

GOERITZ GROUPS OF BRIDGE DECOMPOSITIONS

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ABSTRACT. For a bridge decomposition of a link in the 3-sphere, we define the Goeritz group to be the group of isotopy classes of orientation-preserving homeomorphisms of the 3-sphere that preserve each of the bridge sphere and link setwise. After describing basic properties of this group, we discuss the asymptotic behavior of the minimal pseudo-Anosov entropies. We then give an application to the asymptotic behavior of the minimal entropies for the original Goeritz groups of Heegaard splittings of the 3-sphere and the real projective space.

INTRODUCTION

Every closed orientable 3-manifold M can be decomposed into two handlebodies V^+ and V^- by cutting it along a closed orientable surface Σ of genus g for some $g \geq 0$. Such a decomposition is called a *genus- g Heegaard splitting* of M and denoted by $(M; \Sigma)$. The *Goeritz group* $\mathcal{G}(M; \Sigma)$ of the Heegaard splitting $(M; \Sigma)$ is then defined to be the group of isotopy classes of orientation-preserving self-homeomorphisms of M that preserve each of the two handlebodies setwise. We note that this group can naturally be thought of as a subgroup of the mapping class group of the surface Σ . Indeed, restricting the maps of concern to Σ , we can describe the Goeritz group $\mathcal{G}(M; \Sigma)$ as

$$\mathcal{G}(M; \Sigma) = \text{MCG}(V^+) \cap \text{MCG}(V^-) < \text{MCG}(\Sigma),$$

where $\text{MCG}(\cdot)$ is the mapping class group. The structure of this group is studied by many authors (see e.g. recent papers [JR13, JM13, FS18, CK19, IK20, Zup19] and references therein).

In this paper, we define an analogous group, which we also call the *Goeritz group*, for a bridge decomposition of a link, and study some of its interesting properties. Recall that an *n -bridge decomposition* $(L; S)$ of a link L in the 3-sphere S^3 is a splitting of (S^3, L) into two trivial n -tangles $(B^+, B^+ \cap L)$ and $(B^-, B^- \cap L)$ along a sphere $S \subset S^3$. We define the Goeritz group $\mathcal{G}(L; S)$

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of the bridge decomposition $(L; S)$ to be the group of isotopy classes of orientation-preserving self-homeomorphisms of S^3 that preserve each of the trivial tangles setwise. (See Section 2 for the definition.) Similarly to the case of Heegaard splittings, it is easily seen that this group can be thought of as a subgroup of the mapping class group $\text{MCG}(S, S \cap L)$, that is, the group of isotopy classes of orientation-preserving self-homeomorphisms of S that preserve $S \cap L$ setwise, see for example [HK17, Proposition A.4]. More precisely, restricting the maps of concern to S , the Goeritz group $\mathcal{G}(L; S)$ can be written as

$$\mathcal{G}(L; S) = \text{MCG}(B^+, B^+ \cap L) \cap \text{MCG}(B^-, B^- \cap L) < \text{MCG}(S, S \cap L),$$

where $\text{MCG}(B^\pm, B^\pm \cap L)$ is the group of isotopy classes of orientation-preserving self-homeomorphisms of the 3-ball B^\pm that preserve $B^\pm \cap L$ setwise.

The Goeritz groups of Heegaard splittings and those of bridge decompositions are related as follows. Given an n -bridge decomposition

$$(S^3, L) = (B^+, B^+ \cap L) \cup_S (B^-, B^- \cap L)$$

with $n \geq 2$, let $q : M_L \rightarrow S^3$ be the 2-fold covering of S^3 branched over L . Then that bridge decomposition induces a Heegaard splitting $M_L = V^+ \cup_\Sigma UV^-$, where $V^\pm := q^{-1}(B^\pm)$ and $\Sigma := q^{-1}(S)$. Let T be the non-trivial deck transformation of the covering. Note that $T|_\Sigma$ is a hyperelliptic involution of Σ . We define the *hyperelliptic Goeritz group* $\mathcal{HG}_T(M_L; \Sigma)$ to be the group of orientation-preserving, fiber-preserving self-homeomorphisms of M_L that preserve each of the two handlebodies setwise modulo isotopy through fiber-preserving homeomorphisms. Here a self-homeomorphism of M_L is said to be *fiber-preserving* if it takes the fiber (with respect to the projection q) of each point in S^3 to the fiber of some point in S^3 . Then we prove in Section 2.2 the following theorem, which is the “hyperelliptic Goeritz group version” of the original Birman-Hilden’s theorem [BH71] for hyperelliptic mapping class groups.

Theorem 0.1. *We have $\mathcal{HG}_T(M_L; \Sigma)/\langle [T] \rangle \cong \mathcal{G}(L; S)$.*

It is then natural to think about a relationship between $\mathcal{HG}_T(M_L; \Sigma)$ and the usual Goeritz group $\mathcal{G}(M_L; \Sigma)$. In the same section, we give a rigorous proof of the following naturally expected result.

Theorem 0.2. *We have $\mathcal{HG}_T(M_L; \Sigma) \cong \mathcal{G}(M_L; \Sigma) \cap \mathcal{H}(\Sigma)$, where $\mathcal{H}(\Sigma)$ is the hyperelliptic mapping class group with respect to the hyperelliptic involution $T|_\Sigma$.*

In Theorem 0.2 we regard both $\mathcal{G}(M_L; \Sigma)$ and $\mathcal{H}(\Sigma)$ as subgroups of $\text{MCG}(\Sigma)$.

Using the two isomorphisms in Theorems 0.1 and 0.2, we can obtain finite presentations for the Goeritz groups of 3-bridge decompositions of 2-bridge links, see Example 2.9.

For a Heegaard splitting $M = V^+ \cup_{\Sigma} V^-$, Hempel [Hem01] introduced a measure of complexity called the *distance*. Roughly speaking, this is defined to be the distance between the sets of the boundaries of meridian disks of V^+ and V^- in the curve graph $\mathcal{C}(\Sigma)$. The distance gives a nice way to describe the structure of the Goeritz groups of Heegaard splittings. In fact, by work of Namazi [Nam07] and Johnson [Joh10], it turned out that the Goeritz group is always a finite group if the distance of the Heegaard splitting is at least 4. This is completely different from the case for the splittings of distance at most 1, where the Goeritz group is always an infinite group (see Johnson-Rubinstein [JR13] and Namazi [Nam07]). The notion of distance can naturally be defined for bridge decompositions as well, see for example Bachman-Schleimer [BS05]. It is hence quite natural to expect that the Goeritz groups of bridge decompositions hold similar properties as above. In Example 2.11 we actually see that the Goeritz group is always an infinite group when the distance of the bridge decomposition is at most 1 (except the case of the 2-bridge decomposition of the trivial knot), and in Section 3, we prove the following.

Theorem 0.3. *There exists a uniform constant $N > 0$ such that for each integer n at least 3, if the distance of an n -bridge decomposition $(L; S)$ of a link L in S^3 is greater than N , then $\mathcal{G}(L; S)$ is a finite group.*

We will show that the constant N in the above theorem can actually be taken to be at most 3796. This number is, however, far bigger than the constant 4 in the case of Heegaard splittings. For a 2-bridge decomposition we will see in Example 2.8 that the Goeritz group is always a finite group except for the case of the trivial 2-component link.

In Sections 4 and 5, we will discuss the Goeritz groups of bridge decompositions of links whose orders are infinite. Let $\Sigma_{g,m}$ denote the orientable surface of genus g with m marked points, possibly $m = 0$. By Σ_g we mean $\Sigma_{g,0}$ for simplicity. The mapping class group $\text{MCG}(\Sigma_{g,m})$ is the group of isotopy classes of orientation-preserving self-homeomorphisms of $\Sigma_{g,m}$ which preserve the marked points setwise. Assume $3g - 3 + m \geq 1$. By the Nielsen-Thurston classification, elements in $\text{MCG}(\Sigma_{g,m})$ fall into three types: periodic, reducible, pseudo-Anosov [Thu88, FM12, FLP79]. This classification can also be applied for elements of the Goeritz group $\mathcal{G}(L; S)$ of an n -bridge decomposition $(L; S)$ by regarding $\mathcal{G}(L; S)$ as a subgroup of $\text{MCG}(S, S \cap L) = \text{MCG}(\Sigma_{0,2n})$.

Given a bridge decomposition $(L; S)$ of a link $L \subset S^3$, consider the bridge decomposition $(L; S_{(p,1)})$ obtained by the 1-fold stabilization of $(L; S)$ at a point $p \in S \cap L$ (see Section 1.3 for the definition of a stabilization). This bridge decomposition has distance at most 1 (see Lemma 1.3), and hence the order of its Goeritz group is infinite (except the case of the 2-bridge decomposition of the trivial knot, which is the 1-fold stabilization of the 1-bridge decomposition of the trivial knot). Therefore, it is quite natural to

ask whether it contains a pseudo-Anosov element. In Section 4 we give a complete answer to this question as in the following way.

Theorem 0.4. *Let $(L; S)$ be an n -bridge decomposition of a link L in S^3 with $n \geq 2$. Let p be an arbitrary point in $S \cap L$. If $(L; S)$ is the 2-bridge decomposition of the 2-component trivial link, then $\mathcal{G}(L; S_{(p,1)})$ is an infinite group consisting only of reducible elements. Otherwise, the Goeritz group $\mathcal{G}(L; S_{(p,1)})$ contains a pseudo-Anosov element.*

Consider the sequence

$$(L; S_{(p,1)}), (L; S_{(p,2)}), \dots, (L; S_{(p,k)}), \dots$$

obtained by a k -fold stabilization of a bridge decomposition $(L; S)$ at a point $p \in S \cap L$ for each positive integer k . By Theorem 0.4, the Goeritz groups of the bridge decompositions in this sequence always have pseudo-Anosov elements. We are interested in how big those Goeritz groups can be from the view point of dynamical properties of their elements. In the final section of the paper we actually discuss the asymptotic behavior of the minimal pseudo-Anosov dilatations in the Goeritz groups with respect to stabilizations. To each pseudo-Anosov element $\phi \in \text{MCG}(\Sigma_{g,m})$, there is an associated dilatation (stretch factor) $\lambda(\phi) > 1$. The logarithm $\log \lambda(\phi)$ of the dilatation is called the *entropy* of ϕ .

Fix a surface $\Sigma_{g,m}$ and consider the set of entropies

$$\{\log \lambda(\phi) \mid \phi \in \text{MCG}(\Sigma_{g,m}) \text{ is pseudo-Anosov}\}.$$

This is a closed, discrete subset of \mathbb{R} due to Ivanov [Iva88]. For any subgroup $G \subset \text{MCG}(\Sigma_{g,m})$ containing a pseudo-Anosov element, let $\ell(G)$ denote the minimum of entropies over all pseudo-Anosov elements in G . Note that $\ell(G) \geq \ell(\text{MCG}(\Sigma_{g,m}))$. Penner [Pen91] proved that $\ell(\text{MCG}(\Sigma_g))$ is comparable to $1/g$, and Hironaka-Kin [HK06] proved that $\ell(\text{MCG}(\Sigma_{0,m}))$ is comparable to $\frac{1}{m}$. Here, for real valued functions $f, h : X \rightarrow \mathbb{R}$ on a subset $X \subset \mathbb{N}$, we say that f is *comparable* to h , and write $f \asymp h$, if there exists a constant $c > 0$ such that $h(x)/c \leq f(x) \leq ch(x)$ for all $x \in X$. For the genus- g handlebody group $\text{MCG}(V_g) \subset \text{MCG}(\Sigma_g)$, Hironaka [Hir14] proved that $\ell(\text{MCG}(V_g))$ is also comparable to $\frac{1}{g}$.

In Section 5, we show the following.

Theorem 0.5. *Let $(O; n)$ be the n -bridge decomposition of the trivial knot $O \subset S^3$. Then we have*

$$\ell(\mathcal{G}(O; n)) \asymp \frac{1}{n}.$$

We note that the sequence $(O; 2), (O; 3), \dots$ we consider in the above theorem is nothing but that of finite-fold stabilizations of the 1-bridge decomposition $(O; 1)$ of the trivial knot O (at any point).

Theorem 0.6. *Let $(H; S_{(p,n-2)})$ be the n -bridge decomposition of the Hopf link $H \subset S^3$ obtained from the 2-bridge decomposition $(H; S)$ by the $(n-2)$ -fold stabilization at a point $p \in S \cap H$. Then we have*

$$\ell(\mathcal{G}(H; S_{(p,n-2)})) \asymp \frac{1}{n}.$$

The proofs of Theorems 0.5 and 0.6 are based on the braid-theoretic description (Theorem 2.1) of the Goeritz groups of bridge decompositions derived from the earlier-mentioned identification $\mathcal{G}(L; S) = \text{MCG}(B^+, B^+ \cap L) \cap \text{MCG}(B^-, B^- \cap L)$ for a bridge decomposition $(L; S)$. Theorems 0.5 and 0.6, together with Theorems 0.1 and 0.2, allow us to have the following intriguing application to the Goeritz groups of Heegaard splittings.

Corollary 0.7. *Let $(S^3; g)$ be the genus- g Heegaard splitting of S^3 for $g \geq 0$. Then we have*

$$\ell(\mathcal{G}(S^3; g)) \asymp \frac{1}{g}.$$

Corollary 0.8. *Let $(\mathbb{RP}^3; g)$ be the genus- g Heegaard splitting of the real projective space \mathbb{RP}^3 for $g \geq 1$. Then we have*

$$\ell(\mathcal{G}(\mathbb{RP}^3; g)) \asymp \frac{1}{g}.$$

1. PRELIMINARIES

Throughout the paper, we will work in the piecewise linear category. For a subspace Y of a space X , $N(Y; X)$ denotes a regular neighborhood of Y in X , $\text{Cl}(Y; X)$ the closure of Y in X , and $\text{Int}(Y)$ the interior of Y . When X is a metric space, $N_\varepsilon(Y; X)$ denotes the closed ε -neighborhood of Y . If there is no ambiguity about the ambient space in question, the X is suppressed from the notation. The number of components of X is denoted by $\#X$.

Let X_1, \dots, X_n be possibly empty subspaces of an orientable manifold M . Let $\text{Homeo}_+(M, X_1, \dots, X_n)$ denote the group of orientation-preserving self-homeomorphisms of M that map X_i onto X_i for each $i = 1, \dots, n$. The *mapping class group*, denoted by $\text{MCG}(M, X_1, \dots, X_n)$, is defined by

$$\text{MCG}(M, X_1, \dots, X_n) = \pi_0(\text{Homeo}_+(M, X_1, \dots, X_n)).$$

Here, we do not require that the maps and isotopies fix the points in ∂M . For a compact orientable surface Σ with marked points, by $\text{MCG}(\Sigma)$ we mean $\text{MCG}(\Sigma, \{p_1, \dots, p_m\})$, where $\{p_1, \dots, p_m\}$ is the set of marked points of Σ . We apply elements of mapping class groups from right to left, i.e., the product fg means that g is applied first.

1.1. Heegaard splittings. A *handlebody* of genus g is an oriented 3-manifold obtained from a 3-ball by attaching g copies of a 1-handle. Every closed orientable 3-manifold M can be obtained by gluing together two handlebodies V^+ and V^- of the same genus g for some $g \geq 0$, that is, M can be represented as $M = V^+ \cup V^-$ and $V^+ \cap V^- = \partial V^+ = \partial V^- = \Sigma \cong \Sigma_g$. We denote

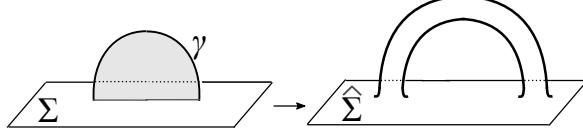


FIGURE 1. A stabilization for a Heegaard surface.

such a decomposition by $(M; \Sigma)$ or $V^+ \cup_{\Sigma} V^-$, and we call it a *genus- g Heegaard splitting* of M . The surface Σ is called the *Heegaard surface* of the splitting. We say that two Heegaard splittings $(M; \Sigma)$ and $(M; \Sigma')$ of M are *equivalent* if the Heegaard surfaces Σ and Σ' are isotopic in M .

We recall the notion of stabilization for a Heegaard splitting $V^+ \cup_{\Sigma} V^-$ of M . Take a properly embedded arc γ in V^+ which is parallel to Σ (Figure 1). We denote the union $V^- \cup N(\gamma)$ by \hat{V}^- , the closure of $V^+ \setminus N(\gamma)$ by \hat{V}^+ , and their common boundary $\partial \hat{V}^+ = \partial \hat{V}^-$ by $\hat{\Sigma}$. Then $\hat{V}^+ \cup_{\hat{\Sigma}} \hat{V}^-$ is again a Heegaard splitting of M , where the genus of $\hat{\Sigma}$ is one greater than that of Σ . Note that $\hat{\Sigma}$ does not depend on γ . We say that $\hat{\Sigma}$ is obtained from Σ by a *stabilization*.

By Waldhausen [Wal68], the 3-sphere S^3 admits a unique genus- g Heegaard splitting up to equivalence for each $g \geq 0$. Similarly, by Bonahon-Otal [BO83], a lens space admits a unique genus- g Heegaard splitting up to equivalence for each $g \geq 1$.

1.2. Goeritz groups of Heegaard splittings. Let $V = V_g$ be a handlebody of genus g . We call $\text{MCG}(V)$ a *handlebody group*. Since the map

$$\text{MCG}(V) \rightarrow \text{MCG}(\partial V)$$

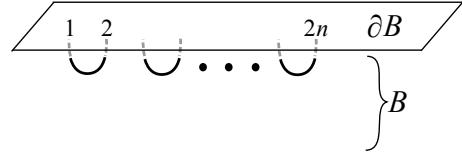
sending $[f] \in \text{MCG}(V)$ to $[f|_{\partial V}] \in \text{MCG}(\partial V)$ is injective, we regard $\text{MCG}(V)$ as a subgroup of $\text{MCG}(\partial V)$.

Suppose that M admits a genus- g Heegaard splitting $(M; \Sigma) = V^+ \cup_{\Sigma} V^-$. We equip the common boundary $\partial V^+ = \partial V^- (= \Sigma)$ with the orientation induced by that of V^- . The *Goeritz group*, denoted by $\mathcal{G}(M; \Sigma)$ or $\mathcal{G}(V^+ \cup_{\Sigma} V^-)$, of the Heegaard splitting is defined by

$$\mathcal{G}(M; \Sigma) = \text{MCG}(M, V^+).$$

We note that $\text{MCG}(M, V^+) = \text{MCG}(M, V^-)$. We can regard $\mathcal{G}(M; \Sigma)$ as a subgroup of both $\text{MCG}(V^+)$ and $\text{MCG}(V^-)$. Further, regarding $\text{MCG}(V^+)$ and $\text{MCG}(V^-)$ as subgroups of $\text{MCG}(\Sigma)$, the group $\mathcal{G}(M; \Sigma)$ is nothing but the intersection $\text{MCG}(V^+) \cap \text{MCG}(V^-)$.

When $(M; \Sigma)$ is the unique genus- g Heegaard splitting of M up to equivalence, we simply call $\mathcal{G}(M; \Sigma)$ the *genus- g Goeritz group* of M , and we denote it by $\mathcal{G}(M; g)$.

FIGURE 2. The standard n -tangle $\mathcal{A} = \mathcal{A}_n$.

1.3. Bridge decompositions. Let $\mathcal{T} = T_1 \cup \dots \cup T_n$ be n disjoint arcs properly embedded in a 3-ball B . We call \mathcal{T} an n -*tangle* or simply a *tangle*. The endpoints of \mathcal{T} , denoted by $\partial\mathcal{T}$, mean the set $\partial T_1 \cup \dots \cup \partial T_n$ of $2n$ points in ∂B .

Suppose that \mathcal{T} and \mathcal{T}' are n -tangles in B such that they share the same endpoints, that is, $\partial\mathcal{T} = \partial\mathcal{T}'$. We say that \mathcal{T} and \mathcal{T}' are *equivalent* if there exists an orientation-preserving homeomorphism $f : B \rightarrow B$ sending \mathcal{T} to \mathcal{T}' with $f|_{\partial B} = \text{id}_{\partial B}$. In this case, we write $\mathcal{T} = \mathcal{T}'$.

In what follows, when we consider an n -tangle in the 3-ball B , we always adopt the following convention.

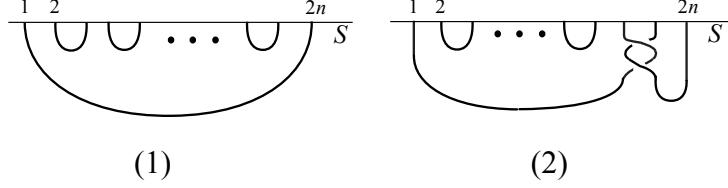
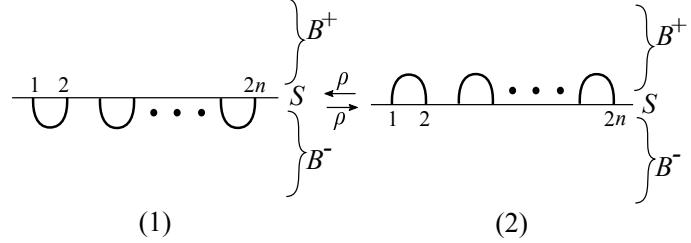
- We identify the boundary ∂B of the 3-ball B with S^2 , and we implicitly fix an oriented great circle C of ∂B .
- We fix $2n$ points labeled $1, 2, \dots, 2n$ on C , where the order of the labels is compatible with the prefixed orientation of C .
- The endpoints $\partial\mathcal{T}$ of a given n -tangle \mathcal{T} is exactly the set $\{1, 2, \dots, 2n\}$.
- When we show a figure of a given n -tangle \mathcal{T} , we draw C as a horizontal line in such a way that the labeled points $1, 2, \dots, 2n$ are ordered from left to right as in Figure 3. The sphere $S := \partial B$ near C is perpendicular to the paper plane in such a figure.

This convention will be particularly important in the arguments in Section 1.7.

Let $\mathcal{A} = \mathcal{A}_n$ be the n -tangle in B as in Figure 2. We say that an n -tangle in B is *standard* if it is equivalent to \mathcal{A} .

We say that an n -tangle \mathcal{T} in B is *trivial* if there exists an orientation-preserving homeomorphism $f : B \rightarrow B$ sending \mathcal{A} to \mathcal{T} . Here, f does not necessarily respect the label of the endpoints. In other words, an n -tangle \mathcal{T} in B is trivial if we can move $N(\partial\mathcal{T}; \mathcal{T})$ by an isotopy keeping $\partial\mathcal{T}$ lying in ∂B so that the resulting tangle is equivalent to \mathcal{A} . For example, Figure 3 shows examples of trivial n -tangles (1) $\mathcal{B} = \mathcal{B}_n$ and (2) $\mathcal{C} = \mathcal{C}_n$.

Consider the genus-0 Heegaard splitting $B^+ \cup_S B^-$ of S^3 , where B^+ and B^- are 3-balls and $S = \partial B^+ = \partial B^-$. Let $\mathbb{R}_+^3 = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$ and $\mathbb{R}_-^3 = \{(x, y, z) \in \mathbb{R}^3 \mid z \leq 0\}$. We identify B^+ with $\mathbb{R}_+^3 \cup \{\infty\}$, B^- with $\mathbb{R}_-^3 \cup \{\infty\}$. Then S is written by $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\} \cup \{\infty\}$. Define an involution $\rho : S^3 \rightarrow S^3$ by $\rho(x, y, z) = (x, y, -z)$. Note that $\rho|_S = \text{id}_S$,

FIGURE 3. The n -tangles (1) $\mathcal{B} = \mathcal{B}_n$ and (2) $\mathcal{C} = \mathcal{C}_n$.FIGURE 4. The involution $\rho : S^3 \rightarrow S^3$ interchanges (1) the n -tangle \mathcal{A} in B^- with (2) the n -tangle $\bar{\mathcal{A}}$ in B^+ .

and ρ interchanges B^+ with B^- . This means that ρ interchanges n -tangles in B^+ with n -tangles in B^- . Consider the standard n -tangle $\mathcal{A} = \mathcal{A}_n$ in B^- . We set $\bar{\mathcal{A}} = \bar{\mathcal{A}}_n = \rho(\mathcal{A}_n)$, which is a trivial n -tangle in B^+ . Hereafter, we illustrate the splitting $B^+ \cup_S B^-$ as in Figure 4: again, the horizontal line indicates the sphere S , the 3-ball B^- lies below S and the other 3-ball B^+ lies above S .

Let L be a link, possibly a knot in S^3 . Suppose that $B^+ \cap L$ and $B^- \cap L$ are trivial n -tangles. Then we have the decomposition

$$(S^3, L) = (B^+, B^+ \cap L) \cup (B^-, B^- \cap L),$$

which is denoted by $(L; S)$ or $(B^+ \cap L) \cup_S (B^- \cap L)$. We call such a decomposition an n -bridge decomposition of L . We also call $S = \partial B^+ = \partial B^-$ a bridge sphere of L .

We say that two n -bridge decompositions $(L; S)$ and $(L; S')$ are equivalent if S and S' are isotopic through bridge spheres of L .

A stabilization of an n -bridge decomposition $(L; S) = (B^+ \cap L) \cup_S (B^- \cap L)$ is defined as follows. Take a point $p \in L \cap S$. We deform the bridge sphere S near p into a sphere $S_{(p,k)}$ so that the cardinality of the intersection $L \cap S_{(p,k)}$ increases by $2k$ as illustrated in Figure 5(2). More precisely, let U be a disk embedded in B^+ whose boundary consists of three arcs α , β and γ , where $\alpha = U \cap L$, $\beta = U \cap S$, see Figure 5(1). Then $\gamma \cap L$ consists of an endpoint of γ , and $\gamma \cap S$ consists of the other endpoint of γ . Let $N(\gamma)$ be a regular neighborhood of γ . We denote the union $B^- \cup N(\gamma)$ by $B_{(p,1)}^-$, the closure of $B^+ \setminus N(\gamma)$ by $B_{(p,1)}^+$, their common boundary by $S_{(p,1)}$. For $k \geq 1$, take k

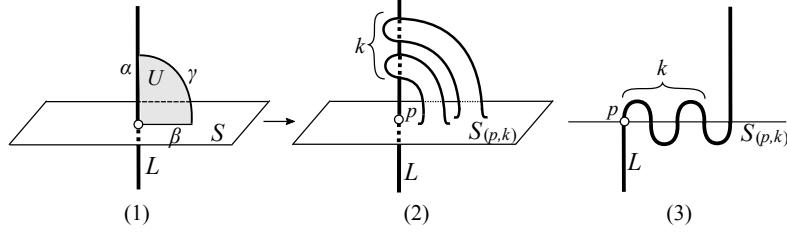


FIGURE 5. A stabilization for a bridge sphere. (1) A bridge sphere S . (2) The bridge sphere $S_{(p,k)}$ obtained from S by a k -fold stabilization, where $k = 2$. (3) Illustration of the bridge decomposition $(L; S_{(p,k)})$.

parallel copies $\gamma'_1, \dots, \gamma'_k$ of γ , and consider the union $\gamma_k = \gamma'_1 \cup \dots \cup \gamma'_k \subset U$. Let $N(\gamma_k)$ be a closed regular neighborhood of γ_k . We denote the union $B^- \cup N(\gamma_k)$ by $B_{(p,k)}^-$, the closure of $B^+ \setminus N(\gamma_k)$ by $B_{(p,k)}^+$, their common boundary by $S_{(p,k)}$. Then

$$(L; S_{(p,k)}) = (B_{(p,k)}^+ \cap L) \cup_{S_{(p,k)}} (B_{(p,k)}^- \cap L)$$

is an $(n+k)$ -bridge decomposition. Note that $S_{(p,k)}$ does not depend on the disk U . It only depends on S , p and k . We say that $S_{(p,k)}$ is obtained from S by a k -fold stabilization (at p). When L is a knot, the stabilized bridge decomposition $(L; S_{(p,k)})$ does not depend on the choice of the point p in $L \cap S$. See Jang-Kobayashi-Ozawa-Takao [JKOT19] for a rigorous proof of this fact.

It is proved by Otal [Ota82] that for each $n \geq 1$, an n -bridge decomposition of the trivial knot O is unique up to equivalence. We denote the n -bridge decomposition of O by $(O; n)$. We note that the same consequence holds as well for the 2-bridge knots by Otal [Ota85], and the torus knots by Ozawa [Oza11].

Consider the 1-bridge decomposition $(O; 1)$ with the bridge sphere S . Let p be a point in $O \cap S$, and let $S_{(p,n-1)}$ be the bridge sphere obtained from S by an $(n-1)$ -fold stabilization, see Figure 6(1). The resulting n -bridge decomposition of O can be expressed by using the trivial tangles $\bar{\mathcal{A}} = \bar{\mathcal{A}}_n$ and $\mathcal{B} = \mathcal{B}_n$ as follows.

$$(O; n) = \bar{\mathcal{A}} \cup_{S_{(p,n-1)}} \mathcal{B}.$$

For the 2-bridge decomposition $(H; S)$ of the Hopf link H , we pick a point $p \in H \cap S$ as in Figure 6(2). Let $S_{(p,n-2)}$ be the bridge sphere obtained from S by an $(n-2)$ -fold stabilization. The resulting n -bridge decomposition of H is of the form

$$(H; S_{(p,n-2)}) = \bar{\mathcal{A}} \cup_{S_{(p,n-2)}} \mathcal{C}$$

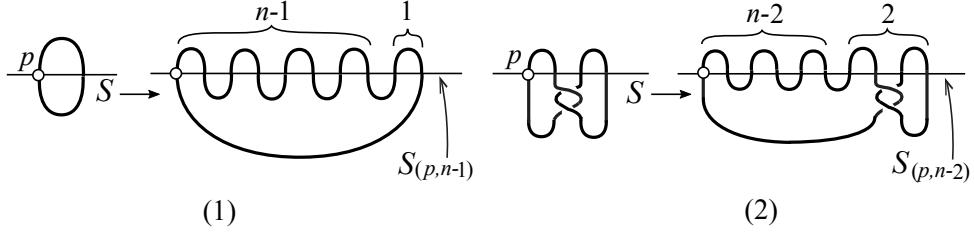


FIGURE 6. (1) The n -bridge sphere of the trivial knot O obtained from the stabilization of the 1-bridge sphere. (2) The n -bridge sphere of the Hopf link H obtained from the stabilization of the 2-bridge sphere.

by using the trivial tangles $\bar{\mathcal{A}} = \bar{\mathcal{A}}_n$ and $\mathcal{C} = \mathcal{C}_n$.

1.4. Curve graphs. Let Σ be a compact orientable surface. A simple closed curve on Σ is said to be *essential* if it does not bound a disk in Σ and it is not parallel to a component of the boundary. A properly embedded arc in Σ is said to be *essential* if it cannot be isotoped (rel. $\partial\Sigma$) into $\partial\Sigma$.

Suppose that Σ is not an annulus. The *curve graph* $\mathcal{C}(\Sigma)$ of Σ is defined to be the 1-dimensional simplicial complex whose vertices are the isotopy classes of essential simple closed curves on Σ and a pair of distinct vertices spans an edge if and only if they admit disjoint representatives. By definition, when Σ is a torus, a 1-holed torus or a 4-holed sphere, $\mathcal{C}(\Sigma)$ has no edges. In these cases, we alter the definition slightly for convenience. When Σ is a torus or a 1-holed torus, two distinct vertices of $\mathcal{C}(\Sigma)$ span an edge if and only if their geometric intersection number is equal to 1. When Σ is a 4-holed sphere, two vertices of $\mathcal{C}(\Sigma)$ span an edge if and only if their geometric intersection number is equal to 2.

Similarly, the *arc and curve graph* $\mathcal{AC}(\Sigma)$ of Σ is the 1-dimensionaional simplicial complex defined as follows. When Σ is not an annulus, the vertices of $\mathcal{AC}(\Sigma)$ are the isotopy classes of essential simple closed curves and isotopy classes of essential arcs (rel. $\partial\Sigma$) on Σ . A pair of distinct vertices spans an edge if and only if they admit disjoint representatives. When Σ is an annulus, the vertices of $\mathcal{AC}(\Sigma)$ are isotopy classes of essential arcs (rel. endpoints). Two distinct vertices spans an edge if and only if they admits disjoint representatives. In this case, we set $\mathcal{C}(\Sigma) := \mathcal{AC}(\Sigma)$ for convenience.

By $\mathcal{C}^{(0)}(\Sigma)$ and $\mathcal{AC}^{(0)}(\Sigma)$ we denote the set of vertices of $\mathcal{C}(\Sigma)$ and $\mathcal{AC}(\Sigma)$, respectively. We can regard $\mathcal{C}(\Sigma)$ (respectively, $\mathcal{AC}(\Sigma)$) as the geodesic metric space equipped with the simplicial metric $d_{\mathcal{C}(\Sigma)}$ (respectively, $d_{\mathcal{AC}(\Sigma)}$).

Let $\delta > 0$. A geodesic metric space is said to be δ -*hyperbolic* if any geodesic triangle is δ -*slim*, that is, each side of the triangle lies in the closed δ -neighborhood of the union of the other two sides.

Recent independent work by Aougab [Aou13], Bowditch [Bow14], Clay-Rafi-Schleimer [CRS14] and Hensel-Przytycki-Webb [HPW15] after a famous work on the hyperbolicity of $\mathcal{C}(\Sigma)$ by Masur-Minsky [MM99] shows the following.

Theorem 1.1. *The curve graph $\mathcal{C}(\Sigma)$ is a δ -hyperbolic space, where δ does not depend on the topological type of Σ .*

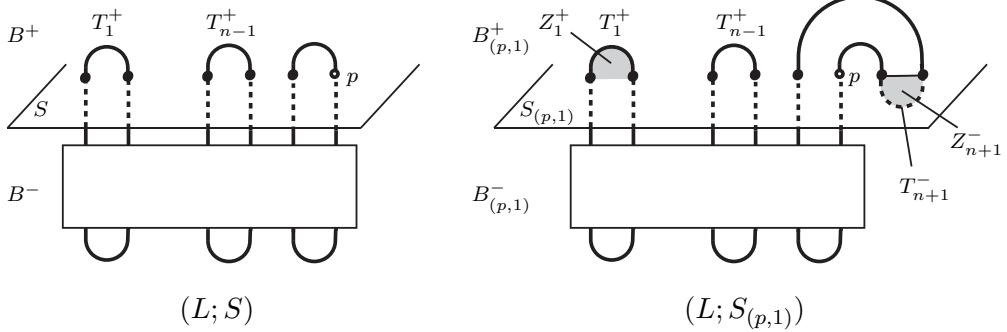
We note that in [HPW15] it was shown that the constant 102¹ is enough for the hyperbolicity constant δ in the above theorem.

Let Σ be a compact orientable surface with a negative Euler characteristic. In what follows, we assume that simple closed curves in a surface are properly embedded, essential, and their intersection is transverse and minimal up to isotopy. Recall that a subsurface Y of Σ is said to be *essential* if each component of ∂Y is not contractible in Σ . We do not allow annuli homotopic to a component of $\partial\Sigma$ to be essential subsurfaces. We always assume that essential subsurfaces are connected and proper. Let Y be an essential subsurface of Σ . The *subsurface projection* $\pi_Y : \mathcal{C}^{(0)}(\Sigma) \rightarrow \mathcal{P}(\mathcal{C}^{(0)}(Y))$, where $\mathcal{P}(\cdot)$ denotes the power set, is defined as follows. First, we consider the case where Y is not an annulus. Define $\kappa_Y : \mathcal{C}^{(0)}(\Sigma) \rightarrow \mathcal{P}(\mathcal{AC}^{(0)}(Y))$ to be the map that takes $\alpha \in \mathcal{C}^{(0)}(\Sigma)$ to $\alpha \cap Y \subset \mathcal{P}(\mathcal{AC}^{(0)}(Y))$. Further, define $\sigma_Y : \mathcal{AC}^{(0)}(Y) \rightarrow \mathcal{P}(\mathcal{C}^{(0)}(Y))$ by taking $\alpha \in \mathcal{AC}^{(0)}(Y)$ to the set of essential simple closed curves that are components of the boundary of $N(\alpha \cup \partial Y; Y)$. The map σ_Y naturally extends to the map $\sigma_Y : \mathcal{P}(\mathcal{AC}^{(0)}(Y)) \rightarrow \mathcal{P}(\mathcal{C}^{(0)}(Y))$. The map π_Y is then defined by $\pi_Y := \sigma_Y \circ \kappa_Y$. Next we consider the case where Y is an annulus. Fix a hyperbolic metric on Σ . Let $p : \tilde{Y} \rightarrow \Sigma$ be the covering map corresponding to $\pi_1(Y)$. Let \hat{Y} be the metric completion of \tilde{Y} . We can identify Y with \hat{Y} . Suppose that $\alpha \in \mathcal{C}^0(\Sigma)$. We can regard $p^{-1}(\alpha)$ as the set of properly embedded arcs in Y and define $\pi_Y(\alpha)$ to be the set of the properly embedded arcs that are essential in Y . The following theorem, called the *bounded geodesic image theorem*, was proved by Masur-Minsky [MM00].

Theorem 1.2. *Let Σ be a compact orientable surface with a negative Euler characteristic. Then there exists a constant $C > 0$ satisfying the following condition. Let $Y \subsetneq \Sigma$ be an essential subsurface that is not a 3-holed sphere. Let c be a geodesic in $\mathcal{C}(\Sigma)$ such that $\pi_Y(\alpha) \neq \emptyset$ for any vertex α of c . Then $\text{diam}_{\mathcal{C}(Y)}(\pi_Y(c)) \leq C$.*

We remark that Webb [Web15] showed that the constant C in the above theorem can be taken to be independent of the topological type of Σ .

¹In [HPW15] the hyperbolicity constant is defined using the *k-centered triangle* condition instead of the δ -slim triangle condition, which we adopt in this paper. The claim of [HPW15] is that any geodesic triangle of $\mathcal{C}(\Sigma)$ is 17-centered. By Bowditch [Bow06, Lemma 6.5], this implies that $\mathcal{C}(\Sigma)$ is $17 \cdot 6 = 102$ -hyperbolic.

FIGURE 7. The bridge decompositions $(L; S)$ and $(L; S_{(p,1)})$.

1.5. The distance of bridge decompositions. Let L be a link in S^3 , and let $(B^+ \cap L) \cup_S (B^- \cap L)$ be an n -bridge decomposition of L with $n \geq 2$. Set $S_L := \text{Cl}(S - N(S \cap L; S))$. We denote by \mathcal{D}^+ (respectively, \mathcal{D}^-) the set of vertices of $\mathcal{C}(S_L)$ that are represented by simple closed curves bounding disks in $B^+ - L$ (respectively $B^- - L$). The *distance* $d(L; S)$ of the bridge decomposition $(S; L)$ is defined to be the (extrinsic) distance of the two subsets \mathcal{D}^+ and \mathcal{D}^- in the curve graph $\mathcal{C}(S_L)$, that is, $d(L; S) := \min d_{\mathcal{C}(S_L)}(\alpha, \beta)$, where the minimum is taken over all $\alpha \in \mathcal{D}^+$ and $\beta \in \mathcal{D}^-$.

Lemma 1.3. *The distance of a stabilized bridge decomposition of a link in S^3 is at most 1.*

Proof. Let $(L; S) = (B^+ \cap L) \cup_S (B^- \cap L)$ be an n -bridge decomposition of L . Take a point $p \in S \cap L$. Consider the $(n+1)$ -bridge decomposition

$$(L; S_{(p,1)}) = (B_{(p,1)}^+ \cap L) \cup_{S_{(p,1)}} (B_{(p,1)}^- \cap L).$$

It suffices to show that $d(L; S_{(p,1)})$ is at most 1. If $n = 1$, then $(L; S) = (O; 1)$ and $(L; S_{(p,1)}) = (O; 2)$. It is thus easily checked that $d(L; S_{(p,1)}) = 1$. See the definition of the curve graph for a 4-holed sphere. Suppose that $n \geq 2$. Let T_1^+, \dots, T_{n-1}^+ (respectively, T_1^-, \dots, T_{n-1}^-) be the components of $B^+ \cap L$ (respectively, $B^- \cap L$) disjoint from p . We note that the arcs T_1^+, \dots, T_{n-1}^+ (respectively, T_1^-, \dots, T_{n-1}^-) remain to be components of $B_{(p,1)}^+ \cap L$ (respectively, $B_{(p,1)}^- \cap L$). See Figure 7. Let T_{n+1}^- be the (unique) component of $(B_{(p,1)}^- \cap L) - \bigcup_{i=1}^{n-1} T_i^-$ disjoint from p . Then there exist disjoint disks $Z_1^+ \subset B_{(p,1)}^+$ and $Z_{n+1}^- \subset B_{(p,1)}^-$ such that $Z_1^+ \cap L = \partial Z_1^+ \cap L = T_1^+$, $\partial Z_1^+ - T_1^+ \subset S_{(p,1)}$, $Z_{n+1}^- \cap L = \partial Z_{n+1}^- \cap L = T_{n+1}^-$ and $\partial Z_{n+1}^- - T_{n+1}^- \subset S_{(p,1)}$. The simple closed curve $\alpha := \partial N(Z_1^+ \cap S_{(p,1)}; S_{(p,1)})$ bounds a disk in $B_{(p,1)}^+ - (B_{(p,1)}^+ \cap L)$ while $\beta := \partial N(Z_{n+1}^- \cap S_{(p,1)}; S_{(p,1)})$ bounds a disk in $B_{(p,1)}^- - (B_{(p,1)}^- \cap L)$. Since both α and β are disjoint essential simple closed

curves in $\text{Cl}(S_{(p,1)} - N(S_{(p,1)} \cap L; S_{(p,1)}))$, the distance $d(L; S_{(p,1)})$ is at most 1. \square

Recall that a subset Y of a geodesic metric space X is said to be *K-quasiconvex* in X for a positive number $K > 0$ if, for any two points x, y in Y , any geodesic segment in X connecting x and y lies in the K -neighborhood of Y . We are going to discuss the quasiconvexity of \mathcal{D}^+ and \mathcal{D}^- in $\mathcal{C}(S_L)$. In fact, the following holds.

Theorem 1.4. *There exist a constant $K > 0$ satisfying the following property. For any n -bridge decomposition of a link L in S^3 with $n \geq 2$, the set \mathcal{D}^+ (respectively, \mathcal{D}^-) is K -quasiconvex in $\mathcal{C}(S_L)$.*

This theorem can actually be proved using Masur-Minsky [MM04] and Hamenstädt [Ham18, Section 3]. In the following we adopt Vokes's arguments in [Vok18] and [Vok19] to explain that the K in the above theorem can be taken to be at most 1796.

A compact orientable genus- g surface Σ with m holes is said to be *non-sporadic* if $3g + m \geq 5$. Let α and β be simple closed curves in a non-sporadic surface Σ . A simple closed curve γ in Σ is called an (α, β) -curve with 0 corners if $\gamma = \alpha$ or $\gamma = \beta$. A simple closed curve γ in Σ is called an (α, β) -curve with 2 corners if there exist subarcs $\alpha' \subset \alpha$ and $\beta' \subset \beta$ such that $\partial\alpha' = \partial\beta'$, $\text{Int}\alpha' \cap \text{Int}\beta' = \emptyset$, and γ is homotopic in Σ to the concatenation $\alpha' * \beta'$, where orientations are chosen in an appropriate way. A simple closed curve γ in Σ is called an (α, β) -curve with 4 corners if there exist subarcs $\alpha'_1, \alpha'_2 \subset \alpha$ and $\beta'_1, \beta'_2 \subset \beta$ satisfying the following.

- $\text{Int}\alpha'_1, \text{Int}\alpha'_2, \text{Int}\beta'_1, \text{Int}\beta'_2$ are mutually disjoint,
- $\partial\alpha'_1 \cup \partial\alpha'_2 = \partial\beta'_1 \cup \partial\beta'_2$ (we allow the case where $\partial\alpha'_1$ and $\partial\alpha'_2$ (and hence $\partial\beta'_1$ and $\partial\beta'_2$) share a single point),
- $\alpha'_1 \cup \alpha'_2 \cup \beta'_1 \cup \beta'_2$ is connected,
- γ is homotopic in Σ to the concatenation $\alpha'_1 * \beta'_1 * \alpha'_2 * \beta'_2$, where orientations are chosen in an appropriate way.

We note that an (α, β) -curve with at most 2 corners are called a *bicorn curve* in Przytycki-Sisto [PS17]. For any simple closed curves α, β in a non-sporadic surface Σ , let $\mathcal{L}_0(\alpha, \beta)$ denote the full subgraph of $\mathcal{C}(\Sigma)$ spanned by the set of (α, β) -curves with at most 4 corners. For a positive number h , we set $R(h) := m - 4h$, where $m > 0$ is the number defined by $2h(6 + \log_2(m+2)) = m$.

The following proposition is a part of Proposition 3.1 of Bowditch [Bow14] that is necessary for our arguments.

Proposition 1.5. *Let $h > 0$ be a constant. Let G be a connected graph equipped with the simplicial metric d_G . Suppose that each pair $\{x, y\}$ (possibly $x = y$) of vertices of G is associated with a connected subgraph $\mathcal{L}(x, y) \subset G$ with $x, y \in \mathcal{L}(x, y)$ satisfying the following.*

- (1) *For any vertices x, y, z of G , $\mathcal{L}(x, y) \subset N_h(\mathcal{L}(x, z) \cup \mathcal{L}(y, z))$.*

- (2) For any vertices x, y of G with $d_G(x, y) \leq 1$, the diameter of $\mathcal{L}(x, y)$ is at most h .

Then for any vertices x, y of G , the Hausdorff distance in G between $\mathcal{L}(x, y)$ and any geodesic segment c in G connecting x and y is at most $R(h)$.

The following lemma is proved in [Vok18, Lemma 5.1.4].

Lemma 1.6. *There exists a constant $h_1 > 0$ such that for any simple closed curves α, β in any non-sporadic surface Σ , $N_{h_1}(\mathcal{L}_0(\alpha, \beta))$ is connected.*

We note that by the remark just before Lemma 5.1.12 in [Vok18] that the constant h_1 in the above lemma can be taken to be at most 7.

The next lemma is due to [Vok18, Lemma 5.1.5].

Lemma 1.7. *There exists a constant $h_2 > 0$ such that for any simple closed curves α, β, γ in any non-sporadic surface Σ , $\mathcal{L}_0(\alpha, \beta) \subset N_{h_2}(\mathcal{L}_0(\alpha, \gamma) \cup \mathcal{L}_0(\beta, \gamma))$.*

By [Vok18, Lemma 5.1.12], the constant h_2 in the above lemma can be taken to be at most 18. For any simple closed curves α, β in a non-sporadic surface Σ , set $\mathcal{L}(\alpha, \beta) := N_{h_1}(\mathcal{L}_0(\alpha, \beta))$. By [Vok19, Claim 10.4.2], for the constant $h_0 := 2h_1 + h_2$, which can be taken to be at most $2 \cdot 7 + 18 = 32$, the curve graph $\mathcal{C}(\Sigma)$ endowed with the associated subgraphs $\mathcal{L}(\alpha, \beta)$ for $\alpha, \beta \in \mathcal{C}^{(0)}(\Sigma)$ satisfies the condition of Proposition 1.5. Using this fact, [Vok19, Lemma 10.4.4] shows the following.²

Lemma 1.8. *Let α, β be simple closed curves in a non-sporadic surface Σ , P a path in $\mathcal{L}(\alpha, \beta)$ from α and β . Then any geodesic segment in $\mathcal{C}(\Sigma)$ connecting α and β lies in the $(2R(h_0) + 2)$ -neighborhood of P .*

We note that the minimum integer greater than or equal to $R(h_0)$ is 897. Therefore, the above constant $2R(h_0) + 2$ can be taken to be at most 1796.

Now, we quickly review the notion of disk surgery. Let D and E be properly embedded disks in a 3-manifold M intersecting transversely and minimally. Suppose that $D \cap E \neq \emptyset$. Let E' be an outermost subdisk of E cut off by $D \cap E$. The arc $\partial E' \cap D$ cuts D into two subdisks. Choose one D' of them. Set $D_1 := D' \cup E'$. By a slight isotopy, the disk D_1 can be moved to be disjoint from $D \cup E'$. We call D_1 a *disk obtained by a surgery on D along E* (with respect to E'). An operation to obtain D_1 from D in the above way is called a *disk surgery*. We note that the number $\#(D' \cap E)$ of components of $D' \cap E$ is less than $\#(D \cap E)$. Therefore, applying disk surgeries repeatedly, we obtain a finite sequence $D = D_0, D_1, D_2, \dots, D_k = E$ of disks in M such that $D_i \cap D_{i+1} = \emptyset$ for $i = 0, 1, \dots, k - 1$.

²Precisely speaking, [Vok19, Lemma 10.4.4] considers only the case of bicorn curves in a closed orientable surface instead of curves with at most four corners in a non-sporadic orientable surface (possibly with boundary). Lemma 1.8, however, can be proved in exactly the same way.

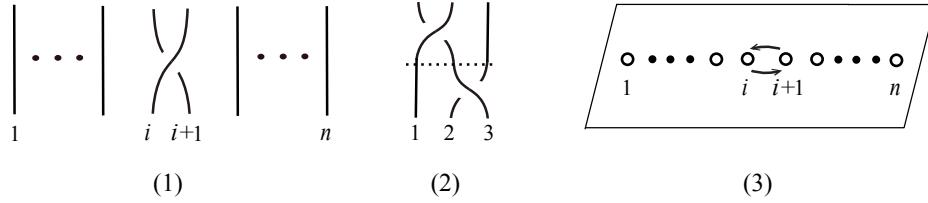


FIGURE 8. (1) The generator σ_i in the spherical/planar braid group. (2) The 3-braid $\sigma_1\sigma_2^{-1}$. (3) The half twist h_i in the mapping class group on the sphere/disk with marked points.

Proof of Theorem 1.4. Let $(L; S)$ be an n -bridge decomposition of a link L in S^3 with $n \geq 2$. Let α and β be arbitrary points in \mathcal{D}^+ . Let $\{\gamma_i\}_{0 \leq i \leq s}$ be a sequence of vertices of \mathcal{D}^+ obtained by disk surgery such that $\alpha = \gamma_0$ and $\beta = \gamma_s$. By definition the path P in $\mathcal{C}(S_L)$ corresponding to this sequence lies within the set $\mathcal{L}(\alpha, \beta)$. It follows from Lemma 1.8 that any geodesic segment c in $\mathcal{C}(S_L)$ connecting α and β lies within the $(2R(h_0)+2)$ -neighborhood of P . Since $\{\gamma_i\}_{0 \leq i \leq s} \subset \mathcal{D}^+$, the geodesic segment c lies within the $(2R(h_0) + 2)$ -neighborhood of \mathcal{D}^+ , which implies the assertion. The argument for \mathcal{D}^- is of course the same. \square

1.6. Braid groups. In this subsection, we fix conventions on the planar/spherical braid groups and their relation to mapping class groups. See [Bir74, Chapter 1] for details.

Let B_n be the (planar) braid group with n strands. The group B_n is generated by braids $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ as shown in Figure 8(1). The product of braids is defined as follows. Given $b, b' \in B_n$, we place b above b' , and concatenate the bottom of b with the top of b' . The product $bb' \in B_n$ is the resulting braid, see Figure 8(2).

We set $\delta_j = \sigma_1\sigma_2 \cdots \sigma_{j-1} \in B_n$. The *half twist* $\Delta \in B_n$ is given by

$$\Delta = \delta_n\delta_{n-1} \cdots \delta_2.$$

The second power Δ^2 is called the *full twist*.

Let SB_n be the spherical braid group with n strands. By abusing notation, we still denote by σ_i , the spherical braid as shown in Figure 8(1). We define the product of spherical braids in the same manner as above.

We recall connections between the planar/spherical braid groups and the mapping class groups on the disk/sphere with marked points. Let D_n be the disk with n marked points. Then we have the surjective homomorphism

$$\Gamma_D : B_n \rightarrow \text{MCG}(D_n)$$

which sends each generator σ_i to the right-handed half twist h_i between the i th and $(i+1)$ th marked points, see Figure 8(3). The kernel of Γ_D is an infinite cyclic group generated by the full twist Δ^2 , that is, $\ker \Gamma_D = \langle \Delta^2 \rangle$.

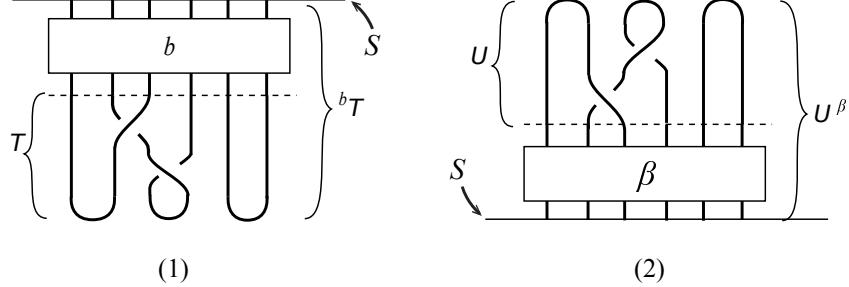


FIGURE 9. (1) The tangle ${}^b\mathcal{T}$ in the 3-ball B^- , where $\mathcal{T} = \sigma_2 \sigma_3^{-1} \mathcal{A}$. (2) The tangle U^β in the 3-ball B^+ , where $U = \bar{\mathcal{A}} \sigma_3 \sigma_2^{-1}$.

Thus, we have

$$B_n / \langle \Delta^2 \rangle \cong \text{MCG}(D_n).$$

We also have the surjective homomorphism

$$\Gamma : \text{SB}_n \rightarrow \text{MCG}(\Sigma_{0,n}),$$

sending each generator σ_i to the right-handed half twist h_i between the i th and $(i+1)$ th marked points in the sphere. The kernel of Γ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ which is generated by the full twist $\Delta^2 \in \text{SB}_n$. Thus we have

$$\text{SB}_n / \langle \Delta^2 \rangle \cong \text{MCG}(\Sigma_{0,n}).$$

1.7. Wicket groups on trivial tangles. Let \mathcal{T} be an n -tangle in B^- and let $b \in \text{SB}_{2n}$. We place b above \mathcal{T} concatenating the bottom endpoints of b with the endpoints $\partial\mathcal{T}$. Then we obtain an n -tangle ${}^b\mathcal{T}$ in B^- , see Figure 9(1). We may assume that ${}^b\mathcal{T}$ share the same endpoints as \mathcal{T} . Observe that SB_{2n} acts on n -tangles in B^- from the left:

$${}^{b'}{}^b\mathcal{T} = {}^{b'}({}^b\mathcal{T}) \text{ for } b, b' \in \text{SB}_{2n}.$$

Similarly, given an n -tangle \mathcal{U} in B^+ and a braid $\beta \in \text{SB}_{2n}$, we obtain an n -tangle \mathcal{U}^β in B^+ , see Figure 9(2). Then SB_{2n} acts on n -tangles in B^+ from the right:

$$\mathcal{U}^{\beta'\beta} = (\mathcal{U}^{\beta'})^\beta \text{ for } \beta, \beta' \in \text{SB}_{2n}.$$

Recall the involution $\rho : S^3 \rightarrow S^3$ defined in Section 1.3. Let \mathcal{A} and $\bar{\mathcal{A}}$ be n -tangles in B^- and B^+ respectively as before. By the definition of ρ , we have $\rho(x, y, z) = (x, y, -z)$ for $(x, y, z) \in \mathbb{R}^3$. This implies that

$$\begin{aligned} \rho({}^b\mathcal{A}) &= \bar{\mathcal{A}}^{b^{-1}}, \\ \rho(\bar{\mathcal{A}}^\beta) &= {}^{\beta^{-1}}\mathcal{A}. \end{aligned}$$

For example, ρ interchanges $\sigma_2 \sigma_3^{-1} \mathcal{A}$ with $\bar{\mathcal{A}} \sigma_3 \sigma_2^{-1}$.

Remark 1.9. Given $\phi \in \text{MCG}(\Sigma_{0,2n})$, there is a braid $b_\phi \in \text{SB}_{2n}$ such that $\Gamma(b_\phi) = \phi$. Take any representative $f : \Sigma_{0,2n} \rightarrow \Sigma_{0,2n}$ of ϕ . We regard f as an orientation-preserving homeomorphism on the sphere $S = \partial B^- = \partial B^+$ with $2n$ marked points. Let $\Phi : B^- \rightarrow B^-$ be an extension of f , that is, $\Phi|_{\partial B^-} = f$. Let \mathcal{A} be the standard n -tangle in B^- . Assume that $\partial \mathcal{A} \subset S$ equals the set of $2n$ marked points. Then $\Phi(\mathcal{A})$ is an n -tangle in B^- . One sees that

$$\Phi(\mathcal{A}) = {}^{b_\phi}\mathcal{A}.$$

for we equip S with the orientation induced by that of B^- . Let us turn to the homeomorphism $\rho\Phi\rho|_{B^+} : B^+ \rightarrow B^+$. Since $\rho|_S = \text{id}_S$, we have $\rho\Phi\rho|_{\partial B^+} = f$. Hence $\rho\Phi\rho|_{B^+} : B^+ \rightarrow B^+$ is an extension of f over B^+ . Then we have

$$\rho\Phi\rho(\bar{\mathcal{A}}) = \rho\Phi(\mathcal{A}) = \rho({}^{b_\phi}\mathcal{A}) = \bar{\mathcal{A}}^{b_\phi^{-1}}.$$

From the above discussion, it is easy to see the following lemma.

Lemma 1.10. Let \mathcal{T} be an n -tangle in B^- , and let \mathcal{U} be an n -tangle in B^+ .

- (1) \mathcal{T} is trivial if and only if $\mathcal{T} = {}^b\mathcal{A}$ for some $b \in \text{SB}_{2n}$.
- (2) \mathcal{U} is trivial if and only if $\mathcal{U} = \bar{\mathcal{A}}^\beta$ for some $\beta \in \text{SB}_{2n}$.

For a trivial n -tangle \mathcal{T} in B^- , we define a subgroup $\text{SW}_{2n}(\mathcal{T}) \subset \text{SB}_{2n}$ as follows.

$$\text{SW}_{2n}(\mathcal{T}) = \{b \in \text{SB}_{2n} \mid {}^b\mathcal{T} = \mathcal{T}\}.$$

Since $\Delta^2 \in \text{SW}_{2n}(\mathcal{T})$, we have

$$\ker \Gamma = \langle \Delta^2 \rangle \subset \text{SW}_{2n}(\mathcal{T}).$$

The group $\text{SW}_{2n}(\mathcal{A})$ for the standard n -tangle \mathcal{A} is called the *wicket group*. We write

$$\text{SW}_{2n} = \text{SW}_{2n}(\mathcal{A}).$$

See Brendle-Hatcher [BH13] for a more extended study on the wicket groups.

Since the map

$$\text{MCG}(B^-, \mathcal{A}) \rightarrow \text{MCG}(\partial B^-, \partial \mathcal{A})$$

sending $[f] \in \text{MCG}(B^-, \mathcal{A})$ to $[f|_{\partial B^-}] \in \text{MCG}(\partial B^-, \partial \mathcal{A})$ is injective, we regard $\text{MCG}(B^-, \mathcal{A})$ as a subgroup of $\text{MCG}(\partial B^-, \partial \mathcal{A})$ ($= \text{MCG}(\Sigma_{0,2n})$). The following theorem is proved in [HK17, Theorem 2.6].

Theorem 1.11. The equation $\Gamma(\text{SW}_{2n}) = \text{MCG}(B^-, \mathcal{A})$ holds. Thus, we have $\text{SW}_{2n}/\langle \Delta^2 \rangle \cong \text{MCG}(B^-, \mathcal{A})$.

Example 1.12. We define $x, y, z \in \text{SB}_6$ as follows. (See Figure 10.)

$$\begin{aligned} x &= \sigma_3^2 \sigma_2 \sigma_3^2 \sigma_2, \\ y &= \sigma_1^2 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_1 \sigma_2 \sigma_3 \sigma_4, \\ z &= \sigma_1^2 \sigma_2 \sigma_3 \sigma_4 \sigma_1 \sigma_2 \sigma_3 \sigma_3 \sigma_4. \end{aligned}$$

The tangles ${}^x\mathcal{A}$, ${}^y\mathcal{A}$ and ${}^z\mathcal{A}$ are equivalent to \mathcal{A} . Hence $x, y, z \in \text{SW}_6$.

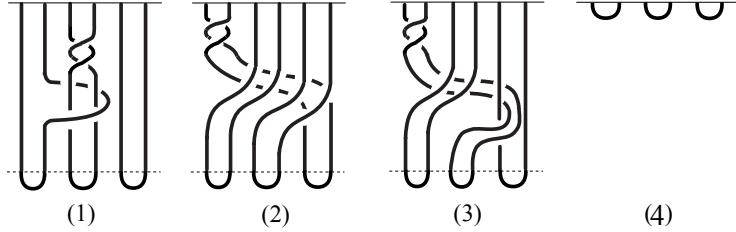


FIGURE 10. (1) ${}^x\mathcal{A}$; (2) ${}^y\mathcal{A}$; and (3) ${}^z\mathcal{A}$ for $x, y, z \in \text{SB}_6$ in Example 1.12. They are equivalent to (4) \mathcal{A} .

Lemma 1.13. *Let \mathcal{T} be a trivial n -tangle in B^- . Let b be a spherical $2n$ -braid such that $\mathcal{T} = {}^b\mathcal{A}$. Then we have $\text{SW}_{2n}(\mathcal{T}) = b(\text{SW}_{2n})b^{-1}$.*

Proof. Take an element $\beta \in \text{SW}_{2n}(\mathcal{T}) = \text{SW}_{2n}({}^b\mathcal{A})$. By the definition of $\text{SW}_{2n}({}^b\mathcal{A})$, we have ${}^\beta({}^b\mathcal{A}) = {}^b\mathcal{A}$. Then ${}^{b^{-1}\beta b}\mathcal{A} = \mathcal{A}$. Hence we have $b^{-1}\beta b \in \text{SW}_{2n}$, which says that $\beta \in b(\text{SW}_{2n})b^{-1}$. Thus $\text{SW}_{2n}(\mathcal{T}) \subset b(\text{SW}_{2n})b^{-1}$. The proof of $b(\text{SW}_{2n})b^{-1} \subset \text{SW}_{2n}(\mathcal{T})$ is similar. \square

For trivial n -tangles \mathcal{T} and \mathcal{U} in B^- , we set

$$\text{SW}_{2n}(\mathcal{T}, \mathcal{U}) = \text{SW}_{2n}(\mathcal{T}) \cap \text{SW}_{2n}(\mathcal{U}).$$

We call the group $\text{SW}_{2n}(\mathcal{T}, \mathcal{U})$ the *wicket group on \mathcal{T} and \mathcal{U}* . Obviously, $\text{SW}_{2n}(\mathcal{A}, \mathcal{A}) = \text{SW}_{2n}(\mathcal{A}) (= \text{SW}_{2n})$.

The following lemma will be used in the proofs of Theorems 0.5 and 0.6.

Lemma 1.14. *Let x, y and z be elements of SW_6 as in Example 1.12. Let $\mathcal{B} = \mathcal{B}_3$ and $\mathcal{C} = \mathcal{C}_3$ be the trivial tangles as in Figure 3. Then we have $x, y \in \text{SW}_6(\mathcal{B})$ and $x, z \in \text{SW}_6(\mathcal{C})$. In particular $x, y \in \text{SW}_6(\mathcal{A}, \mathcal{B})$ and $x, z \in \text{SW}_6(\mathcal{A}, \mathcal{C})$.*

Proof. We see that ${}^x\mathcal{B} = \mathcal{B} = {}^y\mathcal{B}$, and ${}^x\mathcal{C} = \mathcal{C} = {}^z\mathcal{C}$ (Figure 11). We are done. \square

By Lemma 1.13, we immediately have the following corollary.

Corollary 1.15. *Let $b, d \in \text{SB}_{2n}$. Then we have*

$$\text{SW}_{2n}({}^b\mathcal{A}, {}^d\mathcal{A}) = b(\text{SW}_{2n})b^{-1} \cap d(\text{SW}_{2n})d^{-1}.$$

Lemma 1.16. *Let \mathcal{T} and \mathcal{U} be trivial n -tangles in B^- . Let b and d be the spherical $2n$ -braids such that $\mathcal{T} = {}^b\mathcal{A}$ and $\mathcal{U} = {}^d\mathcal{A}$. Then we have*

$$\text{SW}_{2n}(\mathcal{T}, \mathcal{U}) = b(\text{SW}_{2n}(\mathcal{A}, {}^{b^{-1}d}\mathcal{A}))b^{-1}.$$

Proof. By Corollary 1.15, we have

$$\text{SW}_{2n}(\mathcal{T}, \mathcal{U}) = b(\text{SW}_{2n})b^{-1} \cap d(\text{SW}_{2n})d^{-1}.$$

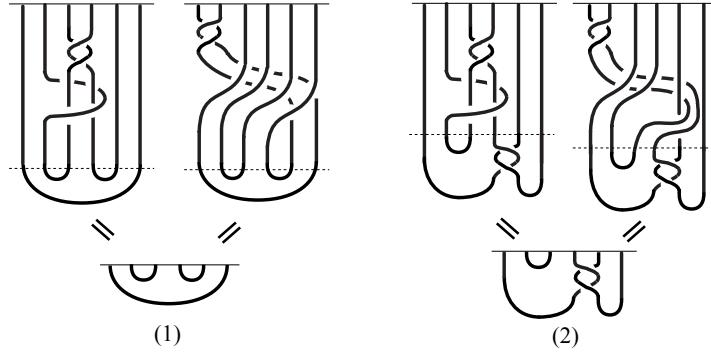


FIGURE 11. (1) ${}^x\mathcal{B} = \mathcal{B} = {}^y\mathcal{B}$ and (2) ${}^x\mathcal{C} = \mathcal{C} = {}^z\mathcal{C}$ for $x, y, z \in \text{SB}_6$ in Example 1.12. See also Figure 3.

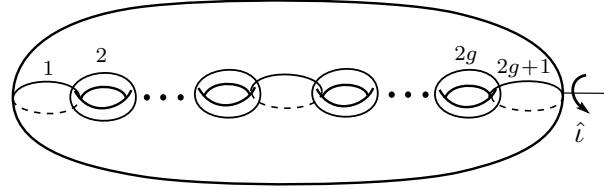


FIGURE 12. A hyperelliptic involution \hat{i} defined on V_g .

By Corollary 1.15 again, we have

$$\text{SW}_{2n}(\mathcal{A}, {}^{b^{-1}d}\mathcal{A}) = \text{SW}_{2n} \cap b^{-1}d(\text{SW}_{2n})d^{-1}b.$$

These two equalities imply the assertion. \square

1.8. Hyperelliptic handlebody groups. In this subsection, we review the notion of the hyperelliptic handlebody group developed in [HK17]. We provide details on properties to be used later.

Let V_g be a handlebody of genus $g \geq 2$ with $\partial V_g = \Sigma_g$. We call an involution $\hat{i} \in \text{Homeo}_+(V_g)$ a *hyperelliptic involution* of V_g if $\hat{i}|_{\Sigma_g}$ is a hyperelliptic involution of Σ_g , that is, $\hat{i}|_{\Sigma_g}$ is an order 2 element of $\text{Homeo}_+(\Sigma_g)$ that acts on $H_1(\Sigma_g; \mathbb{Z})$ by $-I$. The following lemma is straightforward from Pantaleoni-Piergallini [PP11].

Lemma 1.17. *Any two hyperelliptic involutions of V_g are conjugate in the handlebody group $\text{MCG}(V_g)$.*

By this lemma, without loss of generality we can assume that \hat{i} is the map shown in Figure 12, where we think of V_g as being embedded in \mathbb{R}^3 .

Fix a hyperelliptic involution \hat{i} of V_g and set $\iota := \hat{i}|_{\Sigma_g}$. We denote by $\text{SHomeo}_+(V_g)$ (respectively, $\text{SHomeo}_+(\Sigma_g)$) the centralizer in $\text{Homeo}_+(V_g)$

(respectively, $\text{Homeo}_+(\Sigma_g)$) of $\hat{\iota}$ (respectively, ι). We call

$$\mathcal{H}(V_g) := \pi_0(\text{SHomeo}_+(V_g))$$

a *hyperelliptic handlebody group*, and

$$\mathcal{H}(\Sigma_g) := \pi_0(\text{SHomeo}_+(\Sigma_g))$$

a *hyperelliptic mapping class group*. By Birman-Hilden [BH71] the following holds.

Theorem 1.18. *We have a canonical isomorphism*

$$\mathcal{H}(\Sigma_g)/\langle[\iota]\rangle \cong \text{MCG}(\Sigma_{0,2g+2}).$$

We note that this theorem, together with [BH73, Theorem 4], implies that the group $\mathcal{H}(\Sigma_g)$ can naturally be identified with the centralizer in $\text{MCG}(\Sigma_g)$ of the mapping class $[\iota]$. By Theorem 1.18 the quotient map $\mathcal{H}(\Sigma_g) \rightarrow \mathcal{H}(\Sigma_g)/\langle[\iota]\rangle$ gives the surjective homomorphism

$$\Pi : \mathcal{H}(\Sigma_g) \rightarrow \text{MCG}(\Sigma_{0,2g+2}).$$

The map Π sends τ_i to the half twist h_i , where τ_i is the right-handed Dehn twist about the simple closed curve labeled with the number i in Figure 12.

Let $q : V_g \rightarrow V_g/\hat{\iota} =: B$ be the projection. Set $\mathcal{A}(= \mathcal{A}_{g+1}) := q(\text{Fix}(\hat{\iota}))$, where $\text{Fix}(\cdot)$ denotes the set of fixed points. We note that \mathcal{A} is the trivial $(g+1)$ -tangle in the 3-ball B . By basic algebraic topology arguments, we have the following.

Lemma 1.19. (1) Any element in $\text{Homeo}_+(B, \mathcal{A})$ lifts to an element of $\text{SHomeo}_+(V_g)$.
(2) Given a path in $\text{Homeo}_+(B, \mathcal{A})$ with the initial point $\hat{\phi} \in \text{Homeo}_+(B, \mathcal{A})$ and a lift $\hat{f} \in \text{SHomeo}_+(V_g)$ of $\hat{\phi}$, there exists a unique lift in $\text{SHomeo}_+(V_g)$ of the path with the initial point \hat{f} .

The next theorem follows from Theorem 1.18 and Lemma 1.19.

Theorem 1.20 (Theorem 2.11 in [HK17]). *The natural map*

$$\mathcal{H}(V_g) \rightarrow \text{MCG}(B, \mathcal{A}_{g+1})$$

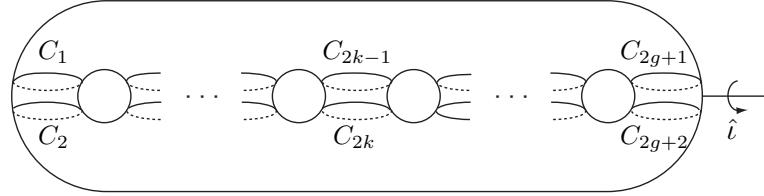
is surjective and its kernel is $\langle[\hat{\iota}]\rangle$. Thus, we have $\mathcal{H}(V_g)/\langle[\hat{\iota}]\rangle \cong \text{MCG}(B, \mathcal{A}_{g+1})$.

We denote by $\text{EHomeo}_+(\Sigma_g)$ the subgroup of $\text{Homeo}_+(\Sigma_g)$ consisting of homeomorphisms of Σ_g that extend to those of V_g . Note that the injectivity of the natural map $\text{MCG}(V_g) \rightarrow \text{MCG}(\Sigma_g)$ implies

$$\pi_0(\text{EHomeo}_+(\Sigma_g)) \cong \text{MCG}(\Sigma_g).$$

We say that two elements of $\text{SHomeo}_+(V_g)$ (respectively, $\text{SHomeo}_+(\Sigma_g)$) are *symmetrically isotopic* if they lie in the same component of $\text{SHomeo}_+(V_g)$ (respectively, $\text{SHomeo}_+(\Sigma_g)$).

Lemma 1.21. (1) For any $f \in \text{EHomeo}_+(\Sigma_g) \cap \text{SHomeo}_+(\Sigma_g)$, there exists an element $\hat{f} \in \text{SHomeo}_+(V_g)$ with $\hat{f}|_{\Sigma_g} = f$.

FIGURE 13. The simple closed curves C_1, \dots, C_{2g+2} on Σ_g .

- (2) Let \hat{f}_0 and \hat{f}_1 be elements of $\text{SHomeo}_+(V_g)$ that are isotopic. Then \hat{f}_0 and \hat{f}_1 are symmetrically isotopic.

Proof. (1) Choose an arbitrary map $f \in \text{EHomeo}_+(\Sigma_g) \cap \text{SHomeo}_+(\Sigma_g)$. Let C_1, \dots, C_{2g+2} be the simple closed curves on Σ_g with $\iota(C_{2k-1}) = C_{2k}$ ($k = 1, \dots, g+1$) as shown in Figure 13. There exist disks D_1, \dots, D_{2g+2} in V_g such that $\partial D_i = C_i$ ($i = 1, \dots, 2g+2$), $D_i \cap D_j = \emptyset$ ($i \neq j$), and $\hat{\iota}(D_{2k-1}) = D_{2k}$ ($k = 1, \dots, g+1$). Let $A_k \subset \Sigma_g$ be the parallelism region, which is an annulus, between C_{k-1} and C_k . Let B_k be the 3-ball bounded by $D_{2k-1} \cup A_k \cup D_{2k}$. Set $C'_i := f(C_i)$. We note that we have

$$\iota(C'_{2k-1}) = \iota(f(C_{2k-1})) = f(\iota(C_{2k-1})) = f(C_{2k}) = C'_{2k}$$

for each $k = 1, \dots, g+1$, and

$$f(\text{Fix}(\iota)) = f(\iota(\text{Fix}(\iota))) = \iota(f(\text{Fix}(\iota))).$$

The latter implies that $f(\text{Fix}(\iota)) = \text{Fix}(\iota)$. By Edmonds [Edm86] there exist disks $D'_1, D'_3, \dots, D'_{2g+1}$ in V_g such that $\partial D'_{2k-1} = C'_{2k-1}$, $D'_{2k-1} \cap \text{Fix}(\hat{\iota}) = \emptyset$, and $D'_{2k-1} \cap \hat{\iota}(D'_{2k-1}) = \emptyset$ ($k = 1, \dots, g+1$). Set $D'_{2k} = \hat{\iota}(D'_{2k-1})$. We note that $\partial D'_{2k} = C'_{2k}$. Since f preserves $\text{Fix}(\iota)$, the self-homeomorphism ϕ of ∂B induced from f preserves $\partial \mathcal{A}$, that is, ϕ is an element of $\text{Homeo}_+(\partial B, \partial \mathcal{A})$. Set $E_k := q(D_{2k-1}) = q(D_{2k})$ and $E'_k := q(D'_{2k-1}) = q(D'_{2k})$ ($k = 1, \dots, g+1$). Since $\partial E'_i \cap \partial E'_j = \emptyset$ ($i \neq j$), the intersection of E'_i and E'_j consists only of simple closed curves. Since $B \setminus \mathcal{A}$ is irreducible, we can move E'_1, \dots, E'_{g+1} by an isotopy in $B \setminus \mathcal{A}$ so as to satisfy $E'_i \cap E'_j = \emptyset$ ($i \neq j$). The disk E_k cuts off from B the 3-ball $q(B_k)$ that contains the single component $q(B_k) \cap \mathcal{A}$ of \mathcal{A} . Similarly, the disk E'_k cuts off from B the 3-ball $q(f(B_k))$ that contains the single component $q(f(B_k)) \cap \mathcal{A}$ of \mathcal{A} . Therefore, we can find an extension $\hat{\phi} \in \text{Homeo}_+(B, \mathcal{A})$ of ϕ with $\hat{\phi}(E_k) = E'_k$. By Lemma 1.19, $\hat{\phi}$ lifts to $\hat{f} \in \text{SHomeo}_+(V_g)$. By replacing \hat{f} with $\hat{i} \circ \hat{f}$, if necessary, we have $\hat{f}|_{\Sigma_g} = f$. (2) For $i = 0, 1$, let $\hat{\phi}_i$ be the elements in $\text{Homeo}_+(B, \mathcal{A})$ induced from \hat{f}_i . Set $\phi_i := \hat{\phi}_i|_{\partial B} \in \text{Homeo}_+(\partial B, \partial \mathcal{A})$. By Theorem 1.18 there exists a symmetric isotopy from $\hat{f}_0|_{\Sigma_g}$ to $\hat{f}_1|_{\Sigma_g}$. This isotopy induces a path in $\text{Homeo}_+(\partial B, \partial \mathcal{A})$ from ϕ_0 to ϕ_1 . By Proposition A.4 in [HK17], there exists

a path in $\text{Homeo}_+(B, \mathcal{A})$ from $\hat{\phi}_0$ to $\hat{\phi}_1$. By Lemma 1.19(2), this path lifts to a path in $\text{SHomeo}_+(V_g)$ with the initial point \hat{f}_0 . Since $\hat{\iota}$ is an involution, the terminal point of the path is either \hat{f}_1 or $\hat{\iota} \circ \hat{f}_1$. The latter is impossible as we assumed that \hat{f}_0 and \hat{f}_1 are isotopic, and so \hat{f}_0 and $\hat{\iota} \circ \hat{f}_1$ cannot be isotopic. This completes the proof. \square

Recall that both $\mathcal{H}(\Sigma_g) (= \pi_0(\text{SHomeo}_+(\Sigma_g)))$ and $\text{MCG}(V_g) (\cong \pi_0(\text{EHomeo}_+(\Sigma_g)))$ can be regarded as subgroups of $\text{MCG}(\Sigma_g)$.

Theorem 1.22. *The map*

$$\text{SHomeo}_+(V_g) \rightarrow \text{EHomeo}_+(\Sigma_g) \cap \text{SHomeo}_+(\Sigma_g)$$

that takes $\hat{f} \in \text{SHomeo}_+(V)$ to $\hat{f}|_{\Sigma_g}$ induces an isomorphism

$$\mathcal{H}(V_g) \xrightarrow{\cong} \text{MCG}(V_g) \cap \mathcal{H}(\Sigma_g).$$

Proof. Recall that $\pi_0(\text{SHomeo}_+(V_g)) = \mathcal{H}(V_g)$. Note that any element of $\text{Homeo}_+(\Sigma_g)$ isotopic to an element of $\text{EHomeo}_+(\Sigma_g)$ is contained in $\text{EHomeo}_+(\Sigma_g)$. This fact, together with Theorem 1.18, allows us to think of $\pi_0(\text{EHomeo}_+(\Sigma_g) \cap \text{SHomeo}_+(\Sigma_g))$ as the subgroup of $\pi_0(\text{EHomeo}_+(\Sigma_g))$ consisting of the mapping classes that contain an element of $\text{SHomeo}_+(\Sigma_g)$. Therefore, we can identify $\pi_0(\text{EHomeo}_+(\Sigma_g) \cap \text{SHomeo}_+(\Sigma_g))$ with $\text{MCG}(V_g) \cap \mathcal{H}(\Sigma_g)$ in a natural way. The surjectivity and injectivity of the map $\mathcal{H}(V_g) \rightarrow \text{MCG}(V_g) \cap \mathcal{H}(\Sigma_g)$ now follow from Lemma 1.21(1) and (2), respectively. \square

In consequence, we have the following canonical identifications from Theorems 1.11, 1.20 and 1.22:

$$\mathcal{H}(V_g) = \text{MCG}(V_g) \cap \mathcal{H}(\partial V_g) = \Pi^{-1}(\Gamma(\text{SW}_{2g+2})).$$

$$\mathcal{H}(V_g)/\langle [\hat{\iota}] \rangle = \text{MCG}(B, \mathcal{A}_{g+1}) = \Gamma(\text{SW}_{2g+2}) = \text{SW}_{2g+2}/\langle \Delta^2 \rangle.$$

2. GOERITZ GROUPS OF BRIDGE DECOMPOSITIONS

Let $(L; S) = (B^+ \cap L) \cup_S (B^- \cap L)$ be an n -bridge decomposition of a link $L \subset S^3$. We define the *Goeritz group*, denoted by $\mathcal{G}(L; S)$ or $\mathcal{G}((B^+ \cap L) \cup_S (B^- \cap L))$, of the n -bridge decomposition by

$$\mathcal{G}(L; S) = \text{MCG}(S^3, B^+, L).$$

Since the map

$$\mathcal{G}(L; S) \rightarrow \text{MCG}(S, S \cap L) \cong \text{MCG}(\Sigma_{0,2n})$$

sending $[f] \in \mathcal{G}(L; S)$ to $[f|_S] \in \text{MCG}(S, S \cap L)$ is injective, we regard $\mathcal{G}(L; S)$ as a subgroup of $\text{MCG}(\Sigma_{0,2n})$. When $(L; S)$ is a unique n -bridge decomposition of L up to equivalence, we simply call $\mathcal{G}(L; S)$ the *n -bridge Goeritz group* of L , and we denote it by $\mathcal{G}(L; n)$. In this section, we discuss several basic properties of this group.

2.1. Relation to wicket groups on tangles. Let $(B^+ \cap L) \cup_S (B^- \cap L)$ be an n -bridge decomposition of a link L . Then there exist braids $b, d \in \text{SB}_{2n}$ such that

$$(B^+ \cap L) \cup_S (B^- \cap L) = \bar{\mathcal{A}}^d \cup_S {}^b\mathcal{A}.$$

We now describe Goeritz groups of the bridge decompositions in terms of wicket groups on tangles.

Theorem 2.1. *Let $b, d \in \text{SB}_{2n}$. Then we have*

$$\mathcal{G}(\bar{\mathcal{A}}^d \cup_S {}^b\mathcal{A}) = \Gamma(\text{SW}_{2n}({}^{d^{-1}}\mathcal{A}, {}^b\mathcal{A})) \cong \text{SW}_{2n}({}^{d^{-1}}\mathcal{A}, {}^b\mathcal{A})/\langle \Delta^2 \rangle.$$

Proof. We prove that $\mathcal{G}(\bar{\mathcal{A}}^d \cup_S {}^b\mathcal{A}) \subset \Gamma(\text{SW}_{2n}({}^{d^{-1}}\mathcal{A}, {}^b\mathcal{A}))$. Take an element $\phi \in \mathcal{G}(\bar{\mathcal{A}}^d \cup_S {}^b\mathcal{A})$. Then there exists a braid $b_\phi \in \text{SB}_{2n}$ with $\Gamma(b_\phi) = \phi$. By abuse of notation, we regard ϕ as a representative of b_ϕ . Notice that ϕ extends to a self-homeomorphism of $S^3 = B^+ \cup_S B^-$ preserving $\bar{\mathcal{A}}^d$ and ${}^b\mathcal{A}$ setwise. By Remark 1.9 and Lemma 1.10, it follows that

$$\bar{\mathcal{A}}^d = \bar{\mathcal{A}}^{db_\phi^{-1}} \text{ and } {}^b\mathcal{A} = {}^{b_\phi b}\mathcal{A}.$$

The second equality implies that $b_\phi \in \text{SW}_{2n}({}^b\mathcal{A})$.

Now, consider the image of $\bar{\mathcal{A}}^d = \bar{\mathcal{A}}^{db_\phi^{-1}}$ under ρ . We have

$${}^{d^{-1}}\mathcal{A} = \rho(\bar{\mathcal{A}}^d) = \rho(\bar{\mathcal{A}}^{db_\phi^{-1}}) = {}^{b_\phi d^{-1}}\mathcal{A}.$$

Hence we have $\mathcal{A} = {}^{db_\phi d^{-1}}\mathcal{A}$, which implies that $db_\phi d^{-1} \in \text{SW}_{2n}$. Thus $b_\phi \in {}^{d^{-1}}(\text{SW}_{2n})d = \text{SW}_{2n}({}^{d^{-1}}\mathcal{A})$ by Lemma 1.13. Putting them together, we have

$$b_\phi \in \text{SW}_{2n}({}^{d^{-1}}\mathcal{A}) \cap \text{SW}_{2n}({}^b\mathcal{A}) = \text{SW}_{2n}({}^{d^{-1}}\mathcal{A}, {}^b\mathcal{A}),$$

which says that $\phi \in \Gamma(\text{SW}_{2n}({}^{d^{-1}}\mathcal{A}, {}^b\mathcal{A}))$. We are done.

The proof of $\mathcal{G}(\bar{\mathcal{A}}^d \cup_S {}^b\mathcal{A}) \supset \Gamma(\text{SW}_{2n}({}^{d^{-1}}\mathcal{A}, {}^b\mathcal{A}))$ is similar. \square

Let $\bar{\mathcal{A}}^d \cup_S {}^b\mathcal{A}$ be an n -bridge decomposition of a link L for some $b, d \in \text{SB}_{2n}$. This is equivalent to the n -bridge decomposition $\bar{\mathcal{A}} \cup_S {}^{db}\mathcal{A}$. By Lemma 1.16 and Theorem 2.1, one sees that their Goeritz groups $\mathcal{G}(\bar{\mathcal{A}}^d \cup_S {}^b\mathcal{A})$ and $\mathcal{G}(\bar{\mathcal{A}} \cup_S {}^{db}\mathcal{A})$ are conjugate to each other in $\text{MCG}(\Sigma_{0,2n})$.

2.2. Relation to hyperelliptic Goeritz groups of Heegaard splittings. Let $(M; \Sigma) = V^+ \cup_\Sigma V^-$ be a genus- g Heegaard splitting with $g \geq 2$. Assume that there exists an involution

$$\hat{\iota} : (M, V^+) \rightarrow (M, V^+)$$

such that $\hat{\iota}|_\Sigma$ is a hyperelliptic involution on the Heegaard surface Σ . By definition $\hat{\iota}|_{V^+}$ and $\hat{\iota}|_{V^-}$ are hyperelliptic involutions of the handlebodies V^+ and V^- , respectively. Let $\text{SHomeo}_+(M, V^+)$ denote the centralizer in $\text{Homeo}_+(M, V^+)$ of $\hat{\iota}$. The *hyperelliptic Goeritz group* $\mathcal{HG}_{\hat{\iota}}(M; \Sigma)$ is then defined by

$$\mathcal{HG}_{\hat{\iota}}(M; \Sigma) = \pi_0(\text{SHomeo}_+(M, V^+)).$$

See Remark 2.4 for this notation. Let $q : M \rightarrow M/\hat{\iota} = S^3$ be the projection. Set $B^\pm := q(V^\pm)$, $S := q(\Sigma)$, $L := q(\text{Fix}(\hat{\iota}))$. We note that $(L; S) = (B^+, B^+ \cap L) \cup_S (B^-, B^- \cap L)$ is a $(g + 1)$ -bridge decomposition of the link $L \subset S^3$.

The following theorem, which implies Theorem 0.1, is again a consequence of Theorem 1.18 and Lemma 1.19 as in Theorem 1.20.

Theorem 2.2. *The natural map*

$$\mathcal{HG}_{\hat{\iota}}(M; \Sigma) \rightarrow \mathcal{G}(L; S)$$

is surjective and its kernel is $\langle [\hat{\iota}] \rangle$. Thus, we have $\mathcal{HG}_{\hat{\iota}}(M; \Sigma)/\langle [\hat{\iota}] \rangle \cong \mathcal{G}(L; S)$.

We denote by $E^\pm \text{Homeo}_+(\Sigma)$ the subgroup of $\text{Homeo}_+(\Sigma)$ consisting of homeomorphisms of Σ that extend to those of V^\pm .

The following theorem, which implies Theorem 0.2, corresponds to Theorem 1.22 for hyperelliptic handlebody groups.

Theorem 2.3. *The map*

$\text{SHomeo}_+(M, V^+) \rightarrow E^+ \text{Homeo}_+(\Sigma) \cap E^- \text{Homeo}_+(\Sigma) \cap \text{SHomeo}_+(\Sigma)$
that takes $\hat{f} \in \text{SHomeo}_+(M, V^+)$ to $\hat{f}|_\Sigma$ induces an isomorphism

$$\mathcal{HG}_{\hat{\iota}}(M; \Sigma) \xrightarrow{\cong} \mathcal{G}(M; \Sigma) \cap \mathcal{H}(\Sigma).$$

Proof. Recall that $\pi_0(\text{SHomeo}_+(M, V^+)) = \mathcal{HG}_{\hat{\iota}}(M; \Sigma)$. As in the proof of Theorem 1.22, we can think of

$$\pi_0(E^+ \text{Homeo}_+(\Sigma) \cap E^- \text{Homeo}_+(\Sigma) \cap \text{SHomeo}_+(\Sigma))$$

as the subgroup of

$$\pi_0(E^+ \text{Homeo}_+(\Sigma) \cap E^- \text{Homeo}_+(\Sigma))$$

consisting of the mapping classes that contain an element of $\text{SHomeo}_+(\Sigma)$. Therefore, we can identify

$$\pi_0(E^+ \text{Homeo}_+(\Sigma) \cap E^- \text{Homeo}_+(\Sigma) \cap \text{SHomeo}_+(\Sigma))$$

with $\mathcal{G}(M; \Sigma) \cap \mathcal{H}(\Sigma)$.

The surjectivity and injectivity of the map $\mathcal{HG}_{\hat{\iota}}(M; \Sigma) \rightarrow \mathcal{G}(M; \Sigma) \cap \mathcal{H}(\Sigma)$ follow from Lemma 1.21(1) and (2), respectively. \square

We can summarize the above discussion as follows. Let L be a link in S^3 admitting an n -bridge decomposition $(L; S) = \bar{\mathcal{A}} \cup_S {}^b \mathcal{A}$ for some $b \in \text{SW}_{2n}$. Let $q : M_L \rightarrow S^3$ be the 2-fold covering branched over L , and set $\Sigma := q^{-1}(S)$. The preimage of q of the genus-0 Heegaard splitting $S^3 = B^+ \cup_S B^-$ gives a genus- $(n - 1)$ Heegaard splitting

$$(M_L; \Sigma) = q^{-1}(B^+) \cup_\Sigma q^{-1}(B^-).$$

We call $(M_L; \Sigma)$ the *Heegaard splitting of M_L associated with the bridge decomposition $(L; S)$* . Let $T : M_L \rightarrow M_L$ be the non-trivial deck transformation of $q : M_L \rightarrow S^3$. We note that $T|_\Sigma : \Sigma \rightarrow \Sigma$ is a hyperelliptic

involution. Let $\mathcal{H}(\Sigma)$ be the hyperelliptic mapping class group associated with $T|_\Sigma$. By Theorems 2.1, 2.2 and 2.3, we have the following canonical identifications:

$$\mathcal{HG}_T(M_L; \Sigma) = \mathcal{G}(M_L; \Sigma) \cap \mathcal{H}(\Sigma) = \Pi^{-1}(\Gamma(\text{SW}_{2n}(\mathcal{A}, {}^b\mathcal{A}))).$$

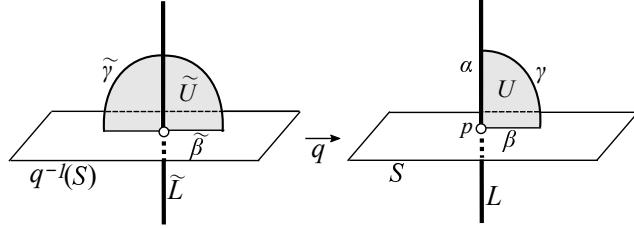
$$\mathcal{HG}_T(M_L; \Sigma)/\langle [T] \rangle = \mathcal{G}(L; S) = \Gamma(\text{SW}_{2n}(\mathcal{A}, {}^b\mathcal{A})) = \text{SW}_{2n}(\mathcal{A}, {}^b\mathcal{A})/\langle \Delta^2 \rangle.$$

Remark 2.4. Let $(M; \Sigma) = V^+ \cup_\Sigma V^-$ be a genus- g Heegaard splitting with $g \geq 2$. Assume that there exists an involution

$$\hat{\iota} : (M, V^+) \rightarrow (M, V^+)$$

such that $\hat{\iota}|_\Sigma$ is a hyperelliptic involution on the Heegaard surface Σ . Some readers might wonder why we write $\mathcal{HG}_{\hat{\iota}}(M; \Sigma)$ rather than $\mathcal{HG}(M; \Sigma)$, whereas a hyperelliptic mapping class group and a hyperelliptic handlebody group are simply denoted by $\mathcal{H}(\Sigma_g)$ and $\mathcal{H}(V_g)$, respectively. The reason for the case of $\mathcal{H}(\Sigma_g)$ is that any two hyperelliptic involutions of a closed surface Σ_g are conjugate in $\text{MCG}(\Sigma_g)$ (see e.g. Farb-Margalit [FM12, Proposition 7.15]). Thus, any two hyperelliptic mapping class groups are conjugate. In particular, the structure of the group $\mathcal{H}(\Sigma_g)$ does not depend on the choice of a particular hyperelliptic involution of Σ_g . The same fact holds for hyperelliptic handlebody groups as well by Lemma 1.17. In the case of hyperelliptic Goeritz groups, however, the situation is more subtle. In fact, the conjugacy class of the above involution $\hat{\iota}$ in the Goeritz group does depend on the choice of $\hat{\iota}$ as we shall see now.

Let (H, S) be the 2-bridge decomposition of the Hopf link $H \subset S^3$. Let K and K' be the components of H . Take $p \in S \cap K$ and $p' \in S \cap K'$, and consider the two 4-bridge decompositions $(H; S_{(p,2)})$ and $(H; (S_{(p,1)})_{(p',1)})$ of H . Since $|S_{(p,2)} \cap K| = 6$ and $|S_{(p,2)} \cap K'| = 2$ whereas $|(S_{(p,1)})_{(p',1)} \cap K| = |(S_{(p,1)})_{(p',1)} \cap K'| = 4$, these bridge decompositions are not equivalent. Let $q : \mathbb{RP}^3 \rightarrow S^3$ be the 2-fold covering branched over H . Let $(\mathbb{RP}^3; \Sigma_1)$ and $(\mathbb{RP}^3; \Sigma_2)$ be the Heegaard splittings of \mathbb{RP}^3 associated with $(H; S_{(p,2)})$ and $(H; (S_{(p,1)})_{(p',1)})$, respectively. Let $T : \mathbb{RP}^3 \rightarrow \mathbb{RP}^3$ be the non-trivial deck transformation of $q : \mathbb{RP}^3 \rightarrow S^3$. Then $T|_\Sigma : \Sigma \rightarrow \Sigma$ and $T|_{\Sigma'} : \Sigma' \rightarrow \Sigma'$ are hyperelliptic involutions on Σ and Σ' , respectively. Recall, by the way, that due to Bonahon-Otal [BO83], the Heegaard splittings $(\mathbb{RP}^3; \Sigma_1)$ and $(\mathbb{RP}^3; \Sigma_2)$ are equivalent. Therefore, for the unique genus-3 Heegaard splitting (\mathbb{RP}^3, Σ) of \mathbb{RP}^3 , we can consider the two involutions: one corresponds to T for (\mathbb{RP}^3, Σ) and the other corresponds to T for (\mathbb{RP}^3, Σ') . We denote the former involution by $\hat{\iota}$ and the latter by $\hat{\iota}'$. Then $\hat{\iota}$ and $\hat{\iota}'$ can no longer be conjugate in the Goeritz group $\mathcal{G}(\mathbb{RP}^3; \Sigma)$, for $(H; S_{(p,2)})$ and $(H; (S_{(p,1)})_{(p',1)})$ are not equivalent. Therefore, it is not necessarily true that the hyperelliptic Goeritz groups $\mathcal{HG}_{\hat{\iota}}(\mathbb{RP}^3; \Sigma)$ and $\mathcal{HG}_{\hat{\iota}'}(\mathbb{RP}^3; \Sigma)$ are conjugate.

FIGURE 14. $\tilde{U} = q^{-1}(U)$ and $\tilde{\gamma} = q^{-1}(\gamma)$.

Let S be an n -bridge sphere of a link $L \subset S^3$. Take a point $p \in L \cap S$. Let

$$(L; S_{(p,k)}) = (B_{(p,k)}^+ \cap L) \cup_{S_{(p,k)}} (B_{(p,k)}^- \cap L)$$

be the bridge decomposition of L , where $S_{(p,k)}$ is the bridge sphere of L obtained from the k -fold stabilization of S near p . Consider the Heegaard splitting

$$q^{-1}(B_{(p,k)}^+) \cup_{q^{-1}(S_{(p,k)})} q^{-1}(B_{(p,k)}^-)$$

of M_L associated with the bridge decomposition $(L; S_{(p,k)})$. We note that the Heegaard surface $q^{-1}(S_{(p,k)})$ of M_L is obtained from the Heegaard surface $q^{-1}(S)$ by k stabilizations. To see this, take a disk U embedded in the 3-ball B^+ together with an arc $\gamma \subset \partial U$ (as in Section 1.3) so that $S_{(p,1)} = \partial(B^- \cup N(\gamma))$ is obtained from S by a stabilization. Let $\tilde{U} = q^{-1}(U)$, $\tilde{\gamma} = q^{-1}(\gamma)$ be the preimages of U , γ , respectively. See Figure 14. Then $\tilde{\gamma}$ is a properly embedded arc in $q^{-1}(B^+)$ which is parallel to $q^{-1}(S)$. Therefore, $\partial(q^{-1}(B^-) \cup N(\tilde{\gamma}))$ is obtained from $q^{-1}(S)$ by a stabilization. By iterating the same argument, we obtain the desired claim.

2.3. Examples. Below we provide several examples of the Goeritz groups of bridge decompositions that can be computed from the definitions. In Examples 2.6 and 2.7 we describe the Goeritz groups of the n -bridge decomposition $(O; n)$ of the trivial knot O and the n -bridge decomposition $(H; S_{p,n-2})$ of the Hopf link H defined in Section 1.3 in terms of wicket groups, which play a key role in Section 5. In Examples 2.8 and 2.9 we give explicit presentations of the Goeritz groups of 2- and 3-bridge decompositions of all 2-bridge links. A sufficient condition for the Goeritz group to be an infinite group in terms of the distance will also be provided.

Example 2.5. Consider the n -bridge decomposition $(O_n; n) = \bar{\mathcal{A}} \cup_S \mathcal{A}$ of the n -component trivial link O_n , where $\bar{\mathcal{A}} = \bar{\mathcal{A}}_n$ and $\mathcal{A} = \mathcal{A}_n$. By Theorems 1.11 and 2.1 we have

$$\mathcal{G}(O_n; n) \cong \Gamma(\text{SW}_{2n}(\mathcal{A}, \mathcal{A})) = \Gamma(\text{SW}_{2n}) = \text{MCG}(B^-, \mathcal{A}).$$

In particular, the group $\mathcal{G}(O_n; n)$ is an infinite group except $\mathcal{G}(O_1; 1) \cong \mathbb{Z}/2\mathbb{Z}$. A finite generating set of $\mathcal{G}(O_n; n)$ is given by work [Hil75] of Hilden

on $\text{MCG}(B^-, \mathcal{A})$. The asymptotic behavior of the minimal pseudo-Anosov dilatations in these groups was studied in [HK17].

Example 2.6. Recall that the n -bridge decomposition $\bar{\mathcal{A}} \cup_{S_{(p,n-1)}} \mathcal{B}$ defined in Section 1.3 is the bridge decomposition $(O; n)$ of the trivial knot O . By Theorem 2.1, we have $\mathcal{G}(O; n) = \Gamma(\text{SW}_{2n}(\mathcal{A}, \mathcal{B}))$. The 3-sphere S^3 is the 2-fold covering of S^3 branched over O . Let $(S^3; \Sigma)$ be the genus- $(n - 1)$ Heegaard splitting of S^3 associated with $(O; n)$. Then by Theorems 2.2 and 2.3 we have

$$\mathcal{H}\mathcal{G}_T(S^3; \Sigma) = \Pi^{-1}(\Gamma(\text{SW}_{2n}(\mathcal{A}, \mathcal{B}))).$$

Example 2.7. Recall that the n -bridge decomposition $\bar{\mathcal{A}} \cup_{S_{(p,n-2)}} \mathcal{C}$ defined in Section 1.3 is the bridge decomposition $(H; S_{(p,n-2)})$ of the Hopf link H . By Theorem 2.1, we have $\mathcal{G}(H; S_{(p,n-2)}) = \Gamma(\text{SW}_{2n}(\mathcal{A}, \mathcal{C}))$. The real projective space \mathbb{RP}^3 is the 2-fold covering of S^3 branched over H . Let $(\mathbb{RP}^3; \Sigma)$ be the genus- $(n - 1)$ Heegaard splitting of \mathbb{RP}^3 associated with $(H; S_{(p,n-2)})$. Then by Theorems 2.2 and 2.3 we have

$$\mathcal{H}\mathcal{G}_T(\mathbb{RP}^3; \Sigma) = \Pi^{-1}(\Gamma(\text{SW}_{2n}(\mathcal{A}, \mathcal{C}))).$$

Example 2.8. Let $L = S(p, r)$ be a 2-bridge link (or the trivial knot) given by the Schubert normal form (Schubert [Sch56], see also Hatcher-Thurston [HT85]). By Ota [Ota82], there exists a unique 2-bridge decomposition $(L; S) = (B^+ \cap L) \cup_S (B^- \cap L)$ of L . If $(p, r) = (0, 1)$, then L is the 2-component trivial link O_2 and the Goeritz group $\mathcal{G}(L; S) = \mathcal{G}(O_2; 2)$ has already been described in Example 2.5. Suppose that $(p, r) \neq (0, 1)$. Since $B^+ \cap L$ (respectively, $B^- \cap L$) is a trivial 2-tangle, there exists a unique essential separating disk D^+ (respectively, D^-) in $B^+ - L$ (respectively, $B^- - L$). This implies that any element of $\mathcal{G}(L; S)$ preserves both D^+ and D^- . Since $(p, r) \neq (0, 1)$, $S - (\partial D^+ \cup \partial D^-)$ consists only of disks, and each of them contains at most one point of $S \cap L$. Therefore, $\mathcal{G}(L; S)$ acts on the pair $(\partial D^+, \partial D^+ \cap \partial D^-)$ faithfully. This implies that $\mathcal{G}(L; S)$ is a subgroup of the dihedral group D_k , where $k = \#(\partial D^+ \cap \partial D^-)$. In fact, it is now easily checked that for any $(p, r) \neq (0, 1)$, the Goeritz group $\mathcal{G}(L; S)$ is isomorphic to $D_2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where the generators are given in Figure 15. In the figure, we think of L and S as being embedded in \mathbb{R}^3 , and the 3-ball bounded by S is B^- . The element shown on the left-hand side in the figure is $\Gamma(\sigma_1^{-1}\sigma_3)$, and the one shown on the right-hand side is $\Gamma(\Delta)$, where $\Gamma : \text{SB}_4 \rightarrow \text{MCG}(\Sigma_{0,4})$.

Example 2.9. Let $L = S(p, r)$ again be a 2-bridge link (or the trivial knot) given by the Schubert normal form. Let $(L; S) = (B^+ \cap L) \cup_S (B^- \cap L)$ be a 3-bridge decomposition of L . Let $q : L(p, r) \rightarrow S^3$ be the 2-fold covering branched over L , where $L(p, r)$ is a lens space. Set $\Sigma := q^{-1}(S)$. Let T be the non-trivial deck transformation of q . Then by Theorem 2.2, we have $\mathcal{G}(L; S) \cong \mathcal{H}\mathcal{G}_T(L(p, r); \Sigma)/\langle [T] \rangle$. Since the genus of Σ is two, it follows

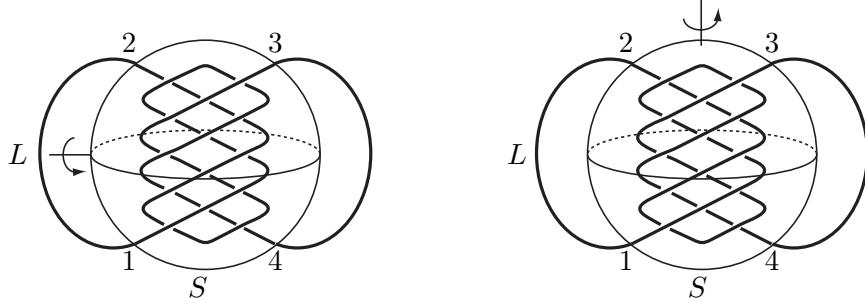


FIGURE 15. The two generators of $\mathcal{G}(L; S) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in the case of $L = S(5, 2)$. They are given as half rotations along the illustrated axes.

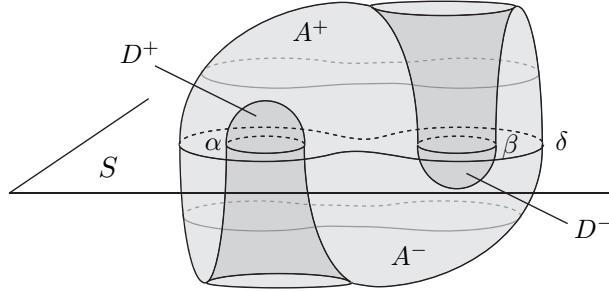
from Theorem 2.3 that the hyperelliptic Goeritz group $\mathcal{HG}_T(L(p, r); \Sigma)$ is canonically isomorphic to the Goeritz group $\mathcal{G}(L(p, r); \Sigma)$ itself, whose finite presentation is given by [Akb08, Cho08, Cho13, CK14, CK16, CK19]. Therefore, we can obtain a finite presentation of the Goeritz group $\mathcal{G}(L; S) \cong \mathcal{G}(L(p, r); \Sigma)/\langle[T] \rangle$ for each (p, r) .

Question 2.10. *Is the Goeritz group of the bridge decomposition of a link in S^3 always finitely generated? In particular, is $\mathcal{G}(O; n)$ finitely generated for any n ?*

Example 2.11. Let $(L; S) = (B^+, B^+ \cap L) \cup_S (B^-, B^- \cap L)$ be an n -bridge decomposition of a link $L \subset S^3$ with $n \geq 3$ and $d(L; S) \leq 1$. Then we can show that the Goeritz group $\mathcal{G}(L; S)$ is an infinite group as follows. Here we regard $\mathcal{G}(L; S)$ as a subgroup of $\text{MCG}(S, S \cap L)$ consisting of elements that extend to both $\text{MCG}(B^+, B^+ \cap L)$ and $\text{MCG}(B^-, B^- \cap L)$. For convenience, we will not distinguish curves and homeomorphisms from their isotopy classes. Similar arguments for Heegaard splittings can be found in Johnson-Rubinstein [JR13, Corollary 6.2] and Namazi [Nam07, Proposition 1].

Suppose first that $d(L; S) = 0$. Then there exists an essential simple closed curve $\alpha \in \mathcal{D}^+ \cap \mathcal{D}^-$, that is, α is a simple closed curve on $S_L = \text{Cl}(S - N(S \cap L; S))$ bounding a disk $D^+ \subset B^+ - L$ and a disk $D^- \subset B^- - L$. Therefore, the Dehn twist τ_α about α is an element of $\mathcal{G}(L; S)$. Indeed, τ_α extends to an element of $\mathcal{G}(L; S)$ as a rotation along the sphere $D^+ \cup D^-$. Since α is essential, the order of τ_α in $\text{MCG}(S, S \cap L)$ is infinite. Thus $\mathcal{G}(L; S)$ is an infinite group.

Suppose that $d(L; S) = 1$. Then there exist disjoint essential simple closed curves $\alpha \in \mathcal{D}^+$ and $\beta \in \mathcal{D}^-$. (Note that here we use the assumption $n \geq 3$. Indeed, in the case of $n = 2$ the definition of the curve graph $\mathcal{C}(S_L)$ is different from the usual case.) The simple closed curve α bounds a disk $D^+ \subset B^+ - L$, and β bounds a disk $D^- \subset B^- - L$. Take a simple arc

FIGURE 16. The 2-sphere $X = D^+ \cup A^- \cup A^+ \cup D^-$.

$\gamma \subset S_L$ connecting α and β . Let δ be the component of the boundary of $N(\alpha \cup \gamma \cup \beta; S)$ that is not isotopic to α or β . Then α and δ cobound an annulus $A^- \subset B^- - L$, while β and δ cobound an annulus A^+ in $D^+ \subset B^+ - L$. In this way, we obtain a 2-sphere $X := D^+ \cup A^- \cup A^+ \cup D^-$ in $S^3 - L$ with $X \cap S = \alpha \cup \beta \cup \delta$, see Figure 16. Consider the map $\phi := \tau_\alpha \circ \tau_\delta^{-1} \circ \tau_\beta : (S, S \cap L) \rightarrow (S, S \cap L)$. By the above construction, ϕ extends to a homeomorphism of B^+ as the composition of the twist about D^+ and the twist about A^+ , while ϕ extends to a homeomorphism of B^- as the composition of the twist about D^- and the twist about A^- . Thus, ϕ extends to an element $\hat{\phi}$ of $\mathcal{G}(L; S)$. Since α, β, δ are pairwise disjoint, pairwise non-parallel, essential simple closed curves on S_L , the order of ϕ in $\text{MCG}(S, S \cap L)$ is infinite. Therefore, $\mathcal{G}(L; S)$ is an infinite group.

3. THE GOERITZ GROUPS OF HIGH DISTANCE BRIDGE DECOMPOSITIONS

As we have seen in Example 2.11, the Goeritz group of a bridge decomposition $(L; S)$ is an infinite group if the distance of $(L; S)$ is at most one. In contrast to Example 2.11, we are going to show that the Goeritz group of $(L; S)$ is a finite group if the distance of $(L; S)$ is sufficiently large. The aim of this section is to prove Theorem 0.3, which is restated below.

Theorem 3.1. *There exists a uniform constant N such that if the distance of an n -bridge decomposition $(L; S)$ of a link L in S^3 with $n \geq 3$ is greater than N , then the Goeritz group $\mathcal{G}(L; S)$ is a finite group.*

We note that an analogous fact was proved for Heegaard splittings by Namazi [Nam07], that is, in that paper he showed that if the distance of a Heegaard splitting is sufficiently large, its Goeritz group is a finite group. If the distance of the Heegaard splitting associated with a bridge decomposition of high distance is also high, then Namazi's result together with Theorems 2.2 and 2.3 immediately implies Theorem 3.1. However, we do not know at present whether there exists a lower bound of the distance of the associated Heegaard splitting in terms of that of a bridge decomposition. Instead, we will give a more direct proof, which shares the same spirit as

Namazi's proof. We also note that Ohshika-Sakuma [OS16] studied another kind of groups related to Heegaard splittings and bridge decompositions. The main result of [Nam07] and the above theorem are comparable with Theorem 2 of [OS16].

Lemma 3.2. *If $\mathcal{G}(L; S)$ contains a non-periodic reducible element, then the distance $d(L; S)$ is at most $2K + 4$. Here K is the uniform constant in Theorem 1.4.*

Proof. Assume that $\mathcal{G}(L; S)$ contains non-periodic reducible element ϕ . Let γ be a curve in the canonical reducing system for ϕ .

Claim 3.3. *Let α be an element of $\mathcal{C}^0(S_L)$, where we recall that $S_L = \text{Cl}(S - N(S \cap L); S)$. If $k > 0$ is sufficiently large, then the distance between γ and any geodesic segment that connects α and $\phi^k(\alpha)$ is at most 2.*

Proof of Claim 3.3. Let $\{\gamma_1, \dots, \gamma_s\}$ be the canonical reducing system for ϕ . Note that $d_{\mathcal{C}(\Sigma)}(\gamma, \gamma_i) \leq 1$ for $1 \leq i \leq s$. By Namazi [Nam07, Proposition 3.2], there exists an essential subsurface Y of S_L , where Y is a pseudo-Anosov component of ϕ or an annular neighborhood of some γ_i , such that $d_{\mathcal{C}(Y)}(\pi_Y(\alpha), \pi_Y(\phi^k(\alpha))) \rightarrow \infty$ as $k \rightarrow \infty$. Let c be a geodesic segment connecting α and $\phi^k(\alpha)$. By Theorem 1.2, if every vertex of c intersects Y , then we have $\text{diam}_{\mathcal{C}(Y)}(\pi_Y(c)) \leq C$. Here $C > 0$ is the constant in Theorem 1.2. Since $d_{\mathcal{C}(Y)}(\pi_Y(\alpha), \pi_Y(\phi^k(\alpha))) \rightarrow \infty$ as $k \rightarrow \infty$, there exists a vertex of c that does not intersect Y for a sufficiently large k . Thus the distance between ∂Y and c is at most 1 in the curve graph $\mathcal{C}(S_L)$. Therefore the distance between γ and c is at most 2. \square

Let α be an arbitrary element of \mathcal{D}^+ . Let $k > 0$ be a sufficiently large integer. Let c be a geodesic segment that connects α and $\phi^k(\alpha)$. By Theorem 1.4, c lies within the K -neighborhood of \mathcal{D}^+ . Combining this fact and Claim 3.3, we conclude that the distance between γ and \mathcal{D}^+ is at most $K + 2$.

Since the same argument can be applied to \mathcal{D}^- , the distance between γ and \mathcal{D}^- is at most $K + 2$. Thus we have $d(L; S) \leq d_{\mathcal{C}(S_L)}(\mathcal{D}^+, \gamma) + d_{\mathcal{C}(S_L)}(\gamma, \mathcal{D}^-) \leq 2K + 4$. \square

Lemma 3.4. *If $\mathcal{G}(L; S)$ contains a pseudo-Anosov element, then the distance $d(L; S)$ is at most $2K + 2\delta$. Here K and δ are the uniform constants in Theorem 1.4 and Theorem 1.1, respectively.*

Proof. Let α (respectively, β) be an arbitrary element of \mathcal{D}^+ (respectively, \mathcal{D}^-). Let c_k be a geodesic segment connecting α and $\phi^k(\alpha)$ for each $k > 0$. Let d_k be a geodesic segment connecting β and $\phi^k(\beta)$ for each $k > 0$. By Theorem 1.1, the distance between c_k and d_k is at most 2δ when k is sufficiently large. By Theorem 1.4, c_k lies within the K -neighborhood of \mathcal{D}^+ . Similarly, d_k lies within the K -neighborhood of \mathcal{D}^- . Thus we have

$$d(L; S) \leq d_{\mathcal{C}(S_L)}(\mathcal{D}^+, c_k) + d_{\mathcal{C}(S_L)}(c_k, d_k) + d_{\mathcal{C}(S_L)}(d_k, \mathcal{D}^-) \leq 2K + 2\delta,$$

which is our assertion. \square

The following lemma is well known, but we include a proof for completeness.

Lemma 3.5. *Any torsion subgroup of $\text{MCG}(\Sigma_{0,2n})$ is finite.*

Proof. By Serre [Ser61], $\text{Mod}(\Sigma_{n-1})$ contains a torsion-free subgroup G of finite index. Set $G' := \mathcal{H}(\Sigma_{n-1}) \cap G$. Then we have

$$[\mathcal{H}(\Sigma_{n-1}) : G'] = [\mathcal{H}(\Sigma_{n-1}) \cdot G : G] \leq [\text{Mod}(\Sigma_{n-1}) : G] < \infty.$$

Thus, the index of $G_0 := \Pi(G')$ in $\text{MCG}(\Sigma_{0,2n})$ is finite, where we recall that $\Pi : \mathcal{H}(\Sigma_{n-1}) \rightarrow \text{MCG}(\Sigma_{0,2n})$ is the natural map. Since G' is torsion free and $\ker \Pi \cong \mathbb{Z}/2\mathbb{Z}$, G_0 is also torsion free.

Suppose that F is a torsion subgroup of $\text{MCG}(\Sigma_{0,2n})$. Since G_0 is torsion free, $F \cap G_0$ is the trivial group. Since $[\text{MCG}(\Sigma_{0,2n}) : G_0]$ is finite, we conclude that F is a finite group. \square

Proof of Theorem 3.1. Set $N := \max\{2K+4, 2K+2\delta\}$, where we recall this is a uniform constant. Let $(L; S)$ be an n -bridge decomposition of a link in S^3 with $n \geq 3$. Suppose that $d(L; S) > N$. By Lemmas 3.2 and 3.4, $\mathcal{G}(L; S)$ contains neither a reducible element nor a pseudo-Anosov element. Thus $\mathcal{G}(L; S)$ is a torsion subgroup of $\text{MCG}(\Sigma_{0,2n})$. By Lemma 3.5, $\mathcal{G}(L; S)$ is a finite group. \square

As we have explained in Sections 1.4 and 1.5, the constant δ can be chosen to be at most 102, and K can be chosen to be at most 1796. Therefore, the above proof shows that the constant 3796 is enough for the constant N in Theorem 3.1.

4. PSEUDO-ANOSOV ELEMENTS IN THE GOERITZ GROUPS OF STABILIZED BRIDGE DECOMPOSITIONS

It follows immediately from Example 2.11 and Lemma 1.3 that the Goeritz group of a stabilized bridge decomposition of a link in S^3 is an infinite group except the case of the 2-bridge decomposition $(O; 2)$ of the trivial knot O . For each of those bridge decompositions, we can find an infinite order element of the Goeritz group looking at a local part of the decomposition as follows. Let $(L; S)$ be an n -bridge decomposition of L with $n \geq 2$. Let p be a point in $S \cap L$. Without loss of generality, we may assume that the point p is labeled by $2n$. Consider the $(n+1)$ -bridge decomposition

$$(L; S_{(p,1)}) = (B_{(p,1)}^+ \cap L) \cup_{S_{(p,1)}} (B_{(p,1)}^- \cap L).$$

Set $S' := S_{(p,1)}$. Recall that the triples (S^3, S, L) and (S^3, S', L) are identical except within a small 3-ball B near the point p shown in Figure 17(1). Set $\alpha := \partial B \cap S'$. Since α bounds a disk $D^+ \subset B^+$ ($D^- \subset B^-$, respectively) with $\#(D^+ \cap L) = 1$ ($\#(D^- \cap L) = 1$, respectively), the Dehn twist $\tau_\alpha : (S', S' \cap L) \rightarrow (S', S' \cap L)$ extends to an element of $\text{MCG}(B_{(p,1)}^+, B_{(p,1)}^+ \cap L)$ ($\text{MCG}(B_{(p,1)}^-, B_{(p,1)}^- \cap L)$, respectively). Therefore, $\tau_\alpha \in \text{MCG}(\Sigma_{0,2n+2})$

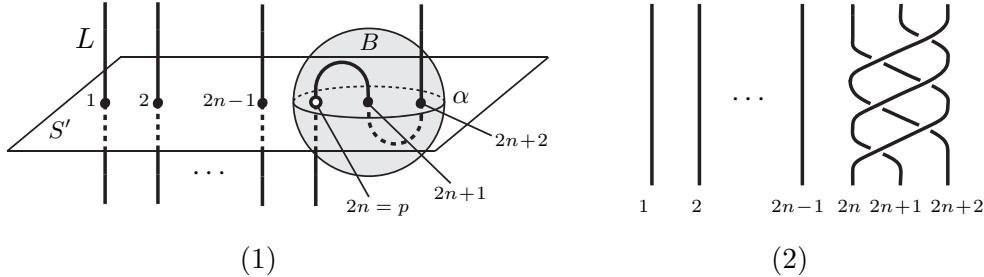


FIGURE 17. (1) A stabilized $(n + 1)$ -bridge decomposition of a link L and the 3-ball B . (2) An element $b \in \text{SW}_{2n+2}$ (which is a ‘full twist’ with 3 strands).

defines an element $\hat{\tau}_\alpha$ of $\mathcal{G}(L; S')$, whose order is clearly infinite. Note that $\tau_\alpha = \Gamma(b)$, where b is the element of SW_{2n+2} shown in Figure 17(2).

The infinite-order elements of the Goeritz group of a stabilized bridge decomposition we have given so far are all reducible: each of them is an extension of either a single Dehn twist (Figure 17) or the composition of the Dehn twists about three disjoint simple closed curves (Example 2.11) in the bridge sphere. In this section, we discuss pseudo-Anosov elements in the Goeritz group. In fact, we prove Theorem 0.4, which is restated below.

Theorem 4.1. *Let $(L; S)$ be an n -bridge decomposition of a link L in S^3 with $n \geq 2$. Let p be an arbitrary point in $S \cap L$. If $(L; S) = (O_2; 2)$, then the Goeritz group $\mathcal{G}(L; S_{(p,1)})$ is an infinite group consisting only of reducible elements. Otherwise, the Goeritz group $\mathcal{G}(L; S_{(p,1)})$ contains a pseudo-Anosov element.*

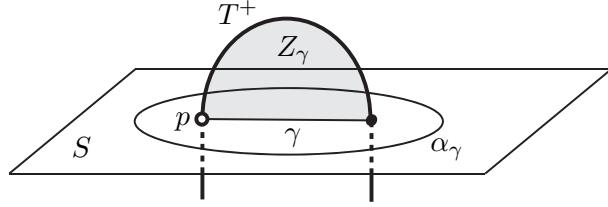
There are two ingredients for the construction of a pseudo-Anosov element in the above theorem. One is a slight modification of the element given in the first paragraph of this section, which corresponds to a Dehn twist about a simple closed curve in S_L . The other is a construction of pseudo-Anosov elements by Penner [Pen88].

In the following arguments, we always assume that curves under consideration in a (marked) surface are properly embedded, and their intersection is transverse and minimal up to isotopy. We will not distinguish curves, surfaces and homeomorphisms from their isotopy classes in this section.

Let

$$(L; S) = (B^+ \cap L) \cup_S (B^- \cap L)$$

be an n -bridge decomposition of a link L in S^3 with $n \geq 2$. Let p be an arbitrary point in $S \cap L$. Then there exists a unique component T^+ (T^- , respectively) of $B^+ \cap L$ ($B^- \cap L$, respectively) one of whose endpoints is p .

FIGURE 18. A reference arc γ for T^+ .

A simple arc γ in the marked sphere $(S, S \cap L)$ with $\partial\gamma = \partial T^+$ is called a *reference arc* for T^+ if there exists a disk Z_γ embedded in B^+ such that $Z_\gamma \cap L = \partial Z_\gamma \cap L = T^+$ and $\partial Z_\gamma - T^+ = \gamma$. A simple closed curve $\alpha \in \mathcal{D}^+$ is said to be *associated with* p if there exists a reference arc γ for T^+ with $\alpha = \partial N(\gamma; S)$. In this case, we write $\alpha = \alpha_\gamma$. See Figure 18. We note that Z_γ and α_γ as above are uniquely determined for each γ . We denote by \mathcal{D}_p^+ the subset of \mathcal{D}^+ consisting of simple closed curves associated with p . The subset $\mathcal{D}_p^- \subset \mathcal{D}^-$ is defined exactly in the same way as above (using “ $-$ ” instead of “ $+$ ”). Two simple closed curves $\alpha, \beta \in \mathcal{C}(S_L)$ are said to *fill* the surface S_L if the union $\alpha \cup \beta$ cuts S_L into open disks and half-open annuli.

Lemma 4.2. *If $(L; S) \neq (O_2; 2)$, then there exist simple closed curves $\alpha^+ \in \mathcal{D}_p^+$ and $\alpha^- \in \mathcal{D}_p^-$ that fill S_L .*

Proof. Suppose first that $n = 2$, that is, $(L; S)$ is a 2-bridge decomposition. In this case, we have $\mathcal{D}_p^+ = \mathcal{D}^+$ ($\mathcal{D}_p^- = \mathcal{D}^-$, respectively), and it consists of only one simple closed curve α^+ (α^- , respectively) (cf. Example 2.8). If $(L; S) \neq (O_2; 2)$, then using its Schubert normal form it is easily seen that α^+ and α^- fill S_L . (In the case $(L; S) = (O_2; 2)$, we have $\alpha^+ = \alpha^-$ and they do not fill S_L .)

In the following we suppose that $n \geq 3$. Choose arbitrary simple closed curves $\alpha^+ \in \mathcal{D}_p^+$ and $\beta \in \mathcal{D}_p^-$. By [HK17, Proposition 1.3] there exists a pseudo-Anosov element ϕ in $\text{MCG}(B^+, B^+ \cap L)$. By replacing ϕ with some positive power, if necessary, we can assume that ϕ is the identity on $\partial B^+ \cap L$. It follows from Masur-Minsky [MM99, Proposition 4.6] that

$$\lim_{k \rightarrow \infty} d_{\mathcal{C}(S_L)}(\alpha^+, \phi^k(\alpha^+)) = \infty,$$

which in particular implies that there exists $k_0 \in \mathbb{N}$ with $d_{\mathcal{C}(S_L)}(\alpha^+, \phi^{k_0}(\alpha^+)) \geq 5$. Since we have assumed that ϕ fixes $\partial B^+ \cap L$, the image $\phi^{k_0}(\alpha^+)$ remains to be contained in \mathcal{D}_p^+ . Now by applying triangle inequalities we have $d_{\mathcal{C}(S_L)}(\alpha^+, \alpha^-) \geq 3$ or $d_{\mathcal{C}(S_L)}(\phi^{k_0}(\alpha^+), \alpha^-) \geq 3$, which implies the assertion. \square

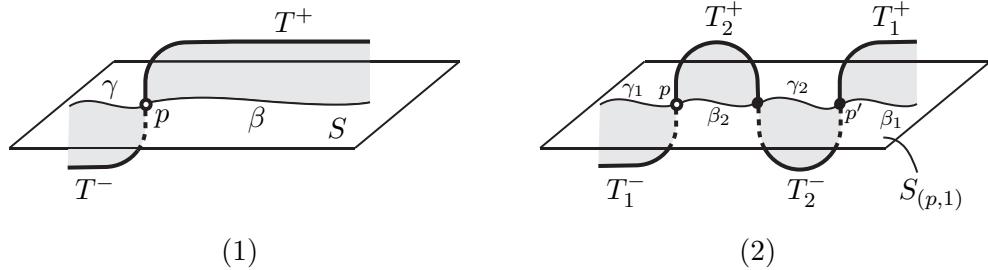


FIGURE 19. (1) The bridge decomposition $(L; S)$ around p .
(2) The bridge decomposition $(L; S_{(p,1)})$ around p .

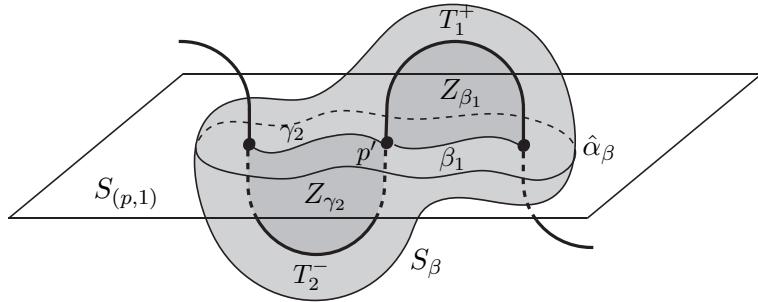


FIGURE 20. The sphere S_β .

Proof of Theorem 4.1. By a 1-fold stabilization of $(L; S)$ at p , we obtain a bridge decomposition

$$(L; S_{(p,1)}) = (B_{(p,1)}^+ \cap L) \cup_{S_{(p,1)}} (B_{(p,1)}^- \cap L).$$

Suppose first that $(L; S) \neq (O_2; 2)$. By Lemma 4.2 there exist reference arcs β and γ for T^+ and T^- , respectively, such that simple closed curves $\alpha_\beta \in \mathcal{D}_p^+$ and $\alpha_\gamma \in \mathcal{D}_p^-$ fill S_L . Let T_1^+ and T_2^+ (T_1^- and T_2^- , respectively) be the components of $B_{(p,1)}^+ \cap L$ ($B_{(p,1)}^- \cap L$, respectively), p' the point of $S_{(p,1)} \cap L$ as shown in Figure 19. Since the component T_1^+ (T_1^- , respectively) of $B_{(p,1)}^+ \cap L$ ($B_{(p,1)}^- \cap L$, respectively) naturally corresponds to T^+ (T^- , respectively), the reference arc β (γ , respectively) defines in a canonical way a reference arc β_1 (γ_1 , respectively) for T_1^+ (T_1^- , respectively). Here we note that under the convention in Section 1.3 (Figure 5), γ_1 is nothing but γ thought of as being embedded in $S_{(p,1)}$. Choose a reference arc β_2 (γ_2 , respectively) for T_2^+ (T_2^- , respectively) so that $\beta_2 \cap \gamma_1 = \{p\}$ ($\gamma_2 \cap \beta_1 = \{p'\}$, respectively). Consider the 2-spheres $S_\beta := \partial N(Z_{\beta_1} \cup Z_{\gamma_2})$ and $S_\gamma := \partial N(Z_{\gamma_1} \cup Z_{\beta_2})$, see Figure 20. Since α_β and α_γ fill the surface S_L , the simple closed curves $\hat{\alpha}_\beta := S_{(p,1)} \cap S_\beta$ and $\hat{\alpha}_\gamma := S_{(p,1)} \cap S_\gamma$ fill $(S_{(p,1)})_L$. Set $D_\beta^+ := S_\beta \cap B_{(p,1)}^+$, $D_\beta^- := S_\beta \cap B_{(p,1)}^-$,

$D_\gamma^+ := S_\gamma \cap B_{(p,1)}^+$ and $D_\gamma^- := S_\gamma \cap B_{(p,1)}^-$. Then each of the disks $D_\beta^+, D_\beta^-, D_\gamma^+$ and D_γ^- intersects L once and transversely. This implies that each of the Dehn twists $\tau_{\widehat{\alpha}_\beta}$ and $\tau_{\widehat{\alpha}_\gamma}$ extends to an element of $\mathcal{G}(L; S_{(p,1)})$. Now, it follows from Penner [Pen88] that the composition $\tau_{\widehat{\alpha}_\beta} \circ \tau_{\widehat{\alpha}_\gamma}^{-1}$ gives rise to a pseudo-Anosov element of $\mathcal{G}(L; S)$.

Finally, suppose that $(L; S) = (O_2; 2)$. Let $q : S^2 \times S^1 \rightarrow S^3$ be the 2-fold covering branched over L . Set $\Sigma := q^{-1}(S_{(p,1)})$, which is the unique genus-2 Heegaard splitting of $S^2 \times S^1$. Let T be the non-trivial deck transformation of q . Then as we have seen in Example 2.9, the group $\mathcal{G}(L; S)$ is isomorphic to $\mathcal{G}(S^2 \times S^1; \Sigma)/\langle [T] \rangle$. The conclusion now follows from Cho-Koda [CK14], which shows that the Goeritz group $\mathcal{G}(S^2 \times S^1; \Sigma)$ is an infinite group consisting only of reducible elements. \square

5. ASYMPTOTIC BEHAVIOR OF MINIMAL PSEUDO-ANOSOV ENTROPIES

In this section, we prove Theorems 0.5 and 0.6.

We say that $\beta \in SB_n$ is *pseudo-Anosov* if $\Gamma(\beta) \in MCG(\Sigma_{0,n})$ is a pseudo-Anosov mapping class. In this case, the *dilatation* $\lambda(\beta)$ (respectively, *entropy* $\log \lambda(\beta)$) of β is defined by the dilatation (respectively, entropy) of $\Gamma(\beta)$.

Collapsing ∂D to a point, we obtain an injective homomorphism

$$\mathfrak{c} : MCG(D_n) \rightarrow MCG(\Sigma_{0,n+1}).$$

We say that $b \in B_n$ is *pseudo-Anosov* if $\mathfrak{c}(\Gamma_D(b))$ is a pseudo-Anosov mapping class. Then the *dilatation* $\lambda(b)$ (respectively, *entropy* $\log \lambda(b)$) of b is defined by the dilatation (respectively, entropy) of $\mathfrak{c}(\Gamma_D(b))$. Let

$$s : B_n \rightarrow SB_n$$

be the surjective homomorphism which sends a braid $b \in B_n$ to the spherical braid in SB_n represented by the same word of letters $\sigma_j^{\pm 1}$'s as b . Let

$$s_+ : B_n \rightarrow SB_{n+1}$$

be the homomorphism which sends a braid $b \in B_n$ to the spherical braid obtained from $s(b)$ with n strands adding the $(n+1)$ th straight strand. Hence $s_+(b)$ is also represented by the same word of letters $\sigma_j^{\pm 1}$'s as b .

Remark 5.1. *By the definition of pseudo-Anosov braids in B_n , we see that $b \in B_n$ is pseudo-Anosov if and only if $s_+(b) \in SB_{n+1}$ is pseudo-Anosov. In this case, $\lambda(b) = \lambda(s_+(b))$ holds.*

For the proofs of Theorems 0.5 and 0.6, we use a result in [HK20], which we recall now. Let z_n be a pseudo-Anosov braid on the plane with d_n strands. We say that a sequence $\{z_n\}$ has a *small normalized entropy* if $d_n \asymp n$ and there is a constant $P > 0$ which does not depend on n such that

$$(5.1) \quad d_n \cdot \log \lambda(z_n) \leq P.$$

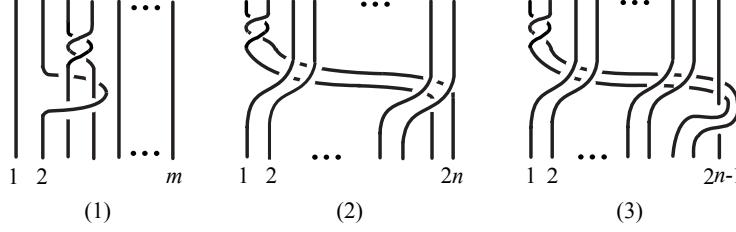


FIGURE 21. (1) $X = X_m \in B_m$. (2) $Y = Y_{2n} \in B_{2n}$. (3) $Z = Z_{2n-1} \in B_{2n-1}$.

Here recall that $d_n \asymp n$ means that d_n is comparable to n . It is known by Penner [Pen91] that if $\phi \in \text{MCG}(\Sigma_{0,n})$ is pseudo-Anosov for $n \geq 4$, then $\log \lambda(\phi) \geq \frac{\log 2}{4n-12}$. This implies that if $\{z_n\}$ satisfies (5.1), then we have

$$\log \lambda(z_n) \asymp \frac{1}{n}.$$

For the definition of *i-increasing planar braids*, see [HK20, Section 3]. (*i* stands for the indices of strands.) If a pseudo-Anosov braid b is *i*-increasing, then one obtains a pseudo-Anosov braid $(b\Delta^{2n})_1$ with more strands than b for each $n \geq 1$ which is well-defined up to conjugate [HK20, Section 4.1]. The number of strands of $(b\Delta^{2n})_1$ can be computed from b . The braid $(b\Delta^{2n})_1$ enjoys the property that the mapping torus of $(b\Delta^{2n})_1$ is homeomorphic to the mapping torus of the original braid b [HK20, Example 4.4(3)]. Furthermore, the sequence of pseudo-Anosov braids $\{(b\Delta^{2n})_1\}$ varying n has a small normalized entropy [HK20, Theorem 5.2(3)].

For each $m \geq 5$ and $n \geq 3$, we define the braids X, Y and Z as follows.

$$\begin{aligned} X = X_m &= (\sigma_2\sigma_3)^3 = \sigma_3^2\sigma_2\sigma_3^2\sigma_2 \in B_m, \\ Y = Y_{2n} &= \sigma_1^2\sigma_2\sigma_3 \cdots \sigma_{2n-1}\sigma_1\sigma_2 \cdots \sigma_{2n-2} \in B_{2n}, \\ Z = Z_{2n-1} &= \sigma_1^2\sigma_2\sigma_3 \cdots \sigma_{2n-2}\sigma_1\sigma_2 \cdots \sigma_{2n-3}\sigma_{2n-3}\sigma_{2n-2} \in B_{2n-1}, \end{aligned}$$

see Figure 21. The spherical 6-braids $s_+(X_5)$, $s_+(Y_6)$ and $s_+(Z_5)$ are equal to x , y , and z , respectively, in Example 1.12. For each $n \geq 3$, we have $s(Y), s_+(Z) \in \text{SB}_{2n}$. For each $m = 2n - 1$ with $n \geq 3$, we have $s_+(X) \in \text{SB}_{2n}$. We write

$$x = s_+(X), \quad y = s(Y), \quad z = s_+(Z) \in \text{SB}_{2n}.$$

It is easy to see the following lemma. (Cf. Lemma 1.14.)

Lemma 5.2. *We have $x, y \in \text{SW}_{2n}(\mathcal{A}, \mathcal{B})$ and $x, z \in \text{SW}_{2n}(\mathcal{A}, \mathcal{C})$.*

For a subgroup $G \subset \text{SB}_n$ containing a pseudo-Anosov element, we write $\ell(G) = \ell(\Gamma(G))$. Example 2.6 tells us that

$$\ell(\mathcal{G}(O; n)) = \ell(\text{SW}_{2n}(\mathcal{A}, \mathcal{B})).$$

We can then restate Theorem 0.5 as follows.

Theorem 5.3. *We have*

$$\ell(\mathcal{G}(O; n)) = \ell(\text{SW}_{2n}(\mathcal{A}, \mathcal{B})) \asymp \frac{1}{n}.$$

Theorem 5.3 implies Corollary 0.7. The reason is that if $\phi \in \text{MCG}(\Sigma_{0,2g+2})$ is pseudo-Anosov, then each element of $\Pi^{-1}(\phi)$ is pseudo-Anosov with the same entropy as ϕ . Theorem 5.3 together with Example 2.6 says that

$$\ell(\mathcal{HG}_T(S^3; q^{-1}(S_{(p,g)}))) \asymp \frac{1}{g}.$$

Since

$$\mathcal{HG}_T(S^3; q^{-1}(S_{(p,g)})) \subset \mathcal{G}(S^3; g) \subset \text{MCG}(\Sigma_g)$$

and $\ell(\text{MCG}(\Sigma_g)) \asymp \frac{1}{g}$ by Penner [Pen91], we conclude that

$$\ell(\mathcal{G}(S^3; g)) \asymp \frac{1}{g}.$$

Proof of Theorem 5.3. In the proof, we regard D_n as the disk with n punctures. For braids $X, Z \in B_5$ as above, we consider the product

$$\alpha := XZ = (\sigma_3^2 \sigma_2 \sigma_3^2 \sigma_2)(\sigma_1^2 \sigma_2 \sigma_3 \sigma_4 \sigma_1 \sigma_2 \sigma_3^2 \sigma_4) \in B_5,$$

see Figure 22(1). We now claim that α is pseudo-Anosov. To see this, we first observe that

$$\alpha \Delta^{-2} = \sigma_2 \sigma_3 \sigma_3 \sigma_2 \sigma_4^{-1} \sigma_3^{-1} \sigma_3^{-1} \sigma_4^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_3^{-1} \sigma_2^{-1}.$$

Let γ denote the following 5-braid

$$\gamma = \sigma_1 \sigma_2 \sigma_4^{-1} \sigma_3^{-1} \sigma_3^{-1} \sigma_4^{-1} \sigma_2^{-1} \sigma_3^{-1}.$$

Then one can check that $\eta^{-1}(\alpha \Delta^{-2}) \eta \gamma^{-1}$ equals the identity element in B_5 , where

$$\eta = \sigma_1 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_1 \sigma_4 \cdot \sigma_1 \sigma_2 \sigma_1 \sigma_4 \sigma_3 \sigma_2 \sigma_1 \cdot \sigma_2 \sigma_3 \sigma_2 \sigma_1 \sigma_4 \sigma_3 \cdot \sigma_3 \sigma_4 \sigma_3 \sigma_2 \sigma_1.$$

This implies that $\Gamma_D(\alpha) = \Gamma_D(\alpha \Delta^{-2})$ is conjugate to $\Gamma_D(\gamma)$ in $\text{MCG}(D_5)$. It is enough to show that γ is pseudo-Anosov, for the Nielsen-Thurston types are preserved under the conjugation.

Removing the third and fourth strands from γ , we obtain $\sigma_1 \sigma_2^{-2} \in B_3$. It is easy to see that $\sigma_1 \sigma_2^{-2}$ is pseudo-Anosov. For instance, see [Han97]. Let $\Phi : D_3 \rightarrow D_3$ be a pseudo-Anosov homeomorphism which represents $\Gamma_D(\sigma_1 \sigma_2^{-2}) \in \text{MCG}(D_3)$. Let \mathcal{O} be a periodic orbit with period k of Φ . Blow up each periodic point in \mathcal{O} . Then we still have a pseudo-Anosov homeomorphism $\Phi^\circ : D_3 \setminus \mathcal{O} \rightarrow D_3 \setminus \mathcal{O}$ defined on the $(3+k)$ -punctured disk with the same entropy as Φ . By using train track maps for pseudo-Anosov 3-braids (see [Han97]), it is not hard to show that there is a periodic orbit \mathcal{O} with period 2 such that the pseudo-Anosov homeomorphism $\Phi^\circ : D_3 \setminus \mathcal{O} \rightarrow D_3 \setminus \mathcal{O}$ represents $\Gamma_D(\gamma) \in \text{MCG}(D_5)$. Thus γ is pseudo-Anosov.

Next, we prove that $\ell(\text{SW}_{4n+2}(\mathcal{A}, \mathcal{B})) \asymp \frac{1}{n}$. We note that the above $\alpha \in B_5$ is a 5-increasing braid. One sees that $(\alpha \Delta^{2n})_1$ is written by

$$(\alpha \Delta^{2n})_1 = XZ^{2n+1} \in B_{4n+7}.$$

(See Figure 22(2) in the case $n = 1$.) Let $(\alpha\Delta^{2n})_1^\bullet$ be the braid with $(4n+6)$ strands obtained from $(\alpha\Delta^{2n})_1$ by removing the last strand. Then we have

$$(\alpha\Delta^{2n})_1^\bullet = XY^{2n+1} \in B_{4n+6}.$$

(See Figure 22(3) in the case $n = 1$.) By Lemma 5.2, we have

$$s((\alpha\Delta^{2n})_1^\bullet) = s(XY^{2n+1}) = xy^{2n+1} \in \text{SW}_{4n+6}(\mathcal{A}, \mathcal{B}).$$

Then [HK20, Lemma 6.3] tells us that for n large, $s((\alpha\Delta^{2n})_1^\bullet)$ is a pseudo-Anosov braid with the same entropy as $(\alpha\Delta^{2n})_1$. Therefore we have

$$\log \lambda(s((\alpha\Delta^{2n})_1^\bullet)) = \log \lambda(xy^{2n+1}) \asymp \frac{1}{n},$$

since the sequence $\{(\alpha\Delta^{2n})_1\}$ varying n has a small normalized entropy [HK20, Theorem 5.2(3)], and we are done.

Finally, we prove $\ell(\text{SW}_{4n}(\mathcal{A}, \mathcal{B})) \asymp \frac{1}{n}$. We consider $\alpha^2 = XZXZ \in B_5$ which is pseudo-Anosov, since α is pseudo-Anosov. This is a 5-increasing braid as well. We consider the sequence of pseudo-Anosov braids $(\alpha^2\Delta^{2n})_1$ varying n . The braid $(\alpha^2\Delta^{2n})_1$ can be written as $(\alpha^2\Delta^{2n})_1 = XZXZ^{2n+1} \in B_{4n+9}$. Then $(\alpha^2\Delta^{2n})_1^\bullet \in B_{4n+8}$ obtained from $(\alpha^2\Delta^{2n})_1$ by removing the last strand is of the form $(\alpha^2\Delta^{2n})_1^\bullet = XYXY^{2n+1} \in B_{4n+8}$. Hence its spherical braid satisfies the following property.

$$s((\alpha^2\Delta^{2n})_1^\bullet) = s(XYXY^{2n+1}) = xyxy^{2n+1} \in \text{SW}_{4n+8}(\mathcal{A}, \mathcal{B}).$$

By [HK20, Lemma 6.3] again, it follows that for n large, $s((\alpha^2\Delta^{2n})_1^\bullet)$ is a pseudo-Anosov braid with the same entropy as $(\alpha^2\Delta^{2n})_1$. The sequence of pseudo-Anosov braids $(\alpha^2\Delta^{2n})_1$ has a small normalized entropy, and hence this property also holds for $s((\alpha^2\Delta^{2n})_1^\bullet) = xyxy^{2n+1} \in \text{SW}_{4n+8}(\mathcal{A}, \mathcal{B})$. This completes the proof. \square

Finally we prove Theorem 0.6. By Example 2.7, we have

$$\ell(\mathcal{G}(H; S_{(p,n-2)})) = \ell(\text{SW}_{2n}(\mathcal{A}, \mathcal{C})).$$

We restate Theorem 0.6 as follows.

Theorem 5.4. *We have*

$$\ell(\mathcal{G}(H; S_{(p,n-2)})) = \ell(\text{SW}_{2n}(\mathcal{A}, \mathcal{C})) \asymp \frac{1}{n}.$$

As in the proof of Corollary 0.7, Corollary 0.8 then follows from Theorem 5.4 and Example 2.7.

Proof of Theorem 5.4. We first prove that $\ell(\text{SW}_{4n}(\mathcal{A}, \mathcal{C})) \asymp \frac{1}{n}$. In the proof of Theorem 5.3, we obtain a sequence of pseudo-Anosov braids $(\alpha\Delta^{2n})_1 = XZ^{2n+1} \in B_{4n+7}$ having a small normalized entropy. We have

$$s_+((\alpha\Delta^{2n})_1) = s_+(XZ^{2n+1}) = xz^{2n+1} \in \text{SB}_{4n+8},$$

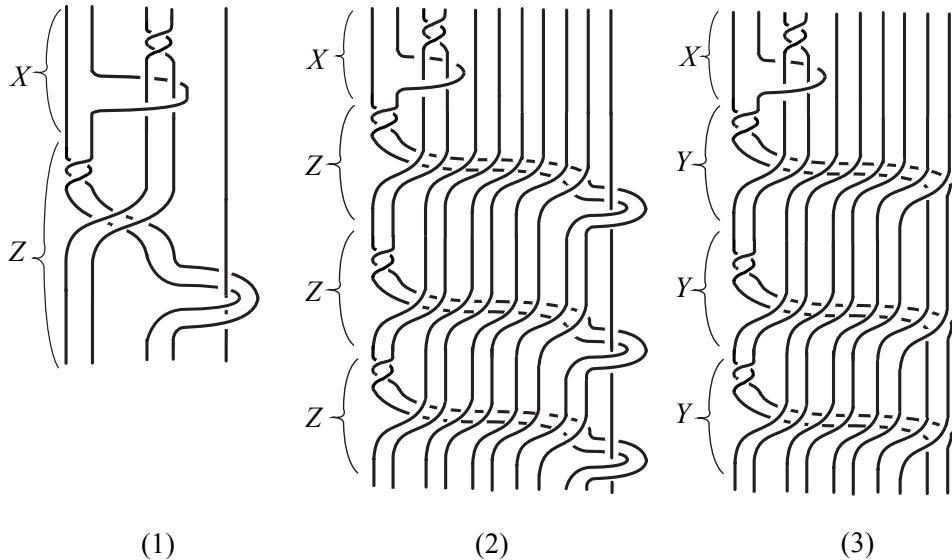


FIGURE 22. (1) $\alpha = X_5Z_5 \in B_5$. (2) $(\alpha\Delta^2)_1 = X_{11}Z_{11}^3 \in B_{11}$. (3) $(\alpha\Delta^2)_1^\bullet = X_{10}Y_{10}^3 \in B_{10}$.

and it is an element of $\text{SW}_{4n+8}(\mathcal{A}, \mathcal{C})$ by Lemma 5.2. Since $s_+((\alpha\Delta^{2n})_1)$ is a pseudo-Anosov braid with the same entropy as $(\alpha\Delta^{2n})_1$ (see Remark 5.1), we are done.

Next, we prove $\ell(\text{SW}_{4n+2}(\mathcal{A}, \mathcal{C})) \asymp \frac{1}{n}$. In the proof of Theorem 5.3, we obtain a sequence of pseudo-Anosov braids $(\alpha^2 \Delta^{2n})_1 = XZXZ^{2n+1} \in B_{4n+9}$ having a small normalized entropy. By Remark 5.1, the spherical braid $s_+(XZXZ^{2n+1}) \in \text{SB}_{4n+10}$ has the same entropy as $XZXZ^{2n+1} \in B_{4n+9}$. We have

$$s_+(XZXZ^{2n+1}) = xzxz^{2n+1}$$

which is an element of $\text{SW}_{4n+10}(\mathcal{A}, \mathcal{C})$ by Lemma 5.2. This completes the proof. \square

Theorems 0.5 and 0.6 motivate us to pose the following question.

Question 5.5. For any bridge decomposition $(L; S)$ of a link $L \subset S^3$ and any point $p \in L \cap S$ do we have $\ell(\mathcal{G}(L; S_{(p,k)})) \asymp \frac{1}{k}$, where $S_{(p,k)}$ is the bridge sphere of L obtained from a k -fold stabilization of S at p ?

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