_2 - C_3), \\ \{C_2, C_2\}_1 &= 8C_1C_2^2, & \{C_2, C_3\}_1 &= 4C_1C_2C_3, & \{C_3, C_3\}_1 &= 2C_1C_3^2, \\ \mathcal{F}_4(\delta^{-k}) &= \frac{k(2k-1)}{2}C_4\delta^{-k}, & \mathcal{F}_4(C_1) &= \frac{1}{2}C_4C_1, & \mathcal{F}_4(C_2) &= \frac{1}{2}C_4(4C_1^2 + 2C_2 - C_3), \\ \mathcal{F}_4(C_3) &= \frac{1}{2}C_4C_3, & \{\delta^{-k}, C_1\}_4 &= 2kC_4C_1\delta^{-k}, & \{\delta^{-k}, C_2\}_4 &= kC_4(C_1^2 + 3C_2 - C_3)\delta^{-k}, \\ \{\delta^{-k}, C_3\}_4 &= 2kC_4C_3\delta^{-k}, & \{C_1, C_1\}_4 &= 2C_4C_1^2, & \{C_1, C_2\}_4 &= C_4C_1(C_1^2 + 3C_2 - C_3), \\ \{C_1, C_3\}_4 &= 2C_4C_1C_3, & \{C_2, C_2\}_4 &= 2C_4C_2(2C_1^2 + 2C_2 - C_3), \\ \{C_2, C_3\}_4 &= C_4C_3(C_1^2 + 3C_2 - C_3), & \{C_3, C_3\}_4 &= 2C_4C_3^2, & \mathcal{F}_5(\delta^{-k}) &= \frac{k(2k-1)}{2}C_5, \\ \mathcal{F}_5(C_1) &= \frac{1}{2}C_5C_1, & \mathcal{F}_5(C_2) &= \frac{1}{2}C_5(4C_1^2 + 2C_2 - C_3), & \mathcal{F}_5(C_3) &= \frac{1}{2}C_5C_3, \\ \mathcal{F}_5(C_4) &= \frac{1}{2}(2C_1^2 - C_2 + C_3), & \{\delta^{-k}, C_1\}_5 &= 2kC_5C_1\delta^{-k}, \\ \{\delta^{-k}, C_2\}_5 &= kC_5(C_1^2 + 3C_2 - C_3)\delta^{-k}, & \{\delta^{-k}, C_3\}_5 &= 2kC_5C_3\delta^{-k}, \\ \{\delta^{-k}, C_4\}_5 &= k(C_1^2 + C_2 - C_3)\delta^{-k}, & \{C_1, C_1\}_5 &= 2C_5C_1^2, \\ \{C_1, C_2\}_5 &= C_5C_1(C_1^2 + 3C_2 - C_3), & \{C_1, C_3\}_5 &= 2C_5C_1C_3, \\ \{C_2, C_2\}_5 &= 2C_5C_2(2C_1^2 + 2C_2 - C_3), & \{C_2, C_3\}_5 &= C_5C_3(C_1^2 + 3C_2 - C_3), \\ \{C_3, C_3\}_5 &= 2C_5C_3^2, & \{C_1, C_4\}_5 &= C_1(C_1^2 + C_2 - C_3), \\ \{C_2, C_4\}_5 &= C_2(3C_1^2 + C_2 - C_3), & \{C_3, C_4\}_5 &= C_3(C_1^2 + C_2 - C_3). \end{aligned}$$

LEMMA B.6. *For a generic g, U, V such that Δ_{ij} are algebraically independent, three variables C_1, C_2 and C_3 are algebraically independent.*

PROOF. Assume that

$$\sum_{p,q,r} C(p, q, r)C_1^pC_2^qC_3^q = 0$$

for some constants $C(p, q, r)$ where the degree of C_1 is the smallest among such relations. If we put $\Delta_{14} = \Delta_{24} = 0$, then we have $C_1 = 0$, and $C_3 = \Delta_{13}^2\Delta_{22}\Delta_{44} + \dots$. Then, since C_2 does not contain any Δ_{13} , this means that $C(0, q, r) = 0$ for any q, r . So we may assume that

$$C_1 \sum_{p \geq 1, q, r} C(p, q, r) C_1^{p-1} C_2^q C_3^r = 0.$$

Since the polynomial ring in Δ_{ij} is UFD, we have $\sum_{p \geq 1, q, r} C(p, q, r) C_1^{p-1} C_2^q C_3^r = 0$. This contradicts to the assumption. \square

LEMMA B.7. *For any integers $a, b, c, p, q, r \geq 0$, there exists a polynomial $P(C_1, C_2, C_3)$ such that*

$$\mathcal{F}_1^a \mathcal{F}_2^b \mathcal{F}_3^c (\delta^{-k}) = \delta^{-k} P(C_1, C_2, C_3).$$

PROOF. By Lemma B.4, it is enough to assume that the left-hand side is $\mathcal{F}_1^a \mathcal{F}_2^b (\delta^{-k} C_3^{r+1})$. Here we have $\mathcal{F}_2 = \mathcal{F}_5 \mathcal{F}_4$. By the fundamental formulas, we see that for i, j with $1 \leq i < j \leq 3$, any of $\mathcal{F}_4(\delta^{-k}), \mathcal{F}_4(C_i), \{\delta^{-k}, C_i\}_4, \{C_i, C_j\}_4$ are $\delta^{-k} C_4$ times a polynomial in C_1, C_2, C_3 , so by Lemma B.5, we see that

$$\mathcal{F}_4(\delta^{-k} C_1^p C_2^q C_3^r) = \delta^{-k} C_4 P_1(C_1, C_2, C_3)$$

for some polynomial $P_1(x, y, z)$ in three variables. We have

$$\begin{aligned} \mathcal{F}_5(\delta^{-k} C_4 P_1(C_1, C_2, C_3)) &= C_4 \mathcal{F}_5(\delta^{-k} P_1(C_1, C_2, C_3)) + \mathcal{F}_5(C_4) \delta^{-k} P_1(C_1, C_2, C_3) \\ &\quad + \{C_4, \delta^{-k} P_1(C_1, C_2, C_3)\}_5. \end{aligned}$$

Here by the same reason as before, the first term is equal to

$$C_4 C_5 \delta^{-k} P_2(C_1, C_2, C_3) = \delta^{-k} C_2 P_2(C_1, C_2, C_3)$$

for some polynomial $P_2(x, y, z)$. Since $\delta^k \{C_4, \delta^{-k}\}_5, \{C_4, C_i\}_5$ for $i = 1, 2, 3$ and $\mathcal{F}_5(C_4)$ are polynomials in C_1, C_2, C_3 , we see that $\mathcal{F}_2(\delta^{-k} C_1^p C_2^q C_3^r)$ is δ^{-k} times a polynomial in C_1, C_2, C_3 . We can show inductively that the same is true for \mathcal{F}_2^b and $\mathcal{F}_1^a \mathcal{F}_2^b$, so we prove the lemma. \square

By Lemma B.2, we need only the power of C_1 part in the polynomial in C_1, C_2, C_3 , so we will study that.

We denote by $\mathfrak{C}_3 = C_3 \mathbb{C}[C_1, C_2, C_3]$ and $\mathfrak{C}_{23} = (C_2, C_3) \mathbb{C}[C_1, C_2, C_3]$ the ideals of $\mathbb{C}[C_1, C_2, C_3]$ generated by C_3 , and by C_2 and C_3 , respectively. We have the following result.

PROPOSITION B.8. (i) *If $c \geq 1$, then we have*

$$\delta^k \mathcal{F}_1^a \mathcal{F}_2^b \mathcal{F}_3^c (\delta^{-k}) \in \mathfrak{C}_3.$$

(ii) *When $c = 0$ in the above, we have*

$$\delta^k \mathcal{F}_1^a \mathcal{F}_2^b (\delta^{-k}) \equiv \frac{1}{2^a} b!(k)_{2b} \left(k - \frac{1}{2}\right)_b (k + 2b)_a (2k + 2b - 1)_a C_1^{a+2b} \pmod{\mathfrak{C}_{23}}.$$

To prove Proposition B.8, we prepare several lemmas.

LEMMA B.9. (i) For any integer $r \geq 1$, we have

$$\delta^k \mathcal{F}_2(\delta^{-k} C_1^p C_2^q C_3^r) \in \mathfrak{C}_3.$$

(ii) For any integer $b \geq 0$, we have

$$\delta^k \mathcal{F}_2^b(\delta^{-k}) \equiv \sum_{p=0}^b p! \binom{b}{p}^2 (k)_{2b} \left(k - \frac{1}{2}\right)_b \left(k + p - \frac{1}{2}\right)_{b-p} C_1^{2p} C_2^{b-p} \pmod{\mathfrak{C}_3}.$$

PROOF. We have

$$\mathcal{F}_4(\delta^{-k} C_1^p C_2^q C_3^r) = \delta^{-k} C_1^p C_2^q \mathcal{F}_4(C_3^r) + C_3^r \mathcal{F}_4(\delta^{-k} C_1^p C_2^q) + \{\delta^{-k} C_1^p C_2^q, C_3^r\}_4.$$

Since $\mathcal{F}_4(C_3) \in C_4 \mathfrak{C}_3$, $\{C_i, C_3\}_4 \in C_4 \mathfrak{C}_3$ for $i = 1, 2, 3$, and $\{\delta^{-k}, C_3\}_4 \in \delta^{-k} C_4 \mathfrak{C}_3$, we see that $\mathcal{F}_4(\delta^{-k} C_1^p C_2^q C_3^r) \in \delta^{-k} C_4 \mathfrak{C}_3$. In the same way, we can show that $\mathcal{F}_5(\delta^{-k} C_4 \mathfrak{C}_3) \subset \mathfrak{C}_3$. So we prove (i). Next we prove (ii) by induction. By direct calculation, we have

$$\delta^k \mathcal{F}_2(\delta^{-k}) = \frac{k(2k-1)(k+1)}{2} C_1^2 + \frac{k(2k-1)^2(k+1)}{4} C_2 - \frac{k(2k-1)^2}{4} C_3.$$

This is nothing but the case $b = 1$ of Lemma B.9 (2). Next we calculate $F_2(\delta^{-k} C_1^p C_2^q)$ in order to calculate $\mathcal{F}_2^b(\delta^{-k})$ inductively. We have

$$\begin{aligned} F_4(\delta^{-k} C_1^p C_2^q) &= F_4(\delta^{-k}) C_1^p C_2^q + \{\delta^{-k}, C_1^p C_2^q\}_4 + \delta^{-k} F_4(C_1^p C_2^q) \\ &= F_4(\delta^{-k}) C_1^p C_2^q + p C_1^{p-1} C_2^q \{\delta^{-k}, C_1\}_4 + q C_1^p C_2^{q-1} \{\delta^{-k}, C_2\}_4 \\ &\quad + \delta^{-k} (F_4(C_1^p) C_2^q + \{C_1^p, C_2^q\}_4 + C_1^p F_4(C_2^q)) \\ &= F_4(\delta^{-k}) C_1^p C_2^q + p C_1^{p-1} C_2^q \{\delta^{-k}, C_1\}_4 + q C_1^p C_2^{q-1} \{\delta^{-k}, C_2\}_4 \\ &\quad + \delta^{-k} \left(p F_4(C_1) C_1^{p-1} C_2^q + \frac{p(p-1)}{2} \{C_1, C_1\}_4 C_1^{p-2} C_2^q \right. \\ &\quad \quad \left. + p q C_1^{p-1} C_2^{q-1} \{C_1, C_2\}_4 + q C_1^p C_2^{q-1} F_4(C_2) \right. \\ &\quad \quad \left. + \frac{q(q-1)}{2} C_1^p C_2^{q-2} \{C_2, C_2\}_4 \right). \end{aligned}$$

So by the fundamental formulas, we have

$$\begin{aligned} F_4(\delta^{-k} C_1^p C_2^q) &= \delta^{-k} C_4 C_1^p C_2^{q-1} \times \left(q(k+p+2q) C_1^2 + \frac{(k+p+2q)(2k+2p+2q-1)}{2} C_2 \right. \\ &\quad \left. - \frac{1}{2} q(2k+2p+2q-1) C_3 \right). \end{aligned} \tag{4}$$

In the same way, we have

$$\begin{aligned} F_5(\delta^{-k} C_4 C_1^p C_2^q) &= \delta^{-k} C_1^p C_2^q \times \left((q+1)(k+p+2q+1) C_1^2 \right. \\ &\quad \left. + \frac{(k+p+2q+1)(2k+2p+2q-1)}{2} C_2 - \frac{(q+1)(2k+2p+2q-1)}{2} C_3 \right). \end{aligned} \tag{5}$$

By applying (4), we have

$$\begin{aligned} \delta^k F_2(\delta^{-k} C_1^p C_2^q) &\equiv q(k+p+2q)F_5(\delta^{-k} C_4 C_1^{p+2} C_2^{q-1}) \\ &\quad + \frac{(k+p+2q)(2k+2p+2q-1)}{2} F_5(\delta^{-k} C_4 C_1^p C_2^q) \pmod{\mathfrak{C}_3}, \end{aligned} \tag{6}$$

and by (5), we have

$$\begin{aligned} \delta^k F_2(\delta^{-k} C_1^p C_2^q) &= q^2(k+p+2q)(k+p+2q+1)C_1^{p+4} C_2^{q-1} \\ &\quad + (k+p+2q)(k+p+2q+1) \left((4p+2) + \left(2p^2 + pq + p + q - \frac{1}{2} \right) \right) C_1^{p+2} C_2^q \\ &\quad + \frac{1}{4}(k+p+2q)(k+p+2q+1)(2k+2p+2q-1)^2 C_1^p C_2^{q+1}. \end{aligned} \tag{7}$$

Now assume that the claim (ii) holds for $b - 1$. Then applying (7), we have (ii). This is of course a straight forward calculation, but this is a bit complicated so we give a precise proof. For the sake of simplicity, we put

$$x_{p,b} = p! \binom{b}{p}^2 (k)_{2b} \left(k - \frac{1}{2}\right)_b \left(k + p - \frac{1}{2}\right)_{b-p}.$$

To see the coefficient of $C_1^{2p} C_2^{b-p}$ in $\mathcal{F}_2(\mathcal{F}_2^{b-1}(\delta^{-k}))$, we should see the linear combination of $\delta^{-k} C_1^{2p-4} C_2^{b-p-1}$, $\delta^{-k} C_1^{2p-2} C_2^{b-p}$ and $\delta^{-k} C_1^{2p} C_2^{b-1-p}$ in $\mathcal{F}_2^{b-1}(\delta^{-k})$ and apply \mathcal{F}_2 on it and see the coefficient at $C_1^{2p} C_2^{b-p}$. We compare each term with $x_{p,b}$. First the coefficient at $C_1^{2p} C_2^{b-p}$ of $\delta^k \mathcal{F}_2(\delta^{-k} C_1^{2p-4} C_2^{b-p+1})$ is given by

$$(k+2b-2)(k+2b-1)(b-p+1)^2.$$

We must multiply $x_{b-1,p-2}$ to this. The product is given by

$$\begin{aligned} &\frac{(b-1)!^2}{(p-2)!(b-p)!^2} (k)_{2b} \left(k - \frac{1}{2}\right)_{b-1} \left(k + p - \frac{5}{2}\right) \left(k + p - \frac{3}{2}\right) \cdots \left(k + b - \frac{5}{2}\right) \\ &= e_1 x_{p,b} \end{aligned}$$

where we put

$$e_1 = \frac{p(p-1)(k+p-\frac{5}{2})(k+p-\frac{3}{2})}{b^2(k+b-\frac{3}{2})^2}.$$

Secondly, the coefficient at $C_1^{2p} C_2^{b-p}$ of $\delta^k \mathcal{F}_2(\delta^{-k} C_1^{2p-2} C_2^{b-p})$ is given by

$$\frac{1}{2}(k+2b-2)(k+2b-1)(2(2b-2p+1)k + (4b^2 - 6b - 4p^2 + 10p - 5)).$$

We must multiply $x_{b-1,p-1}$ to this. The result is $e_2 x_{b,p}$, where we put

$$e_2 = \frac{p(k+p-\frac{3}{2})((2b-2p+1)k + (2b^2 - 3b - 2p^2 + 5p - \frac{5}{2}))}{b^2(k+b-\frac{3}{2})^2}.$$

Finally, the coefficient at $C_1^{2p}C_2^{b-p}$ of $\delta^k \mathcal{F}_2(\delta^{-k}C_1^{2p}C_2^{b-1-p})$ is given by

$$\frac{1}{4}(k + 2b - 2)(k + 2b - 1)(2k + 2b + 2p - 3)^2.$$

Multiplying $x_{b-1,p}$ to this, we have $e_3x_{b,p}$, where we put

$$e_3 = \frac{(b - p)^2(k + b + p - \frac{3}{2})^2}{b^2(k + b - \frac{3}{2})^2}.$$

Since we easily see $c_1 + c_2 + c_3 = 1$, we prove (ii). □

LEMMA B.10. (i) *For any integer $r \geq 1$, we have*

$$\delta^k \mathcal{F}_1(\delta^{-k}C_1^pC_2^qC_3^r) \in \mathfrak{C}_3.$$

(ii) *For any integer $q \geq 1$, we have*

$$\delta^k \mathcal{F}_1(C_1^pC_2^q) \in \mathfrak{C}_{23}.$$

(iii) *For any non-negative integers a, p , we have*

$$\begin{aligned} \delta^k F_1(\delta^{-k}C_1^p) &= \frac{(k + p)(2k + p - 1)}{2}C_1^{p+1} + \frac{p(k + p)}{2}C_1^{p-1}C_2 - \frac{p(2k + p - 1)}{4}C_1^{p-1}C_3, \\ \delta^k \mathcal{F}_1^a(\delta^{-k}C_1^p) &\equiv \frac{(k + p)_a(2k + p - 1)_a}{2^a}C_1^{a+p} \pmod{\mathfrak{C}_{23}}. \end{aligned}$$

PROOF. Since $\mathcal{F}_1(C_3)$, $\delta^k\{\delta^{-k}, C_3\}_1$, $\{C_i, C_3\}_1$ are in \mathfrak{C}_3 for any $i = 1, 2, 3$, the assertion (i) is clear. Since $\mathcal{F}_1(C_2)$, $\delta^k\{\delta^{-k}, C_2\}_1$, $\{C_i, C_2\}_1$ are in \mathfrak{C}_{23} for any $i = 1, 2$, the assertion (ii) is clear. For (iii), the first assertion is obtained by direct calculation. The second assertion is shown by induction by using (i) and (ii). □

PROOF OF PROPOSITION B.8. The assertion (i) is clear from Lemma B.4 and Lemmas B.9 (i), B.10 (i). The assertion (ii) is obvious by Lemmas B.9 (ii) and B.10 (iii). So Proposition B.8 is proved. □

In order to prove Theorem B.3, we fix a non-negative integer l . In order to give $\mathbb{D}_l(\delta^{-k})$, we must sum up each contribution of $\mathcal{F}_1^a\mathcal{F}_2^b\mathcal{F}_3^c(\delta^{-k})$ such that $a + 2b + 2c = l$. By Proposition B.2 and Proposition B.8 (i), the term with $c \geq 1$ does not contribute to the final sum. So we assume $c = 0$ and $a + 2b = l$. We put

$$q_{a,b} = \frac{(-1)^b}{a!}(k)_l(2k + 2b - 1)_a \binom{k - \frac{1}{2}}{b} \binom{k - \frac{3}{2}}{a+b}.$$

By Proposition B.8, we see that this is the contribution from $\mathcal{F}_1^a\mathcal{F}_2^b(\delta^{-k})$ times the coefficient of $F_1^aF_2^b$ in the definition of \mathbb{D}_l , noting that

$$(k)_{2b}(k + 2b)_a = (k)_l.$$

What we want to calculate is $q_0 = \sum_{a+2b=l} q_{a,b}$. Denote by $[l/2]$ the maximum integer which does not exceed $l/2$. To calculate Q inductively, for any integer b such that $0 \leq b \leq [l/2]$, we put

$$q_b = \sum_{0 \leq b \leq b_0 \leq [l/2]} q_{l-2b_0, b_0}.$$

LEMMA B.11. *The notation being as above, for any b with $0 \leq b \leq [l/2]$, we have*

$$q_b = \frac{2(k+b-1)(2k+2l-2b-3)}{(2k+l-2)(2k+l-3)} q_{a,b}. \tag{8}$$

PROOF. We prove this by induction from $b = [l/2]$ to $b = 0$. First we show this for $b = [l/2]$. By definition, we have $q_b = q_{l-2b,b}$, so the problem is if the coefficient of the right-hand side of (8) is 1. For even l , we have $[l/2] = l/2$ and for odd l we have $[l/2] = (l-1)/2$, and in both cases we have

$$2(k + [l/2] - 1)(2k + 2l - 2[l/2] - 3) = (2k + l - 2)(2k + l - 3).$$

So the assertion is clear for $b = [l/2]$. Now assume that the claim holds for some $b \leq [l/2]$ and we calculate q_{b-1} . Calculating the ratio $q_{a,b}/q_{a+2,b-1}$, we have

$$\begin{aligned} q_{b-1} &= q_b + q_{a+2,b-1} = q_{a+2,b-1} \left(1 - \frac{(l-2b+1)(l-2b+2)}{(2k+l-2)(2k+l-3)} \right) \\ &= \frac{2(k+b-2)(2k+2l-2b-1)}{(2k+l-2)(2k+l-3)} q_{l-2b+2,b-1}. \end{aligned}$$

So the claim holds also for $b-1$. □

PROOF OF THEOREM B.3. By Lemma B.11, we have

$$q_0 = \frac{2(k-1)(2k+2l-3)}{(2k+l-2)(2k+l-3)} \times \frac{(k)_l(2k-1)_l(k-\frac{3}{2})_l}{l!}.$$

Here we have

$$(k)_l \left(k - \frac{3}{2} \right)_l (2k+2l-3) = 2^{-2l} (2k-3)(2k-1)_{2l},$$

and

$$2(k-1) \frac{(2k-1)_l}{(2k+l-2)(2k+l-3)} = (2k-2)(2k-1)_{l-2},$$

where we define $(x)_{-1} = 1/(x-1)$ and $(x)_{-2} = 1/(x-1)(x-2)$. So we have

$$q_0 = \frac{1}{2^{2l} l!} (2k-3)_l (2k-1)_{2l}.$$

So we prove Theorem B.3. □

B.0.2. Explicit pullback formula. Based on the results in the last subsection, we can write down the pullback formula for the differential operator \mathbb{D}_l in general for any $n = n_1 + n_2$ with $2 \leq \min(n_1, n_2)$ and $\lambda = (l, l, 0, \dots, 0)$. We use Kozima’s formula in [45, p.247]. We define $Q(X)$ as in Proposition B.2 and Theorem 3.2. Then by Theorem B.3, we have

$$Q(\Delta^{(12)}) = q_0 C_1^{2l}.$$

For an $n \times n$ symmetric matrix, we write a block decomposition as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ {}^t T_{12} & T_{22} \end{pmatrix},$$

where T_{11} and T_{22} are $n_1 \times n_1$ and $n_2 \times n_2$ matrices, respectively. We define a polynomial $\Omega(T)$ in t_{ij} by

$$\Omega(T) = Q(UT_{12} {}^t V).$$

Then we have $\Omega((CZ + D)^{-1}C) = Q(\Delta^{(12)})$. For any $r \leq \min(n_1, n_2)$, we define $n_1 \times n_2$ matrix by $\begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}$, and by abuse of language, we denote this also by 1_r . As in Kozima, we are allowed to write $\Omega(T)$ for $T_{12} = 1_r$ as

$$\Omega \begin{pmatrix} * & 1_r \\ * & * \end{pmatrix},$$

not specifying $*$, since this does not depend on $*$ part by definition. Now we put

$$R_r = \sum_{1 \leq i < j \leq r} (u_{1i}u_{2j} - u_{1j}u_{2i})(v_{1i}v_{2j} - v_{1j}v_{2i}).$$

Then for $\lambda = (l, l, 0, \dots, 0)$, we have

$$\Omega \begin{pmatrix} * & 1_r \\ * & * \end{pmatrix} = q_0 \times R_r^l.$$

We consider two isomorphic realizations of the representation $\det^k \rho_{r,\lambda}$, one is on the space generated by bideterminants in u_{ij} with $i = 1, 2, j \leq r$ and the other is on the space generated by bideterminants in v_{ij} with $i = 1, 2, j \leq r$. We denote the former representation space by V_*^r and the latter by V_{*r} . We identify these representation spaces of $\text{GL}_r(\mathbb{C})$ on U variables and V variables by mapping u_{ij} to v_{ij} . For $v_* \in V_{*r}$, we denote by v^* the corresponding element in V_*^r . Now we define

$$S_r = \{S \in M_r(\mathbb{C}) \mid S = {}^t S, 1_r - \bar{S}\bar{S} > 0\},$$

where $* > 0$ means that the matrix is positive definite. We define a linear map from V_{*r} to V_{r*} by

$$\psi(v_*) = \int_{S_r} \left\langle \rho_r(1_r - \bar{S}S)v^*, \Omega \begin{pmatrix} * & 1_r \\ * & * \end{pmatrix} \right\rangle \det(1_r - \bar{S}S)^{-r-1} dS,$$

where $\rho_r = \det^k \rho_{r,\lambda}$ and $dS = \prod_{h \leq j} dx_{hj} dy_{hj}$ for $S = X + iY$ with $X = (x_{hj})$, $Y = (y_{hj}) \in M_r(\mathbb{R})$. Then we know that $\psi(v_*) = \varphi v_*$ for some constant φ . Then by the result of [45] in page 247, we have

$$c_r = 2^{r(r+1)+1-(rk+2l)} i^{rk+2l} \cdot \varphi.$$

Here φ obviously depends on the inner product. We can explicitly calculate the inner product $\langle *, * \rangle_0$ defined before for the necessary quantity.

LEMMA B.12. *We have*

$$\langle v^*, R_r^l \rangle_0 = (l + 1)! l! v_*.$$

PROOF. For the proof, we quote [11, Theorem 2.16]. Let $X = (x_{ij})$ be an $m \times r$ matrix of variables with $m \leq r$, and let $\partial = (\frac{\partial}{\partial x_{ij}})$. Let A and B be $r \times m$ constant matrices. Then that theorem gives the following general formula.

$$\det(\partial \times B)(\det(XA))^s = \det({}^tAB)(s)_m \det(XA)^{s-1}.$$

So if we put $m = 2$ and assume X and tA to be $2 \times r$ matrices consisting of the first r columns of U and V respectively, and $B = ({}_{0, -2, 2})$, then the above formula means

$$\frac{\partial^2 R_r^l}{\partial u_{11} \partial u_{22}} - \frac{\partial^2 R_r^l}{\partial u_{12} \partial u_{21}} = l(l + 1)(v_{11}v_{22} - v_{12}v_{21})R_r^{l-1}.$$

In the same way we have

$$\frac{\partial R_r^l}{\partial u_{1p} \partial u_{2q}} - \frac{\partial R_r^l}{\partial u_{1q} \partial u_{2p}} = l(l + 1)(v_{1p}v_{2q} - v_{1q}v_{2p})R_r^{l-1}.$$

So iterating these operations l times, we have the assertion. □

By the above results, it is natural to use here the inner product

$$\langle *, * \rangle_l = \langle *, * \rangle_0 / (l + 1)! l!. \tag{9}$$

The remaining part is the following integral

$$I_r = \int_{S_r} \rho_r (1_r - \bar{S}S) \det(1_r - \bar{S}S)^{-r-1} dS,$$

where $\rho_r = \det^k \rho_{r,\lambda}$, to which a dominant integral weight $(k + l, k + l, k, \dots, k)$ corresponds. By Kozima [44, Lemma 2] (and also by [19], [7], [9], [58]) we have

$$I_r = \frac{2^r \pi^{r(r+1)/2}}{\prod_{\nu=2}^4 (2k + 2l - \nu) \prod_{\mu=1}^2 \prod_{\nu=3}^r (2k + l - \mu - \nu) \prod_{3 \leq \mu \leq \nu \leq r} (2k - \mu - \nu)}.$$

THEOREM B.13. *We assume that k is even with $k > n + 1$. Assumption being the same as above, taking the inner product $\langle *, * \rangle_l$ as in (9), the constants c_r for $2 \leq r \leq \min(n_1, n_2)$ in Theorem 5.6 are given by*

$$c_r = \frac{2^{(r+1)^2 - (rk+2l)} (-1)^{rk/2+l} \pi^{r(r+1)/2} (2k-3)_l (2k-1)_{2l}}{2^{2l} l! \prod_{\nu=2}^4 (2k+2l-\nu) \prod_{\mu=1}^2 \prod_{\nu=3}^r (2k+l-\mu-\nu) \prod_{3 \leq \mu \leq \nu \leq r} (2k-\mu-\nu)}.$$

In particular, we have

$$c_2 = \frac{2^{9-2(k+2l)} (-1)^{k+l} \pi^3 (2k-3)_l (2k-1)_{2l-3}}{l!}.$$

We note here that since we assumed that k is even, the number $rk/2$ is an integer.

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