

DETERMINATION OF GENERALIZED HORSESHOE MAPS INDUCING ALL LINK TYPES

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ABSTRACT. Let φ be an orientation preserving automorphism of the 2-disk, and Φ an isotopy from the identity to φ . Then for each periodic orbit P of φ , Φ uniquely determines a link $\mathcal{S}_\Phi P$ in \mathbb{R}^3 by suspension. We say that φ *induces all link types* if there exists such an isotopy Φ that any link is realized as $\mathcal{S}_\Phi P$ for some P . In this paper, we completely determine when generalized horseshoe maps induce all link types.

1. INTRODUCTION

In this paper, we deal with oriented smooth knots and links in \mathbb{R}^3 obtained from periodic orbits of orientation-preserving automorphisms φ of the unit 2-disk D in \mathbb{R}^2 . We say that $x \in D$ is a *periodic point of period k* if $k = \min\{k \in \mathbb{N} \mid \varphi^k(x) = x\}$. For a periodic point $x \in D$, the set $\{\varphi^i(x) \mid i \in \mathbb{N}\}$ is called the periodic orbit. Denote by $\mathcal{O}(\varphi)$ the set of all finite unions of periodic orbits of φ . Let $\Phi = \{\varphi_t\}_{0 \leq t \leq 1}$ be an isotopy of D such that $\varphi_0 = id_D$, $\varphi_1 = \varphi$. Each $P \in \mathcal{O}(\varphi)$ yields a braid $\beta_\Phi P$ lying in the solid cylinder specified by Φ as follows;

$$\beta_\Phi P := \bigcup_{0 \leq t \leq 1} (\varphi_t(P) \times \{t\}) \subset C_\Phi := \bigcup_{0 \leq t \leq 1} (\varphi_t(D) \times \{t\}) \subset \mathbb{R}^3.$$

The link $\mathcal{S}_\Phi P \subset \mathbb{R}^3$, the closure of $\beta_\Phi P$, lies in solid torus C_Φ / \sim , which is obtained by gluing the top and the bottom of C_Φ naturally without twists. Note that $\mathcal{S}_\Phi P$ is naturally oriented by the parametrization by t . From now on, we consider these objects up to isotopies of \mathbb{R}^3 . Obviously, these objects depend on the choice of Φ . Actually, if Φ and Φ' are two such isotopies, the two solid tori are related by $C_{\Phi'} / \sim = h^k(C_\Phi / \sim)$ for some $k \in \mathbb{Z}$, where the homeomorphism h twists the solid torus once along the meridian. Therefore, for any $P \in \mathcal{O}(\varphi)$, we have $\beta_{\Phi'} P = \beta_\Phi P \cdot \Delta^k$, where Δ is a ‘positive’ full-twist of the whole strings. We say that φ *induces all link types* if there exists an isotopy Φ such that any link L in \mathbb{R}^3 belongs to $\{\mathcal{S}_\Phi P \mid P \in \mathcal{O}(\varphi)\}$. However, by the observation above, the following is well-defined.

Definition 1.1. Let φ be an orientation-preserving automorphism of the 2-disk D , and Φ be an arbitrarily fixed isotopy from id_D to φ . Then φ *induces all link types* if there exists an integer i satisfying the following;

$$(*) \text{ For any link } L \subset \mathbb{R}^3, L \in \{h^i(\mathcal{S}_\Phi P) \mid P \in \mathcal{O}(\varphi)\}.$$

Define the universality number by $N(\varphi) = \#\{i \in \mathbb{Z} \mid i \text{ satisfies } (*)\}$.

Note that $N(\varphi)$ measures ‘how persistently’ φ induces all link types.

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The topological entropy $h_{top}(\varphi)$ for φ is a measure of its dynamical complexity ([16]). A result of Gambaudo-van Strien-Tresser ([3, Theorem A]) tells us that if $h_{top}(\varphi) = 0$, then φ does not induce all link types. See also [11] for a parallel work. It is natural to ask the following problem:

Problem 1.2. Which automorphism induces all link types ?

The paper [10] answers Problem 1.2 for the Smale horseshoe map [15]. The Smale horseshoe map H is a fundamental example to study complicated dynamics since the invariant set is a hyperbolic set which is conjugate to the 2-shift and such invariant sets are often observed in many dynamical systems [8].

The following was shown in [10] by using templates.

Theorem 1.3. [10] *Let H be the Smale horseshoe map. Then neither H nor H^2 induces all link types. The third iteration H^3 induces all link types and $N(H^3) = 1$.*

Since $h_{top}(H) = \log 2$ and $h_{top}(H^2) = \log 4$, Theorem 1.3 shows the existence of dynamically complex diffeomorphisms not inducing all link types.

In this paper, we answer Problem 1.2 for generalized horseshoe maps G , by using templates and braids together with the twist signature $t(G)$ (see Definitions 2.7, 2.13). Moreover, in Corollary 3.8, we completely determine the number $N(G)$.

As a direct corollary to Corollary 3.8, our answer is as follows;

Theorem 1.4. *Let G be a generalized horseshoe map with twist signature (a_1, a_2, \dots, a_n) . Then G induces all link types, (i.e., $N(G) \geq 1$) if and only if one of the following is satisfied:*

- (1) *Each $a_i \geq 0$ and $\text{Max}\{a_i\} \geq 3$.*
- (2) *Each $a_i \leq 0$ and $\text{Min}\{a_i\} \leq -3$.*
- (3) *For some i and j , $a_i a_j < 0$.*

We remark that the topological entropy of the above G is equal to $\log n$ (Remark 2.10).

The twist signature was first introduced in [6] for the study of periodic orbits of flows which bifurcate from a Morse-Smale flow to a Smale flow, in particular those which realize all link types. The definition of the twist signature in [6] is different from ours. Our twist signature is determined by *the pleated diagram* (Section 2.2), but the twist signature in [6] is determined by the pleated diagram and the ‘initial value’. (See Remark 2.14.)

In [6, Theorem 4.3], Ghrist-Young gave sufficient conditions for *a pleated vector field* ([6, Definition 3.3]) to support a universal template. By Theorem 3.2, we give computable necessary and sufficient conditions (Theorem 3.9) for a pleated vector field to support a universal template. We state and prove Theorem 3.9 at the end of Section 3.

The paper is organized as follows. In Section 2, we first review some results on the template theory [5] which is our primary machinery. Next, we define generalized horseshoe map G and its twist signature $t(G)$, and construct the key objects, the template $\mathcal{U}(G)$

and the braid $b(G)$. Finally, we reconstruct $\mathcal{U}(G)$ by using $t(G)$ and $b(G)$. In Sections 3 and 4, we state and prove our results.

2. PRELIMINARIES

2.1. Templates.

Definition 2.1. A *template* is a compact branched 2-manifold with boundary and with smooth expansive semiflow built from a finite number of *joining charts* and *splitting charts* as in Figure 1.

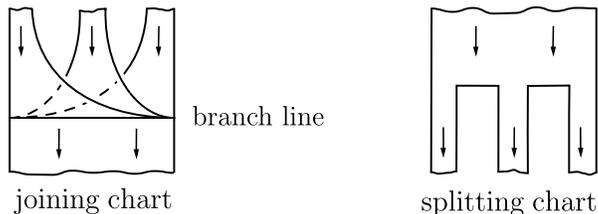


FIGURE 1

In [2, Theorem 2.1], Birman and Williams described the relationship between templates and links of periodic orbits in 3-dimensional flows. In their proof, we see that if the flow has a 1-dimensional hyperbolic chain recurrent set ([5, Definitions 1.2.6, 1..2.11]), then their theorem is described as follows. (Alternatively, see [5, Theorem 2.2.4 and Lemma 2.2.7].)

Theorem 2.2. [2, Theorem 2.1] *Let $f_t : M \rightarrow M$ ($t \in \mathbb{R}$) be a smooth flow on a 3-manifold M . Suppose f_t has a 1-dimensional hyperbolic chain-recurrent set. Then there exists a template \mathcal{W} embedded in M such that the links of the periodic orbits L_f of f_t correspond one-to-one to the links of the periodic orbits $L_{\mathcal{W}}$ on \mathcal{W} . For any finite union of the periodic orbits, the correspondence is via ambient isotopy.*

For $i_1, i_2 \in \mathbb{Z}$, $\mathcal{L}(i_1, i_2)$ denotes the template as in Figure 2 embedded in \mathbb{R}^3 having a single branch line and having two unknotted, unlinked strips with i_1 and i_2 half-twists respectively. We call it a *Lorenz-like template*.

It is known that for a given template \mathcal{W} embedded in \mathbb{R}^3 , there exist an infinite number of distinct knot types as periodic orbits on \mathcal{W} ([5, Chapter 3]).

We say that a template \mathcal{W} embedded in \mathbb{R}^3 is *universal* if for each link L in \mathbb{R}^3 , there exists a finite union of periodic orbits P_L of the semiflow on \mathcal{W} such that $L = P_L$. The following theorem is useful.

Theorem 2.3. [4, Corollary 3] *For each $i_2 < 0$, the Lorenz-like template $\mathcal{L}(0, i_2)$ is universal.*

A subset \mathcal{W}' of a template \mathcal{W} is called a *subtemplate* of \mathcal{W} if \mathcal{W}' with semiflow induced from \mathcal{W} is a template. We use the following to prove Theorem 3.5.

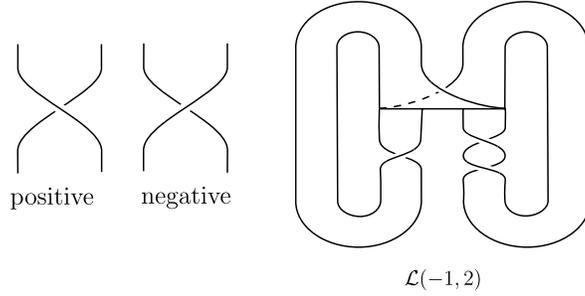


FIGURE 2

Lemma 2.4. *The templates $\mathcal{W}_1, \mathcal{W}_2 \subset \mathbb{R}^3$ depicted in Figure 3 are universal.*

Proof. As in Figure 3, \mathcal{W}_2 contains a subtemplate (shaded) isotopic to $\mathcal{L}(0, -5)$, which by Theorem 2.3 is universal. Therefore \mathcal{W}_2 and its mirror image \mathcal{W}_1 are universal. \square

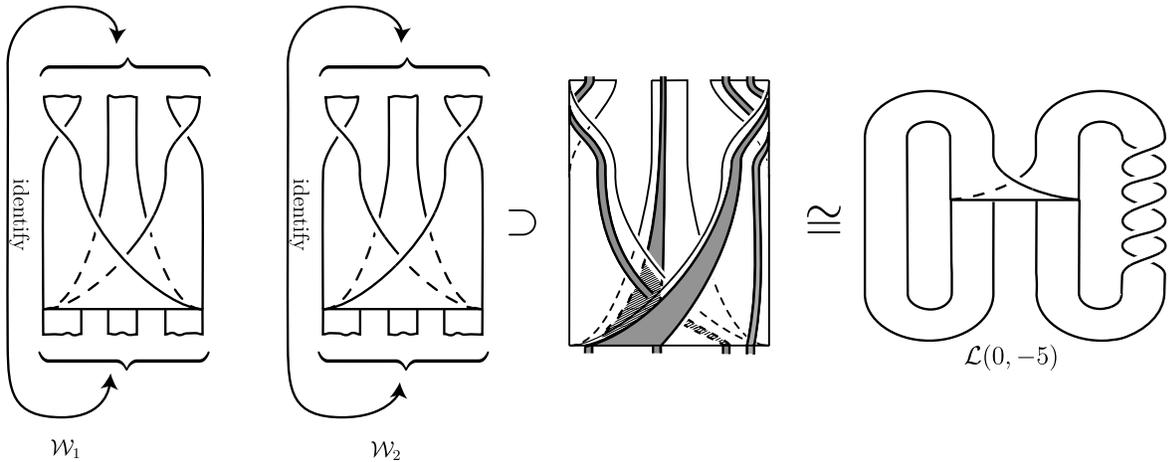


FIGURE 3

Definition 2.5. A link L is said to be *positive* (resp. *negative*) if L has a diagram where all crossings are positive (resp. negative). A template \mathcal{W} is said to be *positive* (resp. *negative*) if any link L of periodic orbits on \mathcal{W} is positive (resp. negative).

Remark 2.6. There exists a knot which is neither positive nor negative (e.g. the figure eight knot). Hence if the template \mathcal{W} is positive or negative, then \mathcal{W} is not universal. This observation is used for the proof of Proposition 3.4.

2.2. Generalized horseshoe map, pleated diagram and twist signature. In this subsection, we define generalized horseshoe map, pleated diagram and twist signature. We first introduce some terminologies used for the definitions. Let $R = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \subset D$, and let S_0, S_1 be half disks as in Figure 4 (a). For $c, c' \in [-\frac{1}{2}, \frac{1}{2}]$, we call $\ell_v = \{c\} \times [-\frac{1}{2}, \frac{1}{2}]$ (resp. $\ell_h = [-\frac{1}{2}, \frac{1}{2}] \times \{c'\}$) a *vertical* (resp. a *horizontal*) line. For $[c, d], [c', d'] \subset [-\frac{1}{2}, \frac{1}{2}]$, we call $B = [c, d] \times [-\frac{1}{2}, \frac{1}{2}]$ (resp. $B' = [-\frac{1}{2}, \frac{1}{2}] \times [c', d']$) a *vertical* (resp. a *horizontal*) rectangle.

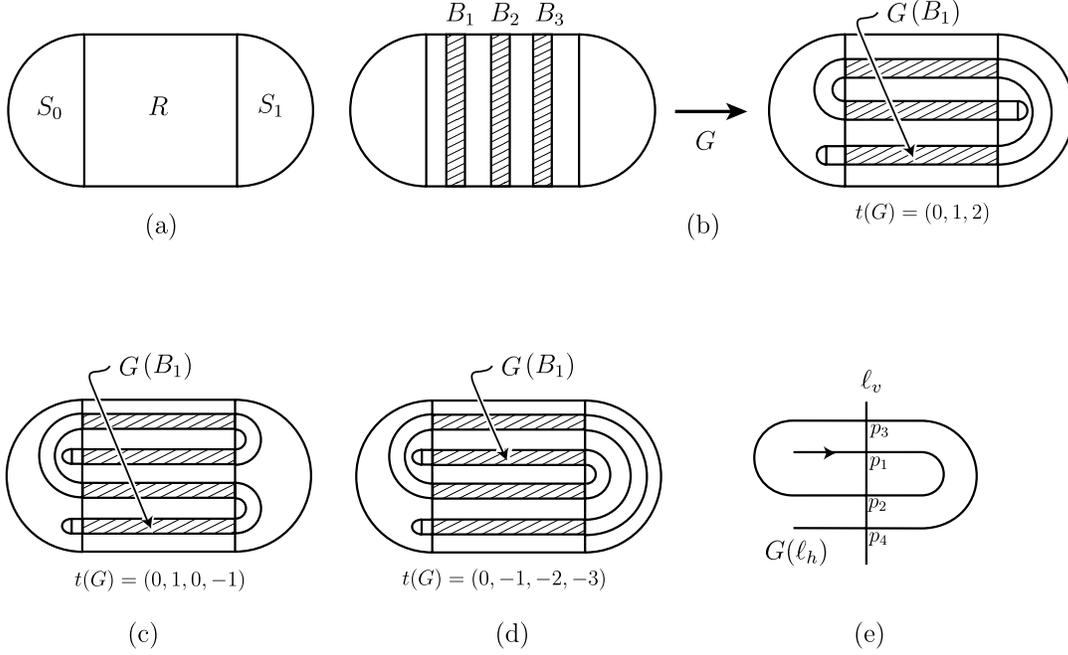


FIGURE 4

Let B_1, B_2 (resp. B'_1, B'_2) be disjoint vertical (resp. disjoint horizontal) rectangles. The notation $B_1 <_1 B_2$ (resp. $B'_1 <_2 B'_2$) means the first (resp. second) coordinate of a point in B_2 (resp. B'_2) is greater than that of B_1 (resp. B'_1). We denote the open rectangle which lies between B_1 and B_2 by (B_1, B_2) .

Definition 2.7. Let $n \geq 2$ be an integer. A *generalized horseshoe map* G of length n is an orientation preserving diffeomorphism of D satisfying the following:

There exist vertical rectangles $B_1 <_1 B_2 <_1 \cdots <_1 B_n$ and horizontal rectangles $B'_1 <_2 B'_2 <_2 \cdots <_2 B'_n$ such that

- (1) for each $1 \leq i \leq n$, $G(B_i) = B'_j$ for some $1 \leq j \leq n$,
- (2) for each $1 \leq i \leq n-1$, $G((B_i, B_{i+1})) \subset S_k$ for some $k \in \{0, 1\}$,
- (3) G expands the part of horizontal lines which intersects each B_i uniformly, and contract the vertical lines in each B_i uniformly,
- (4) $G|_{S_0} : S_0 \rightarrow S_0$ is contractive,
- (5) if n is even (resp. odd), then $G(S_1) \subset \text{Int } S_0$ (resp. $G|_{S_1} : S_1 \rightarrow S_1$ is contractive) and
- (6) G has no periodic points in $D \setminus (S_0 \cup R \cup S_1)$.

Remark 2.8. By (4), G has a unique fixed point α_0 in S_0 . By (5), if n is odd, then G has a unique fixed point α_1 in S_0 . Hence, by (6), if n is even (resp. odd), then any periodic points of G are contained in $\{\alpha_0\} \cup \Lambda$ (resp. $\{\alpha_0, \alpha_1\} \cup \Lambda$). To prove (*) in Definition 1.1 for G , it is enough to show the following:

$$\text{For any link } L \subset \mathbb{R}^3, L \in \{h^i(\mathcal{S}_\Phi P) | P \subset \Lambda, P \in \mathcal{O}(G)\}.$$

Remark 2.9. We note that by the definition of G , $\Lambda = \bigcap_{m \in \mathbb{Z}} G^m(B_1 \cup \cdots \cup B_n)$ is a hyperbolic set and a product of Cantor sets, and that $G|_\Lambda : \Lambda \rightarrow \Lambda$ is conjugate to the (full) n -shift. Therefore, the suspension of Λ by an isotopy Φ from id_D to G is a 1-dimensional hyperbolic chain-recurrent set for the 3-dimensional suspension flow of G in the solid torus C_Φ/\sim , induced by Φ . This is a situation Theorem 2.2 can be applied to.

Remark 2.10. By the formula of [13, Theorem 1.9(b)], the topological entropy for the n -shift is $\log n > 0$. This together with the definition of G shows that the topological entropy for G is also $\log n$.

We introduce the pleated diagram, which represents the “spine” for $G(R)$ and is well-defined up to isotopies of D .

Definition 2.11. Take an oriented horizontal line ℓ_h in R and an oriented vertical line ℓ_v in R . The union of oriented arcs $G(\ell_h) \cup \ell_v$ is called *the pleated diagram of G* , and is denoted by $P(G)$.

Remark 2.12. Our pleated diagram is the same as the pleating signature of [6].

It is easy to see that if G is of length n , then $G(\ell_h)$ intersects ℓ_v n times. We denote the intersections p_1, \cdots, p_n along the orientation of $G(\ell_h)$. For example, Figure 4 (e) is the pleated diagram of the map in Figure 4 (d).

Definition 2.13. Let G be a generalized horseshoe map of length n . *The twist signature $t(G)$ of G* is the array of n integers (a_1, \cdots, a_n) satisfying the following:

- (1) $a_1 = 0$, and (2) for $2 \leq i \leq n$, $a_i = a_{i-1} + 1$ (resp. $a_i = a_{i-1} - 1$) if the oriented segment $[p_{i-1}, p_i]$ goes around in the counterclockwise direction (resp. clockwise direction).

Remark 2.14. The relation of the twist signature in [6] and our twist signature is as follows. Let $\tau = (\tau_1, \cdots, \tau_n)$ be the twist signature in [6] and (a_1, \cdots, a_n) our twist signature. Then $(a_1, \cdots, a_n) = (0, \tau_2 - \tau_1, \tau_3 - \tau_1, \cdots, \tau_n - \tau_1)$.

Example 2.15. Let G be a generalized horseshoe map of length 3 as in Figure 4 (b). Then $t(G) = (0, 1, 2)$. The generalized horseshoe maps of length 4 depicted in Figure 4 (c) and (d) respectively have twist signatures $(0, 1, 0, -1)$ and $(0, -1, -2, -3)$.

Remark 2.16. For any array of integers (a_1, \cdots, a_n) with $a_1 = 0$ and $a_{i+1} = a_i \pm 1$, there exists a generalized horseshoe map G (not necessarily unique) such that $t(G) = (a_1, \cdots, a_n)$.

2.3. The isotopy from id_D to G . In this subsection, we fix an isotopy from id_D to each generalized horseshoe map G . By using the isotopy, we will obtain a template $\mathcal{U}(G)$ of G in the next subsection. We call the rectangles $R \setminus Int(B_1 \cup B_2 \cup \cdots \cup B_n)$ in order E_0, \cdots, E_n , where E_0 touches S_0 .

Let us construct the isotopy $\mathcal{G} = \{G_t\}_{0 \leq t \leq 1}$ from the identity map id_D to G as follows:

Step 1. R is shrunk vertically and stretched horizontally so that the images of E_0 and B_1 respectively coincide with the image of those under G , which hereafter are fixed.

Steps 2 to n . R is further stretched and bent inductively until the images of E_1, \dots, E_n and B_2, \dots, B_n respectively coincide with their images under G . At Step k , the images of E_{k-1} and B_k respectively coincide with $G(E_{k-1})$ and $G(B_k)$, which hereafter are fixed.

Final Step. The other part of D is modified so that the image of D now coincides with $G(D)$.

2.4. **The template $\mathcal{U}(G)$.** Using Φ , we define a template $\mathcal{U}(G)$ for G according to the construction in the proof of Theorem 2.2 so that we have:

$$\begin{aligned} & \{ \text{the link } \mathcal{S}_\Phi P \mid P \subset \Lambda, P \text{ is a finite union of periodic orbits of } G \} \\ &= \{ \text{the link } L \mid L \text{ is a finite union of periodic orbits on } \mathcal{U}(G) \}. \end{aligned}$$

Therefore, by Remark 2.8, we have the following;

Theorem 2.17. *A generalized horseshoe map G induces all link types if and only if $h^i(\mathcal{U}(G))$ is universal for some $i \in \mathbb{Z}$, where $h^i(\mathcal{U}(G))$ is obtained from $\mathcal{U}(G)$ by applying a full twist i times.*

Here we recall the construction: As an example, we use the generalized horseshoe map G in Figure 5 (a). We denote the bottom edges of B_1, \dots, B_n and R by e_1, \dots, e_n and r . Let $\mathcal{T}_i = \bigcup_{0 \leq t \leq 1} (G_t(e_i) \times \{t\}) \subset D \times I$ for $i \in \{1, \dots, n\}$ as in Figure 5 (b). Then we obtain the template $\mathcal{U}(G)$ as in Figure 5 (c) by collapsing the n horizontal lines $G(e_1) \times \{1\}, \dots, G(e_n) \times \{1\}$ to the horizontal line $r \times \{1\}$ along the stable direction of Λ and then identifying $D \times \{0\}$ with $D \times \{1\}$.

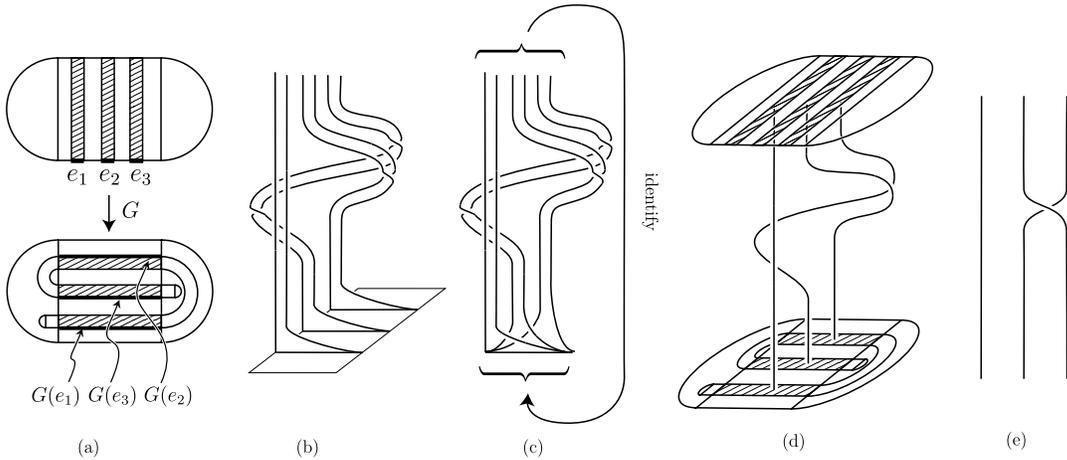


FIGURE 5

We abuse the notation and denote the images of \mathcal{T}_i by the same symbol and call them strips of $\mathcal{U}(G)$. Let $\mathcal{T}_{i_1} \cup \cdots \cup \mathcal{T}_{i_k}$ be a union of strips of $\mathcal{U}(G)$. Then the *subtemplate of $\mathcal{U}(G)$ associated with them* is the subtemplate of $\mathcal{U}(G)$ obtained by omitting other strips than $\mathcal{T}_{i_1} \cup \cdots \cup \mathcal{T}_{i_k}$.

A relation between $t(G)$ and the number of half-twists of each strip is as follows:

Lemma 2.18. *Let G be a generalized horseshoe map with twist signature (a_1, \dots, a_n) . Then the number of half-twists of each \mathcal{T}_i is a_i .*

Proof. Corresponding to Step i in the construction of the isotopy \mathcal{G} , $\mathcal{T}_i, \mathcal{T}_{i+1}, \dots, \mathcal{T}_n$ are added with a positive (resp. a negative) half-twist if $a_i = a_{i-1} + 1$ (resp. $a_i = a_{i-1} - 1$), and corresponding to Steps $i + 1$ to n , \mathcal{T}_i goes down straightly. Hence the number of the total half-twists of each \mathcal{T}_i is a_i . \square

2.5. The braid $b(G)$. In this subsection, we first introduce the braid $b(G)$ of a generalized horseshoe map G . Next, we reconstruct $\mathcal{U}(G)$ by using $b(G)$ and $t(G)$.

First we construct the braid $b(G)$ from G . As an example, we use the generalized horseshoe map in Figure 5 (a). For each B_i , take a point c_i at the center. Then the image of the n points c_1, \dots, c_n under the isotopy \mathcal{G} yields a union of n strings, where the starting points c_1, \dots, c_n are arranged in this order horizontally in the top disk $R \times \{0\}$, and the end points are arranged vertically in the bottom disk $R \times \{1\}$ (Figure 5 (d)). Then by turning the bottom disk $R \times \{1\}$ by $-\frac{\pi}{2}$, we obtain an n -braid $b(G)$ (Figure 5 (e)). We denote by s_i the string of $b(G)$ starting from c_i .

The template $\mathcal{U}(G)$ can be reconstructed from $b(G)$ and $t(G)$ as follows: Replace each string s_i of $b(G)$ by a band with a_i half-twists so that the core of the bands is $b(G)$. Next, join the bottom of the bands to the top of a bunched trapezoids \mathcal{B} as in Figure 6, and attach the top of the bands to the bottom of \mathcal{B} without twists. Then the result with the semiflow is a template. Notice that, by Lemma 2.18 and by the construction of $b(G)$, the template is isotopic to $\mathcal{U}(G)$.

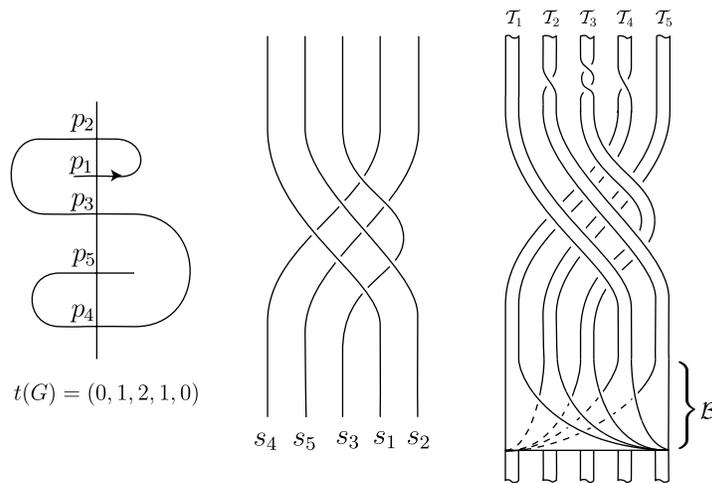


FIGURE 6

3. RESULTS

Definition 3.1. Let G be a generalized horseshoe map with twist signature (a_1, \dots, a_n) . We say that G is *positive* (resp. *negative*) if $a_i \geq 0$ for all $i \in \{1, \dots, n\}$ (resp. $a_i \leq 0$ for all $i \in \{1, \dots, n\}$). We say that G is *mixed* if G is neither positive nor negative.

Example 2.15 gives a positive, a mixed and a negative G .

We are now ready to state the results of this paper. We prove the following theorem in Section 4:

Theorem 3.2. *For a generalized horseshoe map G , the following hold.*

- (1) G is positive if and only if $b(G)$ is positive, and
- (2) G is negative if and only if $b(G)$ is negative.

Remark 3.3. Recall that even though G is positive, we may at first have many negative crossings in the construction of $b(G)$. Theorem 3.2 assures all such negative crossings will be cancelled.

By Theorem 3.2, and the construction of $\mathcal{U}(G)$, we have the following:

Proposition 3.4. *Let G be a generalized horseshoe map. If G is positive or negative, then $\mathcal{U}(G)$ is not universal.*

Proof. We assume G is positive, for the other case is symmetric. Then by Lemma 2.18, each \mathcal{T}_i is non-negatively twisted. Since $b(G)$ is positive by Theorem 3.2, the crossings of the strips of $\mathcal{U}(G)$ does not give rise to a negative crossings for any link carried by $\mathcal{U}(G)$. Finally \mathcal{B} in Figure 6 does not give rise to negative crossings. These show that $\mathcal{U}(G)$ is positive, and hence not universal. (Recall Remark 2.6.) \square

The following are the main theorems in this paper. Note that $h^i(\mathcal{U}(G))$, hereafter abbreviated to $h^i\mathcal{U}(G)$, is a template which is isotopic to $\mathcal{U}(G)$ with i full-twists. (The map h^i is defined in Section 1.)

Theorem 3.5. *Let G be a generalized horseshoe map with twist signature $t(G)$. Then, the template $\mathcal{U}(G)$ is universal if and only if G is mixed.*

Theorem 3.6. *Let G be a generalized horseshoe map with twist signature (a_1, \dots, a_n) . Then, the template $h^i\mathcal{U}(G)$ is universal if and only if the sequence $(2i+a_1, 2i+a_2, \dots, 2i+a_n)$ contains a subsequence $(1, 0, -1)$ or $(-1, 0, 1)$.*

Proof of Theorem 3.5. Recall that G is mixed if and only if $t(G)$ contains a subsequence $(1, 0, -1)$ or $(-1, 0, 1)$.

The only if part. Suppose $t(G)$ does not contain a subsequence $(\pm 1, 0, \mp 1)$. Then by definition G is positive or negative. Therefore, by Proposition 3.4, we see that $\mathcal{U}(G)$ is not universal.

The if part. Suppose $t(G)$ contains a subsequence $(a_{i-1}, a_i, a_{i+1}) = (1, 0, -1)$. (The other case is similar, and hence we omit.) Then $P(G)$ locally appears as in Figure 7 (a) or (b). First we consider the former case.

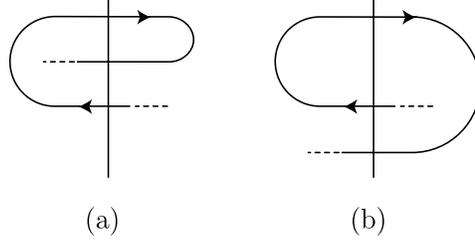


FIGURE 7

Recall the isotopy from the identity to G , where at each step k ($k = 1, 2, \dots, i-1$), the three straps \mathcal{T}_{i-1} , \mathcal{T}_i , \mathcal{T}_{i+1} are half-twisted together positively or negatively according as $a_k > a_{k-1}$ or $a_k < a_{k-1}$. So through steps up to the $(i-1)$ th, \mathcal{T}_{i-1} , \mathcal{T}_i , \mathcal{T}_{i+1} are half-twisted a_{i-1} times and hence only once. See Figure 8. At Step i , \mathcal{T}_i and \mathcal{T}_{i+1} are half-twisted negatively. Then through Step $(i+1)$ to the final step, \mathcal{T}_{i+1} is half-twisted negatively. Note that the three strips are bunched at the bottom in the order \mathcal{T}_{i-1} , \mathcal{T}_{i+1} and \mathcal{T}_i , because of the local picture of $P(G)$ in Figure 7 (a). Therefore, in Figure 8 (b) we see that $\mathcal{U}(G)$ is universal, because it contains the subtemplate \mathcal{W}_1 (in Figure 3) which by Lemma 2.4 is universal. On the other hand, from Figure 7 (b), we obtain another universal template \mathcal{W}_2 , which is also universal. \square

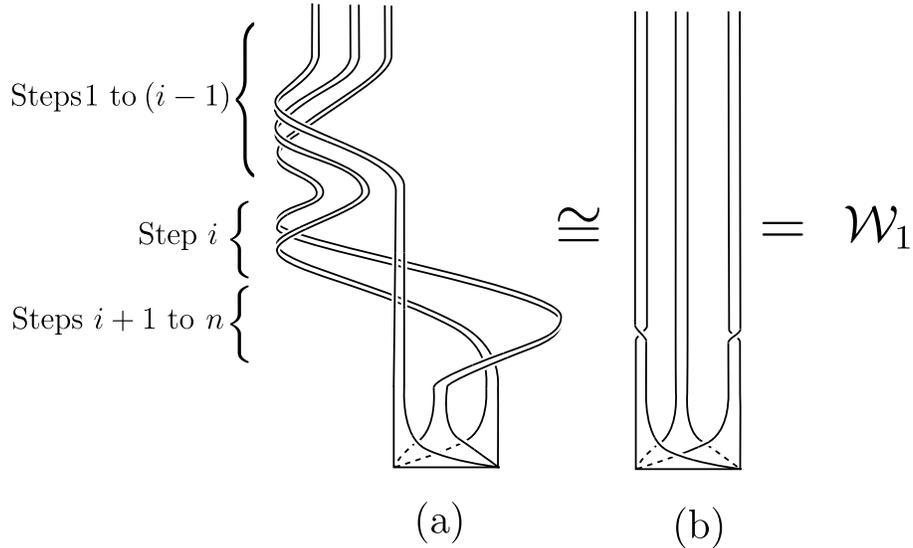


FIGURE 8

To prove Theorem 3.6, we use the following lemma, where we modify G so that $h^i\mathcal{U}(G)$ is a natural subtemplate of the new template.

Lemma 3.7. *Let G be a generalized horseshoe map with $t(G) = (a_1, \dots, a_n)$. Let i be a positive (resp. negative) integer. As in Figure 9, construct G' from G by prepending to $P(G)$ a positive (resp. negative) spiral so that we have;*

$$\begin{cases} t(G') = (0, 1, 2, \dots, 2i-1, \mathbf{2i} + \mathbf{a_1}, \mathbf{2i} + \mathbf{a_2}, \dots, \mathbf{2i} + \mathbf{a_n}) & \text{if } i > 0, \\ t(G') = (0, -1, -2, \dots, 2i+1, \mathbf{2i} + \mathbf{a_1}, \mathbf{2i} + \mathbf{a_2}, \dots, \mathbf{2i} + \mathbf{a_n}) & \text{if } i < 0. \end{cases}$$

Let $\mathcal{T}'_1, \dots, \mathcal{T}'_{2i}, \mathcal{T}'_{2i+1}, \dots, \mathcal{T}'_{2i+n}$ be the strips of $\mathcal{U}(G')$. Then the subtemplate \mathcal{L} of $\mathcal{U}(G')$ consisting of the last n strips $\mathcal{T}'_{2i+1}, \dots, \mathcal{T}'_{2i+n}$ is isotopic to $h^i\mathcal{U}(G)$.

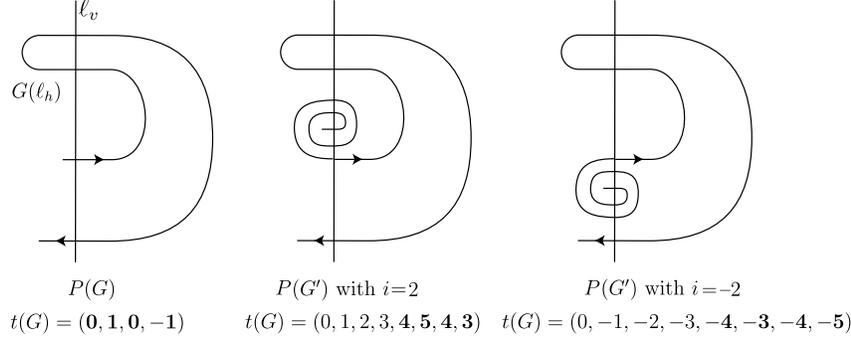


FIGURE 9

Proof of Lemma 3.7. We only prove for the case $i > 0$, for the other case is symmetric. Through Steps 1 to $2i$ of the construction of the isotopy \mathcal{G} , $\mathcal{T}'_{2i+1}, \dots, \mathcal{T}'_{2i+n}$ are half-twisted together $2i$ times, i.e., full-twisted i times. After that, by the construction of $\mathcal{U}(G')$, $\mathcal{T}'_{2i+1}, \dots, \mathcal{T}'_{2i+n}$ behave exactly as $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n$ in $\mathcal{U}(G)$ do. So \mathcal{L} is isotopic to $\mathcal{U}(G)$ with all the strips full-twisted together i times, which by definition is $h^i\mathcal{U}(G)$. \square

Proof of Theorem 3.6. The if part. Suppose $(2i + a_1, 2i + a_2, \dots, 2i + a_n)$ contains a subsequence $(\pm 1, 0, \mp 1)$. Then as in the proof of Theorem 3.5, we see that $\mathcal{U}(G')$ in Lemma 3.7 contains the universal subtemplate \mathcal{W}_1 or \mathcal{W}_2 , which by Lemma 3.7 is also a subtemplate of $h^i\mathcal{U}(G)$. *The only if part.* Suppose $(2i + a_1, 2i + a_2, \dots, 2i + a_n)$ does not contain a subsequence $(\pm 1, 0, \mp 1)$. Then since the former half of $t(G')$ is monotonous starting at 0, $t(G')$ also does not contain $(\pm 1, 0, \mp 1)$, i.e., G' is positive or negative. Therefore, by Proposition 3.4, $\mathcal{U}(G')$ is not universal. Therefore, its subtemplate $h^i\mathcal{U}(G)$ is not universal. \square

Theorem 3.6 tells us whether $h^i(\mathcal{U}(G))$ is universal or not. Hence, by Theorem 2.17, we can determine the number $N(G)$ as follows; For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer which does not exceed x . For a twist signature (a_1, \dots, a_n) , put $M := \max\{a_i | 1 \leq i \leq n\}$ and $m := \min\{a_i | 1 \leq i \leq n\}$.

Corollary 3.8. *For a generalized horseshoe map G , $N(G)$ is given by the following;*

$$N(G) = \begin{cases} \lfloor \frac{M-1}{2} \rfloor, & \text{if } G \text{ is positive:} \\ \lfloor \frac{-m-1}{2} \rfloor, & \text{if } G \text{ is negative:} \\ \lfloor \frac{M-1}{2} \rfloor + \lfloor \frac{-m-1}{2} \rfloor + 1, & \text{if } G \text{ is mixed.} \end{cases}$$

Proof. Suppose G is positive. Then $(2i + a_1, 2i + a_2, \dots, 2i + a_n)$ contains $(\pm 1, 0, \mp 1)$ (i.e., by Theorem 3.6, $h^i\mathcal{U}(G)$ is universal) if and only if one of the following holds:

- (1) M is even and $i \in \{-1, -2, \dots, -(\frac{M}{2} - 1)\}$,
- (2) M is odd and $i \in \{-1, -2, \dots, -\frac{M-1}{2}\}$. Therefore, $N(G) = \lfloor \frac{M-1}{2} \rfloor$.

Suppose G is negative. Then $h^i\mathcal{U}(G)$ is universal if and only if one of the following holds;

- (1) m is even and $i \in \{1, 2, \dots, \frac{-m}{2} - 1\}$,
- (2) m is odd and $i \in \{1, 2, \dots, -\frac{m-1}{2}\}$. Therefore, $N(G) = \lfloor \frac{-m-1}{2} \rfloor$.

If G is mixed, then since $h^0\mathcal{U}(G)$ is already universal, $N(G) = \lfloor \frac{M-1}{2} \rfloor + \lfloor \frac{-m-1}{2} \rfloor + 1$. \square

In the remainder of this section, we give an application of Theorem 3.2 on a class of some vector fields. In [6], R. Ghrist and T. Young analyzed *pleated vector fields* (see Definition 3.3 in [6]) which bifurcate from a Morse-Smale vector field to a Smale vector field realizing all link types.

In [6, Theorem 4.3], they gave sufficient conditions for a *pleated vector field* to contain all links as closed orbits. By Theorem 3.2, we give computable necessary and sufficient conditions.

Theorem 3.9. *Let X_λ be a family of vector fields unfolding a pleated Shil'nikov saddle-node in \mathbb{R}^3 as in [6, Theorem 4.3]. Then for sufficiently small $\lambda > 0$, the induced flow Φ_λ contains all knots and links among the closed orbits if and only if (1) the homoclinic orbits $\{\gamma_i\}$ are unknots, and (2) the twist signature $\tau = (\tau_1, \dots, \tau_n)$ given in [6] has both positive and negative terms.*

Proof. The proof of the if part is given in [6, Theorem 4.3]. We show the only if part, that is, if the homoclinic orbits $\{\gamma_i\}$ are knotted, or $\tau_1, \dots, \tau_n \geq 0$ or $\tau_1, \dots, \tau_n \leq 0$, then induced template for $\lambda > 0$ is not universal. If the homoclinic orbits $\{\gamma_i\}$ are knotted, then induced template is not universal by [6, Lemma 2]. We suppose that the homoclinic orbits $\{\gamma_i\}$ are unknots and $\tau_1, \dots, \tau_n \geq 0$. Let P be a pleated signature of the vector field (see Remark 2.12). By using the same arguments in Lemma 3.7, we define, from P , the pleated diagram P' of some generalized horseshoe map G' such that $t(G') = (0, 1, 2, \dots, \tau_1 - 1, \tau_1, \dots, \tau_n)$. By Proposition 3.4, $\mathcal{U}(G')$ is not universal. Since the subtemplate of $\mathcal{U}(G')$ associated with the last n strips is isotopic to the template \mathcal{T} induced from the vector field, \mathcal{T} is not universal. \square

4. PROOF OF THEOREM 3.2

Proof of the if part. Recall that $a_1 = 0$ by definition. Suppose that for some i , $a_i = 0$ and $a_{i+1} = -1$. As seen in the proof of Lemma 2.18, we see that the strings s_i and s_{i+1} are twisted -1 times. Therefore, $b(G)$ can not be positive. \square

In order to prove the ‘only if’ part, we use the notion of *quasi-braids*. As shown in [12], any braid has the following configuration in a cube $\bigcup_{0 \leq t \leq 1} R_t$; Each string of s_1, \dots, s_n meets all level planes R_t in exactly one point except exactly one plane $R_{1/2}$ where $\bigcup_i s_i \cap R_{1/2}$ consists of mutually disjoint n simple arcs or isolated points. Moreover, while $0 \leq t < 1/2$ and $1/2 < t \leq 1$, the strings are not only monotonous but also straight

(i.e., with no braiding). By construction of $b(G)$, we can place $b(G)$ so that for each s_i of $b(G)$, $s_i \cap R_{1/2}$ is parallel to the subarc of $G(\ell_h)$ between p_1 and p_i . Only for $i = 1$, $s_i \cap R_{1/2}$ is an isolated point. See Figure 10 for example.

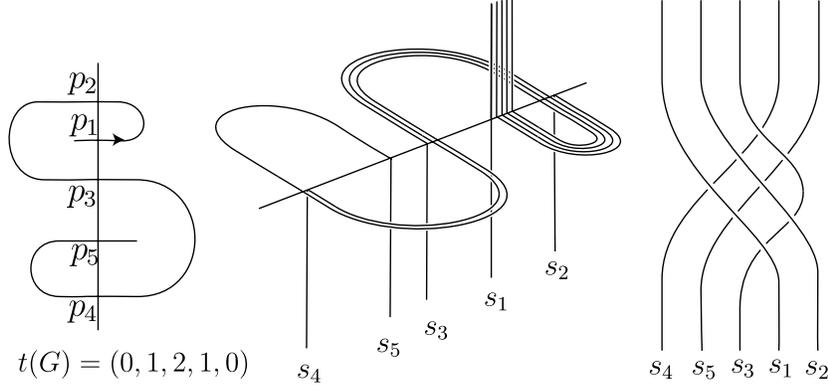


FIGURE 10

We prove the theorem by induction on the length n of the twist signature of G , i.e., the number of strings of braids $b(G)$. If $n = 2$, then it is clear that $b(G)$ is the trivial 2-braid and the claim follows. Suppose the claim holds up to $n - 1$. Let G be a generalized horseshoe map with positive twist signature (a_1, \dots, a_n) .

Case 1: $a_n > a_{n-1}$. In this case, by changing the view point if necessary, we see that the strings s_{n-1} and s_n locally appear as in Figure 11 (a). Note that until s_{n-1} leaves $R_{1/2}$, s_n is contained in a thin neighborhood of s_{n-1} which misses the other strings. Moreover, by a similar argument as in the proof of Lemma 2.18, we see that s_{n-1} and s_n are twisted $a_{n-1}(\geq 0)$ times before s_{n-1} leaves $R_{1/2}$ and not twisted afterward. Let G' be another generalized horseshoe map whose pleated diagram is obtained from that of G by omitting the last turn so that $t(G') = (a_1, \dots, a_{n-1})$. By the induction hypothesis, $b(G')$ is positive. Now by slightly pulling down the last part of $s_n \cap R_{1/2}$, we see that $b(G)$ is a product of positive braids b_1, b_2 , where b_1 is obtained by doubling the sting s_{n-1} in $b(G')$ with $a_{n-1}(\geq 0)$ twists. Therefore, $b(G)$ is positive. See Figure 11 (b) for a typical example.

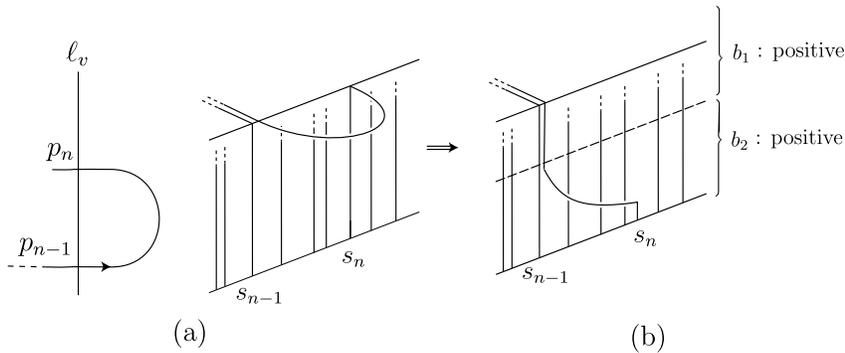


FIGURE 11

Case 2. $a_n < a_{n-1}$. In this case, there exists some $1 < r < n$ such that $a_{r-1} < a_r > a_{r+1}$, since $a_1 = 0$. Then, by changing the view point if necessary, p_{r-1}, p_r, p_{r+1} locally appear as in Figure 12, where the shaded regions may contain other parts of the pleat. Let G', G'' be generalized horseshoe maps which are defined by the pleated diagrams $P(G')$ and $P(G'')$ which only locally differ from $P(G)$ as in Figure 12. Note that the twist signatures $t(G')$ and $t(G'')$ are both positive, and by the induction hypothesis, the braid $b(G'')$ is positive.

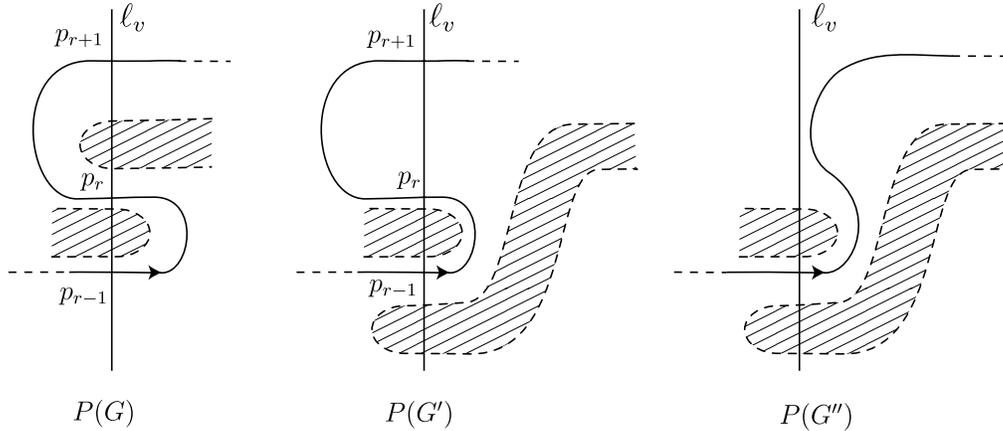


FIGURE 12

Claim 1. *The braid $b(G')$ is positive.*

Proof. As in Figure 13, slightly pull down the last part of $R_{1/2} \cap (s_r \cup s_{r+1})$ of $b(G')$. Then we see that s_{r-1}, s_r, s_{r+1} are twisted $a_{r-1} (\geq 0)$ times before s_{r-1} leaves $R_{1/2}$ in a small neighborhood of s_{r-1} where no other strings meet. By the positivity of $b(G'')$, we see that $b(G')$ is a product of two positive braids, and hence positive.

Claim 2. *The braid $b(G'')$ is positive.*

Proof. Proof is similar to the above: by Figure 14, we see that $b(G)$ is a product of $b(G'')$ and a positive braid. Hence by Claim 1, $b(G'')$ is positive.

Now Theorem 3.2 is completed. \square

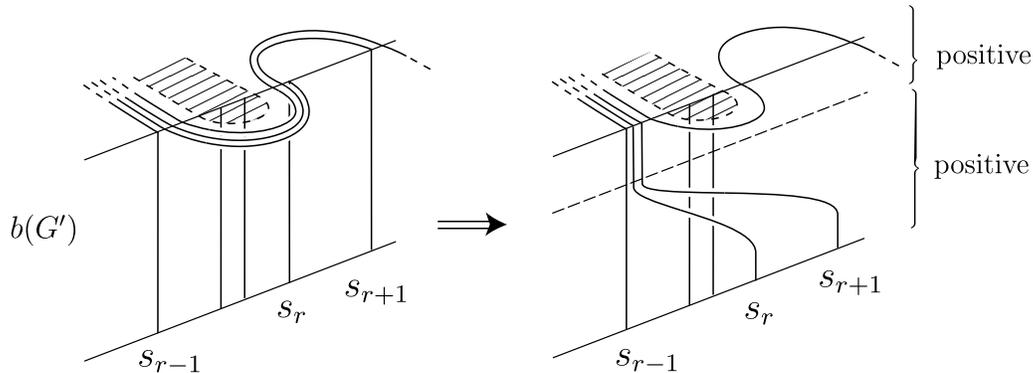


FIGURE 13

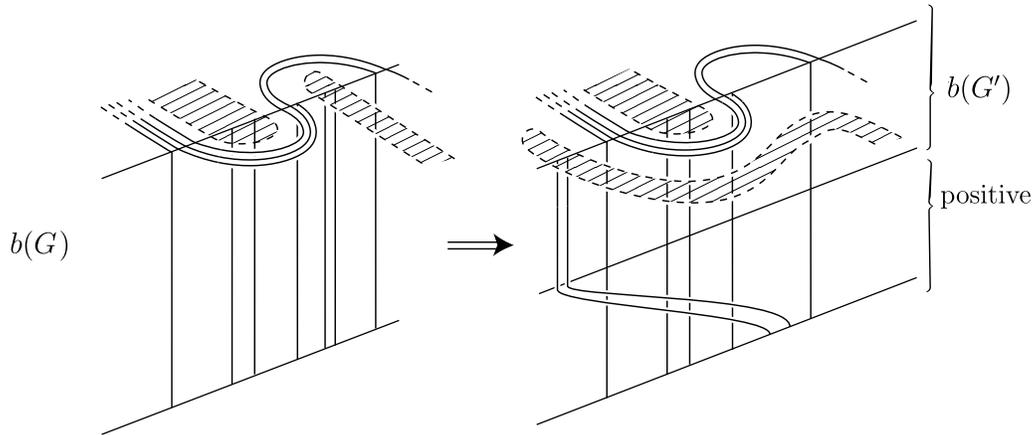


FIGURE 14

5. ACKNOWLEDGMENTS

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