On explicit forms of various zeta functions of quadratic forms

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There are several zeta functions associated with a quadratic form, or a vector space of quadratic forms, partly defined rather classically by Epstein or Siegel or Koecher and Maass, and partly defined in the theory of the prehomogeneous vector spaces. In recent years, we found that the explicit shapes of these zeta functions are unexpectedly simple and gave very concrete formulas. In this short note, we shall illustrate this kind of results for three kinds of general zeta functions with some related results. Some part of these results is a joint work with Hiroshi Saito, or some with Hidenori Katsurada.

1 Zeta functions of symmetric matrices

The zeta functions associated with the vector space of symmetric matrices were defined by Shintani as a part of zeta functions of prehomogeneous vector spaces invented by Mikio Sato. These are very interesting zeta functions, partly because their values at negative integers give the part of the dimension formula of Siegel modular forms (that is, the contribution of central unipotent conjugacy classes). This fact was proved by Shintani in 1976. But except for the case of degree one or two, concrete special values of the zeta functions were not known until 1992 (cf.citeibusaitoduke). Many people imagined that the special values should be simple objects but it was overlooked for a long time that the zeta functions themselves are simple objects. Here we explain a part of our results. (As a whole result, cf. [4].) For any ring R, we denote by $\mathrm{Sym}_n(R)$ the set of $n \times n$ symmmetric matrices, and denote by L^* the lattice of half integral symmetric matrices, that is,

$$L_n^* = \{x = (x_{ij}) \in \mathrm{Sym}_n(\mathbb{Q}); x_{ij} \in (1/2)\mathbb{Z},\ x_{ii} \in \mathbb{Z}\}.$$

We denote by $L_n^{*,+}$ the set of positive definite matrices in L_n^*. For any $x \in L_n^{*,+}$, we put

$$\mathrm{Aut}(x) = \{\gamma \in GL_2(\mathbb{Z});\ {}^t gxg = x\}.$$

This is obviously a finite set, and we denote its cardinality by $\#(\mathrm{Aut}(x))$. The group $GL_n(\mathbb{Z})$ of unimodular matrices acts on $L^{*,+}$ by $x \to {}^t gxg$. We define the zeta function associated with $L_n^{*,+}$ by

$$\zeta(s, L_n^{*,+}) = \sum_{x \in L_n^{*,+}/GL_n(\mathbb{Z})} \frac{1}{\det(x)^s \#(\mathrm{Aut}(x))}.$$

To describe this function explicitly, we introduce several notations. We denote by $\zeta(s)$ the Riemann zeta function. For each non-negative integer m, we denote by B_m the Bernoulli number defined by:

$$\frac{te^t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

For each fundamental discriminant d_K of a quadratic field K over $\mathbb{Q}$, we denote by χ_{d_K} the Kronecker symbol of $\mathbb{Q}(\sqrt{d_K})$: $\chi_{d_K}(a) = \left(\frac{d_K}{a}\right)$. We include the case $d_K = 1$ and in this case χ_{d_K} means the trivial (primitive) character. We denote by $L(s, \chi_{d_K})$ the Dirichlet L-function. When n is even, we need two Dirichlet series $D_n^*(s)$ and $D_n(s)$ defined below. Using Cohen's notation, for each non negative integer d, we define a rational number $H(\frac{n}{2}, d)$ as follows: When $(-1)^{n/2}d \equiv 2$ or $3 \bmod 4$, put $H(\frac{n}{2}, d) = 0$. When $d = 0$, put $H(\frac{n}{2}, 0) = \zeta(1-n)$. When $(-1)^{n/2}d \equiv 1$ or $0 \bmod 4$, then $(-1)^{n/2}d = d_K f^2$ for some positive integer f and the fundamental discriminant d_K or $d_K = 1$. In this case, we put

$$H(\frac{n}{2}, d) = L(1 - \frac{n}{2}, \chi_{d_K}) \sum_{m|f} \mu(m)\chi_{d_K}(m) m^{\frac{n}{2}-1} \sigma_{n-1}(f/m).$$

Then, the Dirichlet series we need are defined by:

$$D_n^*(s) = \sum_{d=1}^{\infty} H(\frac{n}{2}, d) d^{-s}.$$

The following theorem is a joint work with Hiroshi Saito.

Theorem 1.1 ([4]) *When n is an odd integer $n \geq 3$, we get*

$$\begin{aligned}
\zeta(s, L_n^{*,+}) &= \frac{2^{(n-1)s}|B_2 B_4 \cdots B_{n-1}|}{2^{n-1}(\frac{n-1}{2})!} \\
&\times\{\zeta(s-\frac{n-1}{2})\zeta(2s-1)\zeta(2s-3)\cdots\zeta(2s-n+2) \\
&+(-1)^{(n^2-1)/8}\zeta(s)\zeta(2s-2)\zeta(2s-4)\cdots\zeta(2s-n+1)\}.
\end{aligned}$$

When n is an even integer $n \geq 4$, we get

$$\begin{aligned}
\zeta(s, L_n^{*,+}) &= \frac{(-1)^{[\frac{n}{4}]}2^{ns}|B_2 B_4 \cdots B_{n-2}|}{2^{n-1}(\frac{n-2}{2})!} \\
&\times\{D_n^*(s)\zeta(2s-2)\zeta(2s-4)\cdots\zeta(2s-n+2) \\
&+\delta_{n,4}\times\frac{2|B_{n/2}|}{n}\zeta(2s-1)\zeta(2s-3)\cdots\zeta(2s-n+1)\}
\end{aligned}$$

where $\delta_{n,4} = 0$, or 1 if $n \equiv 2 \bmod 4$ or $n \equiv 0 \bmod 4$, respectively.

The proof consists of very long calculation, using a kind of mass formula, Jordan splitting of local quadratic forms, a kind of q-analogue, and careful reconstruction of the global zeta functions from the local data. We omit the details.

As a corollary, we can write values of $\zeta(1-m, L_n^{*,+})$ for all positive integers m explicitly by using the usual Bernoulli numbers (cf. [4] II). This leads us to an explicit conjecture on dimensions of Siegel modular forms of any degree belonging to torsion free congruence subgroup.

2 Zeta functions of an indefinite quadratic form

In the previous section, we explained the results on zeta functions associated with vector spaces of quadratic forms. In this section, we treat another type of zeta function associated with a single quadratic form. We take a non-degenerate $m \times m$ half integral symmetric matrix S of size bigger than 3 with signature (p, q), that is, with p positive and q negative eigenvalues. So, $m = p + q$. We define a cone $\Omega(S)$ by

$$\Omega(S) = \{x \in \mathbb{R}^m; S[x] > 0\},$$

where we use the traditional notation ${}^t xSx = S[x]$. A zeta function associated with S was defined by Siegel.

To define a zeta function associated with S, there are several difficult points. Firstly, for natural number n, there are infinitely many solutions of integral vectors $x \in \mathbb{Z}^m$ such that $S[x] = n$. In most cases, you can overcome this in the following way. We put $\Gamma(S) = \{g \in GL_m(\mathbb{Z});\ {}^t gSg = s\}$. By reduction theory, we know that the number of $\Gamma(S)$-orbits of the solutions is finite. We defnote by $\Gamma(S, x)$ the stabilizer of x in $\Gamma(S)$. We put $O(S) = \{g \in GL_m(\mathbb{R});\ {}^t gSg = S\}$ and $O(S, x) = \{g \in O(S); gx = x\}$. If S is positive definite, the number of solution of $S[x] = n$ can be also written as

$$\sum_y \frac{\#(\Gamma(S))}{\#(\Gamma(S, y))},$$

where y runs over representatives of $\Gamma(S)$ orbit of solutions of $S[x] = n$. So, in our case where S is indefinite, one idea is to replace $\#(\Gamma(S))$ or $\#(\Gamma(S, y))$ in the above definite case by $\mathrm{vol}(O(S)/\Gamma(S))^{-1}$ or $\mathrm{vol}(O(S, y)/\Gamma(S, y))^{-1}$ respectively, if each is finite. In some rare cases, either of these volumes might be infinite, but we omit such pathological cases. Then, roughly speaking, we can define the zeta function by

$$\zeta(s, S) = \mathrm{vol}(O(S)/\Gamma(S))^{-1} \sum_{x \in \Omega(S)/\Gamma(S)} \mathrm{vol}(O(S, x)/\Gamma(S, x)) S[x]^{-s}.$$

But still, this definition has not been well defined yet as we explain below. We must fix a measure to give each volume above. Of course we should take the Haar measure of $O(S, x)$, but this is determined only up to constant and we must give a unified way how to choose this for each x simultaneously. Here it is essential that our vector space is a prehomogeneous vector space. If we put $G = GL(1) \times O(S)$, then $G(\mathbb{C})$ acts transitively on $\mathbb{C}^m - \mathcal{S}$, where $\mathcal{S} = \{x \in \mathbb{C}^m; S[x] = 0\}$. Here $GL(1)$ acts as the scalar multiple. Now, we fix a Haar measure of $G(\mathbb{R})$. For each $x \in \mathbb{R}^m$, $G(\mathbb{R})/G(\mathbb{R}, x)$ is identified with an open subset of $\mathbb{R}^m$, where $G(\mathbb{R}, x)$ is the stabilizer of x. On $\mathbb{R}^m$ we can fix a G-invariant measure, still up to constant. For example, we can take $S[x]^{-m/2}\, dx$ in our case. So, if we fix each G invariant measure dg on $G(\mathbb{R})$ and $\mathbb{R}^m$, we can define the measure dg_x uniquely on each $G(\mathbb{R}, x)$ so that the quotient measure dg/dg_x is the measure on $\mathbb{R}^m$. Still dg_x depends on the choice of dg and the measure of $\mathbb{R}^m$, but there is no ambiguity depending on the choice of x. Actually Siegel's definition is more complicated, since he often gave measures as concretely as possible, but the essence may be explained in the above way. Also, we must fix measures of G and $\mathbb{R}^m$ to give a real definition of zeta functions specifying constant. Since this is more involved, we omit the explanation in detail. We just annouce here that we take Siegel's definition of $\zeta(s,S)$ exactly as in [7]. (Please note that Siegel himself often used the different definitions in various papers.)

Now, our theorem can be stated.

Theorem 2.1 *We take an $m \times m$ half integral indefinite symmetric matrix S. We assume that m is even and $m \geq 4$. Besides, when $m = 4$, we exclude the case where S is either a zero form or* $\det(S)$ *is a square. Then the zeta function $\zeta(s,S)$ is a $\mathbb{Q}$ linear combination of the following zeta functions*

$$a^{-s}L(s,\chi_1)L(s-m/2+1,\chi_2),$$

where a runs over several positive rational numbers and (χ_1, χ_2) are pairs of real characters (i.e. $\chi_1^2 = \chi_2^2 = 1$) such that $\chi_1\chi_2 = \chi_S$. Here $\chi_S(n) = \left(\frac{(-1)^{m/2}\det(S)}{n}\right)$.

Corollary 2.2 *Assumptions and notation being same as above, but here we do not assume that m is even. Then, each value $\zeta(1-n,S)$ is rational for each natural number n.*

3 Koecher Maass series of Siegel Eisenstein series

In the previous section, we treated the zeta function to count the representation number of n by S. The natural generalization is to count something like a number of solution $X \in M_{mn}(\mathbb{Z})$ of $S[X] = T$ for each symmetric matrix T of size $n \leq m$. Indeed, Siegel also defined this type of zeta functions. It seems complicated to obtain the general explicit formula of zeta functions in this case, but an explicit result is obtained in the following case where $S = H_k$ is the $2k \times 2k$ matrix defined by

$$H_k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here the notation plus means to arrange the small matrices diagonally. On the other hand, for $k > n+1$, we can define the Siegel Eisenstein series $E_k(Z)$ of weight k belonging to the full modular group of degree n by

$$E_k(Z) = \sum_{\{C,D\}} \det(CZ+D)^k,$$

where $\{C, D\}$ runs over the so called non-associated coprime symmetric pairs and Z is in the Siegel upper half space of degree n. We have the Fourier expansion

$$E_k^{(n)}(Z) = \sum_T a(T) e^{2\pi i tr(TZ)}$$

of this function, where T runs over $n \times n$ half integral positive semi-definite symmetric matrices T and $a(T)$ is the Fourier coefficients at T. The formula to connect the Fourier coefficients $a(T)$ with the product of local densities

$$\prod_p \alpha_p(H_k, T)$$

is known by Siegel. By this relation, we can regard the Siegel zeta function associated with H_k representing size n symmetric matrices as the Koecher Maass series of $E_k(Z)$, which is defined to be the following series

$$\xi_{n,k}(s) = \sum_{T \in L_n^{*,+}/GL_n(\mathbb{Z})} \frac{a(T)}{\det(T)^s \#(\mathrm{Aut}(T))}.$$

To state the theorem, we prepare notation. We define convolution product of $D_n^*(s)$ and $D_{2k-n}^*(s)$ by

$$D_n^*(s) \otimes D_{2k-n}^*(s) = \zeta(2s-k+1) \sum_d H(\frac{n}{2}, d) H(\frac{2k-n}{2}, d) d^{-s}.$$

The following theorem is a joint work with H. Katsurada.

Theorem 3.1 (cf. [2]) *When n is odd, then*

$$\begin{aligned}
&\xi_{n,k}(s) \\
= \;& (-1)^{nk/2} 2^{(n-1)s} \frac{\prod_{i=0}^{(n-1)/2}(k-i)}{(\frac{n-1}{2})!} \times \frac{\prod_{i=1}^{(n-1)/2} |B_{2i}|}{|B_k| \prod_{i=1}^{(n-1)/2} |B_{2k-2i}|} \\
\times \;& \{\zeta(s)\zeta(s-k+1) \prod_{i=1}^{(n-1)/2} ((\zeta(2s-2i)\zeta(2s-2k+2i+1)) \\
+ \;& (-1)^{(n^2-1)/8} \zeta(s - \frac{n-1}{2}) \zeta(s-k+\frac{n+1}{2}) \prod_{i=1}^{(n-1)/2} (\zeta(2s-2i+1)\zeta(2s-2k+2i))\}.
\end{aligned}$$

When n is even and $n \geq 4$, we get

$$\begin{aligned}
&\xi_{n,k}(s) \\
= \quad & (-1)^{nk/2}2^{ns+n/2} \times \frac{\prod_{i=0}^{n/2}(k-i)\prod_{i=1}^{n/2-1}|B_{2i}|}{(\frac{n}{2}-1)!|B_k|\prod_{i=1}^{n/2}|B_{2k-2i}|} \\
\times \quad & \{(-1)^{(n+k)/2}(D_n^*(s)\otimes D_{2k-n}^*(s))\prod_{i=1}^{n/2-1}\zeta(2s-2i)\zeta(2s-2k+2i+1) \\
+ \quad & \delta_{4,n}(-1)^{n(n+2)/8} \times \frac{|B_{n/2}B_{k-n/2}|}{(n/2)(k-n/2)}\prod_{i=1}^{n/2}\zeta(2s-2i+1)\zeta(2s-2k+2i)\},
\end{aligned}$$

where $\delta_{n,4} = 0$, or 1 if $n \equiv 2 \bmod 4$ or $n \equiv 0 \bmod 4$, respectively.

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