The boundary of a fibered face of the magic 3-manifold and the asymptotic behavior of minimal pseudo-Anosov dilatations

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Abstract. Let $\delta_{g,n}$ be the minimal dilatation of pseudo-Anosovs defined on an orientable surface of genus $g$ with $n$ punctures. It is proved by Tsai that for any fixed $g \geq 2$, there exists a constant $c_g$ depending on $g$ such that
\[
\frac{1}{c_g} \frac{\log n}{n} < \log \delta_{g,n} < c_g \frac{\log n}{n}
\]
for any $n \geq 3$.
This means that the logarithm of the minimal dilatation $\log \delta_{g,n}$ is on the order of $\log n/n$. We prove that if $2g + 1$ is relatively prime to $s$ or $s + 1$ for each $0 \leq s \leq g$, then
\[
\limsup_{n \to \infty} \frac{n(\log \delta_{g,n})}{\log n} \leq 2
\]
holds. In particular, if $2g + 1$ is prime, then the above inequality on $\delta_{g,n}$ holds.
Our examples of pseudo-Anosovs $\phi$'s which provide the upper bound above have the following property: The mapping torus $M_\phi$ of $\phi$ is a single hyperbolic 3-manifold called the magic manifold, or the fibration of $M_\phi$ comes from a fibration of $N$ by Dehn filling cusps along the boundary slopes of a fiber.

1. Introduction

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus $g$ with $n$ punctures and $\text{Mod}(\Sigma)$ the mapping class group of $\Sigma$. By Thurston’s classification theorem of surface automorphisms, elements of $\text{Mod}(\Sigma)$ are either periodic, reducible, or pseudo-Anosov, see [20]. Pseudo-Anosov mapping classes have rich dynamical properties. The hyperbolization theorem by Thurston [21] relates the dynamics of pseudo-Anosovs and the geometry of hyperbolic fibered 3-manifolds. The theorem asserts that $\phi \in \text{Mod}(\Sigma)$ is pseudo-Anosov if and only if the mapping torus $M_\phi$ of $\phi$ admits a complete hyperbolic metric of finite volume.

Each pseudo-Anosov element $\phi \in \text{Mod}(\Sigma)$ has a representative $\Phi : \Sigma \to \Sigma$ called a pseudo-Anosov homeomorphism. Such a homeomorphism is equipped with a constant $\lambda = \lambda(\Phi) > 1$ called the dilatation of $\Phi$. If we let $\text{ent}(\Phi)$ be the topological entropy of $\Phi$, then the equality $\text{ent}(\Phi) = \log \lambda(\Phi)$ holds. Moreover

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ent(Φ) attains the minimal entropy among all homeomorphisms which are isoto-
opic to Φ, see [3, Exposé 10]. The dilatation \( \lambda(\phi) \) of \( \phi \) is defined to be \( \lambda(\Phi) \). We
call the quantities \( \text{ent}(\phi) = \log \lambda(\phi) \) and \( \text{Ent}(\phi) = |\chi(\Sigma)| \log \lambda(\phi) \) the entropy
and normalized entropy of \( \phi \) respectively, where \( \chi(\Sigma) \) is the Euler characteristic
of \( \Sigma \).

If we fix \( \Sigma \), the set of dilatations of pseudo-Anosovs defined on \( \Sigma \) is a closed
discrete subset of \( \mathbb{R} \), see [7] for example. In particular there exists a minimum.
We denote by \( \delta(\Sigma) > 1 \), the minimal dilatation of pseudo-Anosov elements in
\( \text{Mod}(\Sigma) \). The minimal dilatations are determined in only a few cases. (See for
example [9] which is a survey on minimal pseudo-Anosov dilatations.)

Let us set \( \delta_{g,n} = \delta(\Sigma_{g,n}) \) and \( \delta_g = \delta_{g,0} \). We write \( A \asymp B \) if there exists
a universal constant \( c \) such that \( A \leq cB \leq cA \). Penner proved in [17] that
\( \log \delta_g \asymp \frac{1}{g} \). This work by Penner was a starting point for the study of the
asymptotic behavior of the minimal dilatations on surfaces varying topology.
Later it was proved by Hironaka-Kin [6] that \( \log \delta_{1,n} \asymp \frac{1}{n} \), and by Tsai [22] that
\( \log \delta_{1,n} \asymp \frac{1}{n} \). See also Valdivia [23]. The following theorem of Tsai is in contrast
with the cases of genera 0 and 1.

**Theorem 1.1 ([22]).** For any fixed \( g \geq 2 \), there exists a constant \( c_g \) depend-
ing on \( g \) such that
\[
\frac{1}{c_g} \cdot \frac{\log n}{n} < \log \delta_{g,n} < c_g \cdot \frac{\log n}{n} \quad \text{for any } n \geq 3.
\]
In particular for any fixed \( g \geq 2 \), we have
\[
\log \delta_{g,n} \asymp \frac{\log n}{n}.
\]
The following question is due to Tsai.

**Question 1.2.** What is the optimal constant \( c_g \) in Theorem 1.1?

One can also ask the following.

**Question 1.3.** Given \( g \geq 2 \), does \( \lim_{n \to \infty} \frac{n \log \delta_{g,n}}{\log n} \) exist? What is its
value?

This is an analogous question, posed by McMullen, which is asking whether
\( \lim g \log \delta_g \) exists or not, see [15]. Toward Questions 1.2 and 1.3, we prove the
following.

**Theorem 1.4.** Given \( g \geq 2 \), there exists a sequence \( \{n_i\}_{i=0}^\infty \) with \( n_i \to \infty \)
such that
\[
\limsup_{i \to \infty} \frac{n_i \log \delta_{g,n_i}}{\log n_i} \leq 2.
\]
Theorem 1.4 improves the previous upper bound on \( \log \delta_{g,n} \) by Tsai. In fact for
any \( g \geq 2 \), Tsai’s examples in [22] yield the upper bound \( \limsup_{n \to \infty} \frac{n \log \delta_{g,n}}{\log n} \leq \)
2(2g + 1), which is proved by a similar computation in the proof of Theorem 1.4. As a corollary of Theorem 1.4, we have the following.

**Corollary 1.5.** Given \( g \geq 2 \), the following set

\[
\left\{ \frac{n}{\log n} \cdot \text{ent}(\phi) \mid \phi \in \text{Mod}(\Sigma_{g,n}) \text{ is pseudo-Anosov}, \ n \geq 1 \right\}
\]

has an accumulation point 2.

To state other results which are related to Questions 1.2 and 1.3, we define a polynomial \( B_{(g,p)}(t) \) for nonnegative integers \( g \) and \( p \):

\[
B_{(g,p)}(t) = t^{2p+1}(t^{2g+1} - 1) + 1 - 2t^{p+g+1} - t^{2g+1}.
\]

We shall see that there exists a unique real root \( r_{(g,p)} \) greater than 1 of \( B_{(g,p)}(t) \), and these satisfy

\[
\lim_{p \to \infty} \frac{p \log r_{(g,p)}}{\log p} = 1
\]

(Lemma 4.1). The root \( r_{(g,p)} \) gives the following upper bound.

**Theorem 1.6.** For \( g \geq 2 \) and \( p \geq 0 \), suppose that \( \gcd(2g+1, p+g+1) = 1 \). Then

\[
\delta_{g,2p+i} \leq r_{(g,p)} \quad \text{for each } i \in \{1, 2, 3, 4\}.
\]

If \( g \) satisfies \((\ast)\) in the next Theorem 1.7, then one can take the sequence \( \{n_i\}_{i=0}^{\infty} \) in Theorem 1.4 to be the sequence \( \{n\}_{n=1}^{\infty} \) of natural numbers.

**Theorem 1.7.** Suppose that \( g \geq 2 \) satisfies

\[
(\ast) \quad \gcd(2g+1, s) = 1 \text{ or } \gcd(2g+1, s+1) = 1 \text{ for each } 0 \leq s \leq g.
\]

Then

\[
\limsup_{n \to \infty} \frac{n(\log \delta_{g,n})}{\log n} \leq 2.
\]

For example, \((\ast)\) holds for \( g = 4 \) since 9 is relatively prime to 1, 2, 4 and 5; \((\ast)\) does not hold for \( g = 7 \) because \( \gcd(15, 5) = 5 \) and \( \gcd(15, 6) = 3 \). We point out that infinitely many \( g \)'s satisfy \((\ast)\). In fact if \( 2g + 1 \) is prime, then \( 2g + 1 \) is relatively prime to \( s' \) for each \( 1 \leq s' \leq g + 1 \).

**Corollary 1.8.** If \( 2g + 1 \) is prime for \( g \geq 2 \), then

\[
\limsup_{n \to \infty} \frac{n(\log \delta_{g,n})}{\log n} \leq 2.
\]

**Remark 1.9.** One can simplify \((\ast)\) in Theorem 1.7, since \( 2g + 1 \) is relative prime to 1, 2 and \( g \). In the case \( g \geq 5 \), \((\ast)\) is equivalent to

\[
(\ast\ast) \quad \gcd(2g+1, s) = 1 \text{ or } \gcd(2g+1, s+1) = 1 \text{ or each } 3 \leq s \leq g - 2.
\]
Our results are proved by using the theory of fibered faces of hyperbolic and fibered 3-manifolds $M$, developed by Thurston [19], Fried [4], Matsumoto [14] and McMullen [15], see Section 2. We focus on a fibered face of a particular hyperbolic fibered 3-manifold, called the magic manifold $N$. This manifold is the exterior of the 3 chain link $C_3$, see Figure 1. Our examples of pseudo-Anosovs $\phi$’s which provide the upper bounds in Theorems 1.4, 1.6 and 1.7 have the following property: The mapping torus $M_\phi$ of $\phi$ is homeomorphic to $N$, or the fibration of $M_\phi$ comes from a fibration of $N$ by Dehn filling cusps along the boundary slopes of a fiber. An explicit construction of these examples is given by the first author, see [8, Example 4.8].

![Figure 1.](image)

We turn to hyperbolic volumes of hyperbolic 3-manifolds. The set of volumes of hyperbolic 3-manifolds is a well-ordered closed subset in $\mathbb{R}$ of order type $\omega^\omega$, see [18]. In particular if we fix a surface $\Sigma$, then there exists a minimum among volumes of hyperbolic $\Sigma$-bundles over the circle. The proofs of Theorems 1.4, 1.7 immediately imply the following.

**Proposition 1.10.** For each $g \geq 2$, there exists a sequence $\{n_i\}_{i=0}^\infty$ with $n_i \to \infty$ such that the minimal volume of $\Sigma_{g,n_i}$-bundles over the circle is less than or equal to $\text{vol}(N) \approx 5.3334$, the volume of the magic manifold $N$. In particular, for any $g \geq 2$ satisfying $(\ast)$ and any $n \geq 3$, the minimal volume of $\Sigma_{g,n}$-bundles over the circle is less than or equal to $\text{vol}(N)$.

We close the introduction by asking

**Question 1.11 (cf. Theorems 1.4 and 1.7).** Does $$\limsup_{n \to \infty} \frac{n(\log \delta_{g,n})}{\log n} \leq 2$$ hold for all fixed $g \geq 2$?

2. The Thurston norm and fibered 3-manifolds

Let $M$ be an oriented hyperbolic 3-manifold with boundary $\partial M$ (possibly $\partial M = \emptyset$). We recall the Thurston norm $\| \cdot \| : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$. Let $F$ be a
finite union of oriented, connected surfaces. We define $\chi_-(F)$ to be

$$\chi_-(F) = \sum_{F_i \subset F} \max \{0, -\chi(F_i)\},$$

where $F_i$’s are the connected components of $F$ and $\chi(F_i)$ is the Euler characteristic of $F_i$. The Thurston norm $\| \cdot \|$ is defined for an integral class $a \in H_2(M, \partial M; \mathbb{Z})$ by

$$\|a\| = \min_{F} \{\chi_-(F) \mid a = [F]\},$$

where the minimum ranges over all oriented surfaces $F$ embedded in $M$. A surface $F$ which realizes this minimum is called a minimal representative $F_a$, denoted by $F_a$. Then $\| \cdot \|$ defined on integral classes admits a unique continuous extension $\| \cdot \| : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$ which is linear on the ray through the origin. The unit ball $U_M$ with respect to the Thurston norm is a compact, convex polyhedron. See [19] for more details.

Suppose that $M$ is a surface bundle over the circle and let $F$ be its fiber. The fibration determines a cohomology class $a^* \in H^1(M; \mathbb{Z})$, and hence a homology class $a \in H_2(M, \partial M; \mathbb{Z})$ by Poincaré duality. Thurston proved in [19] that there exists a top dimensional face $\Omega$ on $\partial U_M$ such that $a = [F]$ is an integral class of $int(C_\Omega)$, where $C_\Omega$ is the cone over $\Omega$ with the origin and $int(C_\Omega)$ is its interior. Moreover the minimal representative $F_a$ for any integral class $a$ in $int(C_\Omega)$ becomes a fiber of the fibration associated to $a$. Such a face $\Omega$ is called a fibered face, and an integral class $a \in int(C_\Omega)$ is called a fibered class. This work of Thurston tells us that if $M$ has second Betti number greater than 1, then $M$ provides infinitely many pseudo-Anosov monodromies defined on surfaces with variable topology.

The set of integral and rational classes of $int(C_\Omega)$ are denoted by $int(C_\Omega(\mathbb{Z}))$ and $int(C_\Omega(\mathbb{Q}))$ respectively. When $a \in int(C_\Omega(\mathbb{Z}))$ is primitive, the associated fibration on $M$ has a connected fiber represented by $F_a$. Let $\Phi_a : F_a \to F_a$ be the monodromy. Since $M$ is hyperbolic, $\phi_a = [\Phi_a]$ is pseudo-Anosov. The dilatation $\lambda(a)$ and entropy $\text{ent}(a) = \log \lambda(a)$ are defined as the dilatation and entropy of $\phi_a$ respectively. The entropy defined on primitive fibered classes can be extended to rational classes by homogeneity. It is shown by Fried in [4] that $\frac{1}{\text{ent}} : int(C_\Omega(\mathbb{Q})) \to \mathbb{R}$ is concave, and in particular $\text{ent} : int(C_\Omega(\mathbb{Q})) \to \mathbb{R}$ admits a unique continuous extension

$$\text{ent} : int(C_\Omega) \to \mathbb{R}.$$
These properties give us the following observation: Fix a hyperbolic fibered 3-manifold $M$ with a fibered face $\Omega$ as above. For any compact set $\mathcal{D} \subset \text{int}(\Omega)$, there exists a constant $C = C_\mathcal{D} > 0$ satisfying the following. Let $a \in \text{int}(C_\mathcal{D})$ be any integral class of $H_2(M, \partial M; \mathbb{Z})$. The normalized entropy $\text{Ent}(a)(= \text{Ent}(\phi_a))$ is bounded by $C$ from above whenever $\pi \in \mathcal{D}$, where $\pi$ is the projection of $a$ into $\text{int}(\Omega)$.

This observation enables us to investigate the following asymptotic behaviors of minimal dilatations.

1. $\limsup_{n \to \infty} n \log \delta_{0,n} \leq 2 \log(2 + \sqrt{3})$, see [6, 11].
2. $\limsup_{n \to \infty} n \log \delta_{1,n} \leq 2 \log \lambda_0$, where $\lambda_0 \approx 2.2966$ is the largest real root of $t^4 - 2t^3 - 2t + 1$, see [10].
3. $g \log \delta_g \leq \log\left(\frac{1+\sqrt{5}}{2}\right)$, see [2, Appendix] and [5, 1, 12].

We note that for fixed $g \geq 2$, different methods for investigating the asymptotic behavior of $\delta_{g,n}$ varying $n$ are necessary. Theorem 1.1 says that there exists no constant $C > 0$, independent of $n$ so that $|\chi(\Sigma_{g,n})| \log \delta_{g,n} < C$. Thus if, for fixed $g \geq 2$, there exists a sequence of fibered classes $\{a_i\}$ with $a_i \in \text{int}(C_\mathcal{D}) \cap H_2(M, \partial M; \mathbb{Z})$ such that the fiber of the fibration associated to $a_i$ is a surface of genus $g$ having $n_i$ boundary components with $n_i \to \infty$, then the accumulation points of the sequence of projective classes $\{\pi_i\}$ must lie on the boundary of $\Omega$. To prove Theorems 1.4, 1.6 and 1.7, we pay special attention to the magic manifold $N$. In Section 3.4, we choose such a sequence of fibered classes $\{a_i\}$ of $N$ carefully. We analyze the asymptotic behavior of $\lambda(a_i)$’s by using a technique given in Section 3.

The Teichmüller polynomial, developed by McMullen[15] is a certain element $\Theta_\Omega$ (associated to the fibered face $\Omega$) in the group ring $\mathbb{Z}G$, where $G = H_1(M; \mathbb{Z})/\text{torsion}$, i.e, $\Theta_\Omega$ is a finite sum

$$\Theta_\Omega = \sum_{g \in G} c_g g,$$

where $c_g$ is an integer. For every fibered class $a \in \text{int}(C_\mathcal{D})$, the specialization of $\Theta_\Omega$ at the cohomology class $a^* \in H^1(M; \mathbb{Z})$ is defined by

$$\Theta^{(a^*)}_\Omega(t) = \sum_{g \in G} c_g t^{a^*(g)}$$

which is a polynomial with a variable $t$. It is a result in [15] that for all fibered class $a \in \text{int}(C_\mathcal{D})$, the dilatation $\lambda(a)$ is equal to the largest real root of $\Theta^{(a^*)}_\Omega(t)$.

3. Roots of polynomials

This section concerns the asymptotic behavior of roots of families of polynomials. Let

$$g(t) = a_n t^{b_n} + a_{n-1} t^{b_{n-1}} + \cdots + a_1 t^{b_1} + a_0$$

be a polynomial of degree $d$. The roots of $g(t)$ are denoted by $\alpha_1, \alpha_2, \ldots, \alpha_d$. The dilatation $\lambda(a)$ is equal to the largest real root of $\Theta^{(a^*)}_\Omega(t)$.
be a polynomial with real coefficients \(a_0, a_1, \ldots, a_n\) \((a_1, a_2, \ldots, a_n \neq 0)\), where \(g(t)\) is arranged in the order of descending powers of \(t\). Let \(\mathcal{D}(g)\) be the number of variations in signs of the coefficients \(a_n, a_{n-1}, \ldots, a_0\). For example if \(g(t) = +t^4 + t^3 - 2t^2 + t - 1\), then \(\mathcal{D}(g) = 3\); if \(h(t) = +t^5 + t^3 - 2t^2 + t + 1\), then \(\mathcal{D}(h) = 2\). Descartes’s rule of signs (see [24]) says that the number of positive real roots of \(g(t)\) (counted with multiplicities) is equal to either \(\mathcal{D}(g)\) or less than \(\mathcal{D}(g)\) by an even integer.

**Lemma 3.1.** Let \(r > 0\), \(s > 0\) and \(u > 0\) be integers. Let

\[
P_m(t) = t^{2m+r}(t^s - 1) + 1 - Q(t)t^m - t^u
\]

be a polynomial for each \(m \in \mathbb{N}\), where \(Q(t)\) is a polynomial whose coefficients are positive integers. \((Q(t)\) could be a positive constant.\)

1. Suppose that \(t^{2m+r+s}\) is the leading term of \(P_m(t)\). Then \(P_m(t)\) has a unique real root \(\lambda_m\) greater than 1.
2. Given \(0 < c_1 < 1\) and \(c_2 > 1\), we have

\[
m^{c_1} < \lambda_m < m^{c_2}
\]

for \(m\) large.

In particular

\[
\lim_{m \to \infty} \frac{m \log \lambda_m}{\log m} = 1.
\]

3. For any real numbers \(q \neq 0\) and \(v\), we have

\[
\lim_{m \to \infty} \frac{(qm + v) \log \lambda_m}{\log(qm + v)} = q.
\]

**Proof.** (1) Under the assumption on \(P_m(t)\), we have \(\mathcal{D}(P_m) = 2\). By Descartes’s rule of signs, the number of positive real roots of \(P_m(t)\) is either 2 or 0. Since \(P_m(0) = 1\) and \(P_m(1) = -Q(1) < 0\), the number of positive real roots of \(P_m(t)\) is exactly 2. Because \(P_m(t)\) goes to \(\infty\) as \(t\) does, \(P_m(t)\) has a unique real root \(\lambda_m > 1\).

(2) We have

\[
P_m(t)t^{-(2m+r)} = t^s - 1 + t^{(2m+r)} - Q(t)t^{-(m+r)} - t^{-(2m+r-u)}.
\]

We define \(f_m(t)\) and \(g_m(t)\) such that \(P_m(t)t^{-(2m+r)} = f_m(t) + g_m(t)\) as follows.

\[
f_m(t) = t^s - 1 + t^{(2m+r)} , \quad \text{and}
\]

\[
g_m(t) = Q(t)t^{-(m+r)} + t^{-(2m+r-u)}.
\]

We let \(t = m^{\frac{c}{m}}\) for \(c > 0\). Then

\[
f_m(m^{\frac{c}{m}}) = (m^{\frac{c}{m}})^s - 1 + (m^{\frac{c}{m}})^{-(2m+r)}
\]

\[
= (e^{\log m^{1/c}})^s - 1 + m^{-c(2 + \frac{s}{c})}
\]

\[
= e^{\frac{sc}{m} \log m} - 1 + m^{-c(2 + \frac{c}{m})}.
\]
By Maclaurin expansion of $e^{\frac{sc \log m}{m}}$, we have

$$e^{\frac{sc \log m}{m}} = 1 + \frac{sc \log m}{m} + R_2,$$

where

$$R_2 = \frac{ew}{2} \left(\frac{sc \log m}{m}\right)^2 \text{ for some } 0 < w < \frac{sc \log m}{m}.$$

Since $\frac{sc \log m}{m}$ goes to 0 as $m$ goes to $\infty$, we may assume that $\frac{ew}{2} < B$ for some constant $B > 0$. Then

$$f_m(m^\frac{1}{m}) = \frac{sc \log m}{m} + R_2 + m^{1-c(2+\frac{1}{m})}$$

$$< \frac{sc \log m}{m} + B\left(\frac{sc \log m}{m}\right)^2 + \frac{m^{1-c(2+\frac{1}{m})}}{m}$$

$$= \frac{sc \log m}{m} + Bs^2c^2\left(\frac{\log m}{m}\right)^2 + \frac{m^{1-c(2+\frac{1}{m})}}{m}$$

$$< \frac{sc \log m}{m} + Bs^2c^2\left(\frac{\log m}{m}\right) + \frac{m^{1-c(2+\frac{1}{m})}}{m}$$

$$= \frac{(sc + Bs^2c^2) \log m + m^{1-c(2+\frac{1}{m})}}{m}.$$

(The last inequality comes from $0 < \frac{\log m}{m} < 1$ for $m$ large.) Thus

$$f_m(m^\frac{1}{m}) < \frac{(sc + Bs^2c^2) \log m + m^{1-c(2+\frac{1}{m})}}{m}. \quad (1)$$

The first equality $f_m(m^\frac{1}{m}) = \frac{sc \log m}{m} + R_2 + m^{1-c(2+\frac{1}{m})}$ above together with $R_2 > 0$ and $m^{1-c(2+\frac{1}{m})} > 0$ tells us that

$$f_m(m^\frac{1}{m}) > \frac{sc \log m}{m}. \quad (2)$$

Recall that all coefficients of $Q(t)$ (appeared in $P_m(t)$) are positive integers.

If we write $Q(t) = \sum_{j=0}^{\ell} a_j t^j$, where $a_j \geq 0$, then

$$g_m(m^\frac{1}{m}) = Q(m^\frac{1}{m})m^{-c(1+\frac{1}{m})} + m^{-c(2+\frac{1}{m} - \frac{1}{m})}$$

$$= \left(\sum_{j=0}^{\ell} a_j m^{-c(1+\frac{1}{m} - \frac{1}{m})}\right) + m^{-c(2+\frac{1}{m} - \frac{1}{m})}.$$ 

Thus we obtain

$$g_m(m^\frac{1}{m}) = \left(\sum_{j=0}^{\ell} a_j m^{-c(1+\frac{1}{m} - \frac{1}{m})}\right) + m^{-c(2+\frac{1}{m} - \frac{1}{m})}. \quad (3)$$

For the proof of the claim (1), it is enough to prove that for $0 < c_1 < 1$ and $c_2 > 1$, we have $f_m(m^\frac{1}{m}) < g_m(m^\frac{1}{m})$ and $f_m(m^\frac{1}{m}) > g_m(m^\frac{1}{m})$ for $m$ large.
First, suppose that $0 < c < \frac{1}{2}$. Let us consider how the following four terms grow:

$$\log m, \ m^{1-c(2+\frac{r}{m})}, \ m^{1-c(1+\frac{j}{m})} \text{ and } m^{1-c(2+\frac{r}{m}-\frac{j}{m})}. \quad (4)$$

The first two terms appear in (1), and the last two are coming from (3). All four terms go to $\infty$ as $m$ does, since the last three terms have the positive powers of $m$. Note that for any $C > 0$, we have $\log m < m^C$ for $m$ large. Keeping this in mind, we observe that among the four terms in (4), $m^{1-c(1+\frac{j}{m})}$ is dominant.

This is because

$$1 - c(1 + \frac{r}{m} - \frac{j}{m}) > 1 - c(2 + \frac{r}{m} - \frac{u}{m}) \geq 1 - c(2 + \frac{r}{m})$$

for $m$ large. These imply that $f_m(m^{\frac{1}{m}}) < g_m(m^{\frac{1}{m}})$ holds for $m$ large, since $m^{1-c(1+\frac{j}{m})}$ appears in the numerator of $g_m(m^{\frac{1}{m}})$, see (3).

Next, we suppose that $\frac{1}{2} < c < 1$. We can check that $m^{1-c(1+\frac{j}{m})}$ is still dominant among the four in (4). (The second and fourth terms are bounded as $m$ goes to $\infty$.) Therefore we still have $f_m(m^{\frac{1}{m}}) < g_m(m^{\frac{1}{m}})$ for $m$ large.

Finally we suppose that $c > 1$. Clearly, the last three terms in (4) go to $0$ as $m$ goes to $\infty$. Thus the numerator of $g_m(m^{\frac{1}{m}})$, see (3), goes to $0$ as $m$ tends to $\infty$. On the other hand, $f_m(m^{\frac{1}{m}}) > \frac{sc\log m}{m}$ holds (see (2)), and hence the numerator of

$$\frac{sc\log m + mR_2 + m^{1-c(2+\frac{r}{m})}}{m} \quad (= f_m(m^{\frac{1}{m}}))$$

goess to $\infty$ as $m$ does. Thus $f_m(m^{\frac{1}{m}}) > g_m(m^{\frac{1}{m}})$ for $m$ large. This completes the proof of the first part of the claim (2).

Taking logarithms on both sides of $m^{\frac{1}{m}} < \lambda_m < m^{\frac{2}{m}}$ yields

$$c_1 < \frac{m\log \lambda_m}{\log m} < c_2 \quad \text{for } m \text{ large.}$$

Since $0 < c_1 < 1$ and $c_2 > 1$ are arbitrary, we have the desired limit. This completes the proof of the second half of the claim (2).

(3) By the claim (2),

$$\frac{c_1\log m}{m} < \log \lambda_m < \frac{c_2\log m}{m} \quad \text{for } m \text{ large.}$$

Let us set $n = qm + v$. We substitute $m = \frac{n-v}{q}$ above:

$$\frac{c_1\log \left(\frac{n-v}{q}\right)}{\frac{n-v}{q}} < \log \lambda_m < \frac{c_2\log \left(\frac{n-v}{q}\right)}{\frac{n-v}{q}}.$$

Hence

$$\frac{qc_1(\log(n-v) - \log q)}{n-v} < \log \lambda_m < \frac{qc_2(\log(n-v) - \log q)}{n-v}.$$
We multiply all sides above by \( \frac{n}{\log n} > 0 \) (for \( n \) large). Then
\[
\frac{qc_1n(\log(n-v) - \log q)}{(n-v)\log n} < \frac{n(\log \lambda_m)}{\log n} < \frac{qc_2n(\log(n-v) - \log q)}{(n-v)\log n}.
\]
Note that \( \frac{n(\log(n-v) - \log q)}{(n-v)\log n} \) goes to 1 as \( n \) (and hence \( m \)) goes to \( \infty \). Since \( 0 < c_1 < 1 \) and \( c_2 > 1 \) are arbitrary, it follows that
\[
\lim_{m \to \infty} \frac{n(\log \lambda_m)}{\log n} = \lim_{m \to \infty} \frac{(qm + v)\log \lambda_m}{\log(qm + v)} = q.
\]

\[\square\]

4. The magic 3-manifold \( N \)

Monodromies of fibrations on \( N \) have been studied in [10, 11, 12]. In Sections 4.1 and 4.2, we recall some results which tell us that the topology of fibered classes \( a \) and the actual value of \( \lambda(a) \). In Section 4.3, we find a family of fibered classes \( a_{(g,p)} \) of \( N \) with two variables \( g \) and \( p \), and we shall prove that it is a suitable family to prove theorems in Section 1 (cf. Remark 4.4).

Recall that \( \Sigma_{g,n} \) is an orientable surface of genus \( g \) with \( n \) punctures. Abusing the notation, we sometimes denote by \( \Sigma_{g,n} \), an orientable surface of genus \( g \) with \( n \) boundary components.

4.1. Fibered face \( \Delta \). Let \( K_\alpha, K_\beta \) and \( K_\gamma \) be the components of the 3 chain link \( C_3 \). They bound the oriented disks \( F_\alpha, F_\beta \) and \( F_\gamma \) with 2 holes, see Figure 1. Let \( \alpha = [F_\alpha], \beta = [F_\beta], \gamma = [F_\gamma] \in H_2(N, \partial N; \mathbb{Z}) \). The set \( \{\alpha, \beta, \gamma\} \) is a basis of \( H_2(N, \partial N; \mathbb{Z}) \). Figure 1 illustrates the Thurston norm ball \( U_N \) for \( N \) which is the parallelepiped with vertices \( \pm\alpha, \pm\beta, \pm\gamma, \pm(\alpha + \beta + \gamma) \) ([19, Example 3 in Section 2]). Because of the symmetry of \( C_3 \), every top dimensional face of \( U_N \) is a fibered face.

We denote a class \( x\alpha + y\beta + z\gamma \in H_2(N, \partial N; \mathbb{R}) \) by \( (x, y, z) \). We pick a fibered face \( \Delta \) with vertices \( \alpha = (1, 0, 0), \beta + \gamma = (1, 1, 1), \beta = (0, 1, 0) \) and \( -\gamma = (0, 0, -1) \), see Figure 1. The open face \( \text{int}(\Delta) \) is written by
\[
\text{int}(\Delta) = \{(X, Y, Z) \mid X + Y - Z = 1, X > 0, Y > 0, X > Z, Y > Z\}.
\]
A class \( a = (x, y, z) \in H_2(N, \partial N; \mathbb{R}) \) is an element of \( \text{int}(C_\Delta) \) if and only if \( x > 0, y > 0, x > z \) and \( y > z \). In this case, we have \( \|a\| = x + y - z \).

Let \( a = (x, y, z) \) be a fibered class in \( \text{int}(C_\Delta) \). The minimal representative of this class is denoted by \( F_a \) or \( F_{(x,y,z)} \). We recall some formula which tells us that the number of the boundary components of \( F_a \). We denote the tori \( \partial N(K_\alpha), \partial N(K_\beta), \partial N(K_\gamma) \) by \( T_\alpha, T_\beta, T_\gamma \) respectively, where \( N(K) \) is a regular neighborhood of a knot \( K \) in \( S^3 \). Let us set \( \partial_r F_{(x,y,z)} = \partial F_{(x,y,z)} \cap T_\alpha \) which consists of the parallel simple closed curves on \( T_\alpha \). We define the subsets \( \partial_\beta F_{(x,y,z)}, \partial_\gamma F_{(x,y,z)} \subset \partial F_{(x,y,z)} \) in the same manner. By [11, Lemma 3.1], the number of the boundary components
\[
\sharp(\partial F_{(x,y,z)}) = \sharp(\partial_\alpha F_{(x,y,z)}) + \sharp(\partial_\beta F_{(x,y,z)}) + \sharp(\partial_\gamma F_{(x,y,z)})
\]
is given by
\[ z(\partial F_{(x,y,z)}) = \gcd(x, y + z) + \gcd(y, z + x) + \gcd(z, x + y) \] (5)
where \( z(\partial_1 F_{(x,y,z)}) = \gcd(x, y + z), z(\partial_2 F_{(x,y,z)}) = \gcd(y, z + x), z(\partial_3 F_{(x,y,z)}) = \gcd(z, x + y) \) and \( \gcd(0, w) \) is defined by \( |w| \).

4.2. Dilatations and stable foliations of fibered classes \( a \)'s. The Teichmüller polynomial associated to the fibered face \( \Delta \) is computed in [11, Section 3.2], and it tells us that the dilatation \( \lambda_{(x,y,z)} \) of a fibered class \( (x, y, z) \in \text{int}(C_\Delta) \) is the largest real root of
\[ f_{(x,y,z)}(t) = t^x + y - z - t^y - t^x - t^y + t^y - z + 1, \]
see [11, Theorem 3.1]. (In fact, \( \lambda_{(x,y,z)} \) is a unique real root greater than 1 of \( f_{(x,y,z)}(t) \) by Descartes’s rule of signs.)

Let \( \Phi_{(x,y,z)} : F_{(x,y,z)} \to F_{(x,y,z)} \) be the monodromy of the fibration associated to a primitive class \( (x, y, z) \in \text{int}(C_\Delta) \). Let \( F_{(x,y,z)} \) be the stable foliation of the pseudo-Anosov \( \Phi_{(x,y,z)} \). The components of \( \partial_\alpha F_{(x,y,z)} \) (resp. \( \partial_2 F_{(x,y,z)}, \partial_3 F_{(x,y,z)} \)) are permuted cyclically by \( \Phi_{(x,y,z)} \). In particular the number of prongs of \( F_{(x,y,z)} \) at a component of \( \partial_\alpha F_{(x,y,z)} \) (resp. \( \partial_2 F_{(x,y,z)}, \partial_3 F_{(x,y,z)} \)) is independent of the choice of the component. By [12, Proposition 3.3], the stable foliation \( F_{(x,y,z)} \) has the following properties.

- Each component of \( \partial_\alpha F_{(x,y,z)} \) has \( x/\gcd(x, y + z) \) prongs.
- Each component of \( \partial_2 F_{(x,y,z)} \) has \( y/\gcd(y, x + z) \) prongs.
- Each component of \( \partial_3 F_{(x,y,z)} \) has \( (x + y - 2z)/\gcd(z, x + y) \) prongs.
- \( F_{(x,y,z)} \) does not have singularities in the interior of \( F_{(x,y,z)} \).

4.3. Proofs of theorems. Let \( a = (1, 1, 0) \) and \( b = (0, 1, 1) \). For \( g \geq 0 \) and \( p \geq 0 \), define a fibered class \( a_{(g,p)} \) as follows.

\[ a_{(g,p)} = (p + g + 1)a + (p - g)b = (p + g + 1, 2p + 1, p - g) \in \text{int}(C_\Delta). \]

The class \( a_{(g,p)} \) is primitive if and only if \( 2g + 1 \) and \( p + g + 1 \) are relatively prime. One can check the identity
\[ B_{(g,p)}(t) = f_{(p+g+1,2p+1,p-g)}(t) \]
(see Section 1 for the definition of \( B_{(g,p)}(t) \)). We denote by \( r_{(g,p)} \), the dilatation \( \lambda_{a_{(g,p)}} \) of the fibered class \( a_{g,p} \). (Thus the dilatation \( r_{(g,p)} = \lambda_{a_{(g,p)}} \) of \( a_{(g,p)} \) is a unique real root of \( B_{(g,p)}(t) \) which is greater than 1, see Section 4.2.)

**Lemma 4.1.** We fix \( g \geq 0 \). Given \( 0 < c_1 < 1 \) and \( c_2 > 1 \), we have
\[ p^{c_1^p} < r_{(g,p)} < p^{c_2^p} \] for \( p \) large.

**In particular**
\[ \lim_{p \to \infty} \frac{p \log r_{(g,p)}}{\log p} = 1. \]

**Proof.** Apply Lemma 3.1 to the polynomial \( B_{(g,p)}(t) \).
Lemma 4.2. Suppose that \( a_{(g,p)} \) is primitive. The minimal representative \( F_{a_{(g,p)}} \) is a surface of genus \( g \) with \( 2p + 4 \) boundary components, and the stable foliation \( F_{a_{(g,p)}} \) has the following properties. If \( p + g \) is odd (resp. even), then \( \sharp(\partial F_{a_{(g,p)}}) = 2 \) (resp. 1) and \( \sharp(\partial F_{a_{(g,p)}}) = 1 \) (resp. 2). A component of \( \partial_1 F_{a_{(g,p)}} \) has \( \frac{3g+1}{2} \) prongs (resp. \( p + g + 1 \) prongs), and a component of \( \partial_2 F_{a_{(g,p)}} \) has \( p + 3g + 2 \) prongs (resp. \( p + 4 \) prongs).

Proof. By (5), we have that \( \sharp(\partial F_{a_{(g,p)}}) = 2p + 1 \). We have
\[
\sharp(\partial F_{a_{(g,p)}}) = \gcd(p + g + 1, 3p - g + 1) = \gcd(p + g + 1, 2(2g + 1)).
\]
Since \( a_{(g,p)} \) is primitive, \( p + g + 1 \) and \( 2g + 1 \) must be relatively prime. Hence \( \sharp(\partial F_{a_{(g,p)}}) = 1 \) (resp. 2) if \( p + g \) is even (resp. odd). Let us compute \( \sharp(\partial_1 F_{a_{(g,p)}}) \).
We have
\[
\sharp(\partial_1 F_{a_{(g,p)}}) = \gcd(3p + g + 2, p - g) = \gcd(2(2g + 1), p - g).
\]
Since \( \gcd(2g + 1, p - g) = \gcd(2g + 1, p + g + 1) = 1 \), we have that \( \sharp(\partial_1 F_{a_{(g,p)}}) = 2 \) (resp. 1) if \( p - g \) is even (resp. odd), equivalently \( p + g \) is even (resp. odd).
The genus of \( F_{a_{(g,p)}} \) is computed from the identities \( \|a_{(g,p)}\| = (\chi(F_{a_{(g,p)}})) = 2p + 2g + 2 \) and \( \sharp(\partial F_{a_{(g,p)}}) = 2p + 4 \).

The singularity data of \( F_{a_{(g,p)}} \) is obtained from the formula at the end of Section 4.2.

By Lemma 4.2, it is straightforward to prove the following.

Lemma 4.3. Suppose that \( a_{(g,p)} \) is primitive. Then \( (g,p) \not\in \{(0,0),(0,1),(1,0)\} \) if and only if \( F_{a_{(g,p)}} \) does not have a 1 prong on each component of \( \partial_1 F_{a_{(g,p)}} \cup \partial_2 F_{a_{(g,p)}} \). In particular if \( g \geq 2 \) and \( p \geq 0 \), then \( F_{a_{(g,p)}} \) does not have a 1 prong on each component of \( \partial_1 F_{a_{(g,p)}} \cup \partial_2 F_{a_{(g,p)}} \).

We are now ready to prove theorems in Section 1.

Proof of Theorem 1.4. There exists a sequence of primitive fibered classes \( \{a_{(g,p)}\}_{i=0}^{\infty} \) with \( p_i \to \infty \). (In fact, if we take \( p_i = (g + 1) + (2g + 1)i \), then \( 2g + 1 \) and \( p_i + g + 1 \) are relatively prime. Hence \( a_{(g,p_i)} \) is primitive.) Then \( N \) is a \( \Sigma_{g,2p+4} \)-bundle over the circle whose monodromy of the fibration has the dilatation \( r_{(g,p_i)} \). Therefore \( \delta_{g,2p+4} \leq r_{(g,p_i)} \). If we set \( n_i = 2p_i + 4 \), then
\[
n_i \log \delta_{g,n_i} \leq n_i \log r_{(g,p_i)} = \frac{(2p_i + 4)r_{(g,p_i)}}{\log(2p_i + 4)}.
\]
The right hand side goes to 2 as \( i \) goes to \( \infty \), see Lemmas 3.1(3) and 4.1. This completes the proof.

Proof of Theorem 1.6. The monodromy \( \Phi_{a_{(g,p)}} \) of the fibration associated to the primitive fibered class \( a_{(g,p)} \) is defined on the surface of genus \( g \) with \( 2p + 4 \) boundary components. It has the dilatation \( r_{(g,p)} \), and hence \( \delta_{g,2p+4} \leq r_{(g,p)} \).
Now let us prove \( \delta_{g,2p+1} \leq r_{(g,p)} \). The fibration associated to \( a_{(g,p)} \) extends naturally to a fibration on the manifold obtained from \( N \) by Dehn filling two
cusps specified by the tori $T_\alpha$ and $T_\gamma$ along the boundary slopes of the fiber. Then $\Phi_{a_{(g,p)}} : F_{a_{(g,p)}} \to F_{a_{(g,p)}}$ extends to the monodromy $\tilde{\Phi} : \tilde{F} \to \tilde{F}$ of the extended fibration, where the extended fiber $\tilde{F}$ is obtained from $F_{a_{(g,p)}}$ by filling each disk bounded by each component of $\partial_\alpha F_{a_{(g,p)}} \cup \partial_\gamma F_{a_{(g,p)}}$. Thus $\tilde{F}$ has the genus $g$ with $2p+1$ boundary components, see Lemma 4.2. By Lemma 4.3, $F_{a_{(g,p)}}$ does not have 1 prong at each component of $\partial_\alpha F_{a_{(g,p)}} \cup \partial_\gamma F_{a_{(g,p)}}$. Hence $\tilde{F}_{a_{(g,p)}}$ extends canonically to the stable foliation $\tilde{F}$ of $\tilde{\Phi}$. Therefore $\tilde{\Phi} = [\tilde{\Phi}]$ is pseudo-Anosov with the same dilatation as $\Phi_{a_{(g,p)}}$. This implies that $\delta_{g,2p+1} \leq r_{(g,p)}$.

The proofs of the rest of the bounds $\delta_{g,2p+2} \leq r_{(g,p)}$ and $\delta_{g,2p+3} \leq r_{(g,p)}$ are similar. In fact, the extended fiber of the fibration on the manifold obtained from $N$ by Dehn filling a cusp specified by $T_\alpha$ or $T_\gamma$ along the boundary slope of the fiber has the genus $g$ with $2p+2$ or $2p+3$ boundary components, see Lemma 4.2. Lemma 4.3 ensures that the extended monodromy is pseudo-Anosov with the same dilatation as $\Phi_{a_{(g,p)}}$.

Proof of Theorem 1.7. By Theorem 1.6 together with the assumption $(\ast)$ in Theorem 1.7, we have that for any $p \geq 0$ and for $j \in \{3, 4\}$,

$$\delta_{g,2p+j} \leq r_{(g,p)} \text{ or } \delta_{g,2p+j} \leq r_{(g,p+1)}.$$ 

Thus

$$\frac{(2p+j) \log \delta_{g,2p+j}}{\log(2p+j)} \leq \frac{(2p+j) \log r_{(g,p)}}{\log(2p+j)} \text{ or } \frac{(2p+j) \log \delta_{g,2p+j}}{\log(2p+j)} \leq \frac{(2p+j) \log r_{(g,p+1)}}{\log(2p+j)}.$$ 

By Lemma 3.1, it is easy to see that the both right hand sides in the above two inequalities go to 2 as $p$ goes to $\infty$. Thus

$$\limsup_{p \to \infty} \frac{(2p+j) \log \delta_{g,2p+j}}{\log(2p+j)} \leq 2.$$ 

Since this holds for $j \in \{3, 4\}$, the proof is done. 

Proof of Proposition 1.10. We prove the claim in the second half. (The proof in the first half is similar.) If $g \geq 2$ satisfies $(\ast)$, then for any $p \geq 0$ there exist a $\Sigma_{g,2p+3}$-bundle and a $\Sigma_{g,2p+4}$-bundle over the circle obtained from $N$, see proof of Theorem 1.7. More precisely such a bundle is homeomorphic to $N$ or it is obtained from $N$ by Dehn filling cusps along the boundary slopes of the fiber. Thus Proposition 1.10 holds from the result which says that the hyperbolic volume decreases after Dehn filling, see [16, 18]. 

Remark 4.4. To address Question 1.3, we explored fibered classes of the magic manifold whose dilatations have a suitable asymptotic behavior. We found a family of primitive fibered classes $a_{(g,p)}$ by computer. By Lemma 4.2, most of the components of $\partial F_{a_{(g,p)}}$ lie on the torus $T_\beta$. The pseudo-Anosov stable foliation associated to $a_{(g,p)}$ has the property that each component of $\partial_\beta F_{a_{(g,p)}}$
has 1 prong. The striking property of \( a_{(g,p)} \) is that the slope of the components of \( \partial F_{a_{(g,p)}} \) is exactly equal to \(-1\). Moreover, for any fixed \( g \), the projective class \( a_{(g,p)} \) goes to a single point \( (\frac{1}{2}, 1, \frac{1}{2}) \in \partial \Delta \) as \( p \) goes to \( \infty \). It is proved by Martelli and Petronio[13] that the manifold \( N(-1) \) obtained from \( N \) by Dehn filling a cusp along the boundary slope \(-1\) is not hyperbolic. The property that each component of \( \partial F_{a_{(g,p)}} \) has 1 prong can also be seen from the fact that \( N(-1) \) is a non hyperbolic manifold.

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The boundary of a fibered face of the magic 3-manifold

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