VOLUMES OF FIBERED 2-FOLD BRANCHED COVERS OF 3-MANIFOLDS

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Abstract. We prove that for any closed, connected, oriented 3-manifold M, there exists an infinite family of 2-fold branched covers of M that are hyperbolic 3-manifolds and surface bundles over the circle with arbitrarily large volume.

1. Introduction

Sakuma [17] proved that every closed, connected, oriented 3-manifold M with a Heegaard splitting of genus g admits a 2-fold branched cover of M that is a genus g surface bundle over the circle S^1. See also Koda-Sakuma [11, Theorem 9.1]. Brooks [3] showed that the 2-fold branched cover of M in Sakuma's theorem can be chosen to be hyperbolic if $g \geq \max(2, g(M))$, where $g(M)$ is the Heegaard genus of M.

Montesinos gave different proofs of results by Sakuma and Brooks by using open book decompositions of M. To state his theorem, let $\Sigma = \Sigma_{g,m}$ be a compact, connected, oriented surface of genus g with m boundary components, and let $\Sigma_g = \Sigma_{g,0}$. The mapping class group $\mathrm{MCG}(\Sigma)$ is the group of isotopy classes of orientation-preserving self-homeomorphisms on Σ. By the Nielsen-Thurston classification, elements in $\mathrm{MCG}(\Sigma)$ fall into three types: periodic, reducible, pseudo-Anosov [21]. For $f \in \mathrm{MCG}(\Sigma)$, we consider the mapping torus

$$T_f = \Sigma \times [-1,1]/_{(x,1)\sim(f(x),-1)}.$$

We call Σ the fiber of T_f. The 3-manifold T_f is a Σ-bundle over the circle S^1 with the monodromy f. It is known by Thurston [22] that T_f admits a hyperbolic structure of finite volume if and only if f is pseudo-Anosov. The following result is a starting point of our paper.

Theorem 1 (Montesinos [15]). *Let M be a closed, connected, oriented 3-manifold containing a hyperbolic fibered knot of genus $g_0 \geq 1$. Then there exists a 2-fold branched cover of M branched over a 2-component link that is a hyperbolic 3-manifold and a Σ_{2g_0}-bundle over S^1.*

In this paper, building on the approach of Montesinos, we prove the following result. Here $\mathrm{vol}(W)$ denotes the volume of a hyperbolic 3-manifold W.

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Theorem 2. *Let M be a closed, connected, oriented 3-manifold containing a hyperbolic fibered knot of genus $g_0 \geq 2$. Then for any $g \geq g_0$ and $j \in \{1, 2\}$, there exists an infinite family $\{N_\ell\}_{\ell \in \mathbb{N}}$ of hyperbolic 3-manifolds such that*

(a) *N_ℓ is a Σ_{2g+j-1}-surface bundle over S^1,*
(b) *N_ℓ is a 2-fold branched cover of M branched over a $2j$-component link, and*
(c) *the inequalities*
$$\frac{1}{2}\, g < \mathrm{vol}(N_\ell) < \mathrm{vol}(N_{\ell+1}) \quad \text{for } \ell \in \mathbb{N}$$
hold.

By Soma [18], every closed oriented, connected 3-manifold M contains a hyperbolic fibered knot of genus g_0 for some $g_0 \geq 1$. Equivalently there exists an open book decomposition $(\Sigma_{g_0,1}, h)$ of M, where the monodromy h is isotopic to a pseudo-Anosov homeomorphism. By stabilizing open book decompositions along suitable arcs, one may assume that M contains a hyperbolic fibered knot of genus g for some $g \geq 2$, see Colin-Honda [4], Detcherry-Kalfagianni [5] for example. Hence Theorem 2 applies to all 3-manifolds M.

Let $\mathcal{D}_g(M)$ be the subset of $\mathrm{MCG}(\Sigma_g)$ on the closed surface of genus g consisting of elements f such that its mapping torus T_f is homeomorphic to a 2-fold branched cover of M branched over a link. The above result by Sakuma tells us that $\mathcal{D}_g(M) \neq \emptyset$ if $g \geq g(M)$, and there exist infinitely many pseudo-Anosov elements in $\mathcal{D}_g(M)$ if $g \geq \max(2, g(M))$, see [3]. For a study of stretch factors of pseudo-Anosov elements of $\mathcal{D}_g(M)$, see [10]. As an immediate corollary of Theorem 2, we have following.

Corollary 3. *Let M be a closed, connected, oriented 3-manifold containing a hyperbolic fibered knot of genus $g_0 \geq 2$. Then there exists an infinite family $\{\phi_g\}_{g=1}^{\infty}$ of pseudo-Anosov elements $\phi_g \in \mathcal{D}_{2g_0+g-1}(M)$ such that the volume $\mathrm{vol}(T_{\phi_g})$ of the mapping torus of ϕ_g goes to ∞ as $g \to \infty$.*

We ask the following question.

Question 4. *For g sufficiently large, does the set $\mathcal{D}_g(M)$ contain an infinite family of pseudo-Anosov elements whose mapping tori have arbitrarily large volume?*

Note that Futer, Purcell and Schleimer [8] give a positive answer to Question 4 when $M = S^2 \times S^1$. Using the results of [8], in Corollary 11, we also obtain a positive answer to Question 4 when $M = S^3$ and g is even.

2. Fathi's theorem and volume variation

This section is devoted to the proof of a result which is a generalization of a theorem by Fathi [6]. Given a surface $\Sigma = \Sigma_{g,m}$ of genus g with m boundary components, let $\mathrm{MCG}(\Sigma)$ be the group of isotopy classes of orientation preserving self-homeomorphisms of Σ. In this section, we do not require that the maps and isotopies fix the boundary $\partial\Sigma$ of Σ pointwise. In Sections 3 and 4, we restrict our attention to the open book decompositions (Σ, h) of closed 3-manifolds, where the monodromy $h : \Sigma \to \Sigma$ preserves $\partial\Sigma$ pointwise. By abuse of notations, we denote a representative of a mapping class $f \in \mathrm{MCG}(\Sigma)$ by the same notation f.

A simple closed curve γ in Σ is *essential* if it is not homotopic to a point or a boundary component. For simplicity, we may not distinguish between a simple closed curve γ and its isotopy class $[\gamma]$. Let τ_γ denote the positive (i.e. right-handed) Dehn twist about γ.

Let $\gamma_1, \dots, \gamma_k$ be essential simple closed curves in Σ. We say that the set $\{\gamma_1, \dots, \gamma_k\}$ *fills* Σ if for each essential simple closed curve γ' in Σ, there exists some $j \in \{1, \dots, k\}$ such that $i_\Sigma(\gamma', \gamma_j) > 0$, where $i_\Sigma(\cdot, \cdot)$ is the geometric intersection number on Σ. In this case, we also say that $\gamma_1, \dots, \gamma_k$ *fill* Σ.

Given $f \in \mathrm{MCG}(\Sigma)$, we call $O_f(\gamma) = \{f^\ell(\gamma) \mid \ell \in \mathbb{Z}\}$ the *orbit of γ under f*. Strictly speaking, this is the set of isotopy classes $[f^\ell(\gamma)]$ of simple closed curves $f^\ell(\gamma)$. We say that orbits of $\gamma_1, \dots, \gamma_k$ under f are *distinct* if $O_f(\gamma_i) \neq O_f(\gamma_j)$ for any $i, j \in \{1, \dots, k\}$ with $i \neq j$. Notice that $O_f(\gamma_i) = O_f(\gamma_j)$ if and only if there exists an integer $\ell \in \mathbb{Z}$ such that $f^\ell(\gamma_i) = \gamma_j$. We say that the orbits of $\gamma_1, \dots, \gamma_k$ under f *fill* Σ if there exists an integer $n > 0$ such that the set $\{f^\ell(\gamma_j) \mid j \in \{1, \dots, k\},\ \ell \in \{0, \pm 1, \dots, \pm n\}\}$ fills Σ.

Suppose that $\partial\Sigma \neq \emptyset$. A properly embedded arc α in Σ is *essential* if it is not parallel to $\partial\Sigma$. As in the case of simple closed curves, we do not distinguish between an arc α and its isotopy class $[\alpha]$. We allow that endpoints of the arcs are free to move around $\partial\Sigma$, and an arc α' that is isotopic to α may have the different endpoints from the ones of α. Given $f \in \mathrm{MCG}(\Sigma)$, we call $O_f(\alpha) = \{f^\ell(\alpha) \mid \ell \in \mathbb{Z}\}$ the *orbit of α under f*.

Let $\alpha_1, \dots, \alpha_k$ be essential arcs. We say that the orbits of $\alpha_1, \dots, \alpha_k$ under f are *distinct* if $O_f(\alpha_i) \neq O_f(\alpha_j)$ for any $i, j \in \{1, \dots, k\}$ with $i \neq j$.

Theorem 5. *Let $\gamma_1, \dots, \gamma_k$ be essential simple closed curves in $\Sigma = \Sigma_{g,m}$, where $k \geq 1$ and $3g - 3 + m > 0$ (possibly $m = 0$). For any mapping class $f \in \mathrm{MCG}(\Sigma)$, suppose that the orbits of $\gamma_1, \dots, \gamma_k$ under f are distinct and fill Σ. (i.e. $O_f(\gamma_i) \neq O_f(\gamma_j)$ for any $i, j \in \{1, \dots, k\}$ with $i \neq j$, and the orbits of $\gamma_1, \dots, \gamma_k$ under f fill Σ.) Then there exists $n \in \mathbb{N}$ which satisfies the following.*

(a) *For any $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k$ with $|n_i| \geq n$ for $i = 1, \dots, k$, the mapping class*
$$f_{\boldsymbol{n}} = \tau_{\gamma_k}^{n_k} \dots \tau_{\gamma_1}^{n_1} f \in \mathrm{MCG}(\Sigma)$$
is pseudo-Anosov.

(b) *There exists a sequence $\{\boldsymbol{n}_\ell\}_{\ell \in \mathbb{N}}$ of the k-tuple of integers $\boldsymbol{n}_\ell = (n_{\ell_1}, \dots, n_{\ell_k}) \in \mathbb{Z}^k$ with $|n_{\ell_i}| \geq n$ for $i = 1, \dots, k$ such that the mapping tori $T_{f_{\boldsymbol{n}_\ell}}$ of $f_{\boldsymbol{n}_\ell} = \tau_{\gamma_k}^{n_{\ell_k}} \dots \tau_{\gamma_1}^{n_{\ell_1}} f$ are hyperbolic 3-manifolds with strictly increasing volumes:*
$$\frac{1}{2}\, k < \mathrm{vol}(T_{f_{\boldsymbol{n}_\ell}}) < \mathrm{vol}(T_{f_{\boldsymbol{n}_{\ell+1}}}) \quad \textit{for} \quad \ell \in \mathbb{N}.$$

In the case of $\partial\Sigma = \emptyset$, Theorem 5(a) is due to Fathi [6, Theorem 0.2]. The result by Fathi is a generalization of a theorem by Long-Morton [12]. The argument we give below follows the line of the proof in [12]. In our setting, the mapping tori of the pseudo-Anosovs obtained in Theorem 5(a) are given by the Dehn filling along hyperbolic 3-manifolds. This allows us to use results on volume variation under the Dehn filling to prove Theorem 5(b).

Proof of Theorem 5. We take numbers $0 < t_1 < t_2 < \cdots < t_k < 1$. Let $\delta_i = \gamma_i \times \{t_i\}$ be a curve lying on the fiber $F_i = \Sigma \times \{t_i\}$ for $i = 1, \ldots, k$ of the mapping torus T_f. Now $L_k = \delta_1 \cup \cdots \cup \delta_k$ is a link in T_f.

Claim 1. Let $\mathcal{N}(L_k)$ denote a regular neighborhood of the link L_k. Then the 3-manifold $N = \overline{T_f \setminus \mathcal{N}(L_k)}$ is hyperbolic.

Proof of Claim 1. Since $\gamma_1, \ldots, \gamma_k$ are essential simple closed curves in Σ, the 3-manifold N is irreducible and boundary irreducible.

We first show that N is atoroidal, i.e. N contains no essential embedded tori. Assume that there exists a torus T embedded in N that is incompressible and not peripheral. Since the fundamental group of a thickened surface $\Sigma \times I$, where I is an interval, does not contain free abelian subgroups of rank 2, the torus T must intersect some of the fibers F_i of T_f, where the curves δ_i lie.

Without loss of generality, we may suppose that T intersects the fiber F_1, where $\delta_1 = \gamma_1 \times \{t_1\}$ lies. We identify Σ with the t_1-level $F_1 = \Sigma \times \{t_1\}$ in T_f. Let W_Σ denote the manifold obtained by cutting T_f open along Σ. Since we assumed that $0 < t_1 < t_2 < \cdots < t_k < 1$, in the beginning of the proof, and Σ is identified with the t_1-level $F_1 = \Sigma \times \{t_1\}$, the level surface $\Sigma \times \{1\} = \Sigma \times \{-1\}$ (as a set) is disjoint from F_1. We can view W_Σ as the identification space

$$W_\Sigma = (\Sigma \times [t_1, 1] \coprod \Sigma \times [-1, t_1])/_{(x,1)\sim(f(x),-1)}, \tag{2.1}$$

and T_f is obtained from W_Σ by identifying the two copies of $\Sigma \times \{t_1\}$ in ∂W_Σ by the identity map.

By using the irreducibility of T_f and the incompressibility of Σ, we may isotope the torus T so that all components of $T \setminus \Sigma$ are annuli, and each component A of $T \setminus \Sigma$ is either vertical with respect to the I-product (i.e. A runs around the S^1 factor of T_f), or there exists an annulus $\widehat{A}$ in one copy of $\Sigma \subset \partial W_\Sigma$ such that $A \cup \widehat{A}$ bounds a solid torus in W_Σ and ∂A lies in the same copy of Σ, where the annulus $\widehat{A}$ sits. The former and latter annuli are called the *vertical* and *horizontal* annuli respectively. There are two types (A1), (A2) for a horizontal annulus A.

- (A1) There exist no curves δ_i which is contained in the solid torus bounded by $A \cup \widehat{A}$.
- (A2) There exists a curve δ_i which is contained in the solid torus bounded by $A \cup \widehat{A}$.

If A is a horizontal annulus of type (A2), then the curve δ_i in the condition of (A2) is unique: If δ_i and δ_j $(i \neq j)$ are contained in the solid torus bounded by $A \cup \widehat{A}$, then δ_i and δ_j are isotopic in W_Σ, which implies that $O_f(\gamma_i) = O_f(\gamma_j)$. This contradicts the assumption that the orbits of $\gamma_1, \ldots, \gamma_k$ under f are distinct. Hence the curve δ_i in (A2) is unique. In particular ∂A consists of two curves which are parallel to $\gamma_i \times \{t_1\}$ since Σ is identified with the t_1-level $F_1 = \Sigma \times \{t_1\}$.

Notice that each horizontal annulus of type (A1) can be removed by an isotopy of the torus T, and hence we may suppose that each component of $T \setminus \Sigma$ is a vertical annulus or a horizontal annulus of type (A2).

If there exists a horizontal annulus of type (A2), then by replacing the fiber F_i (containing the curve δ_i) with F_1 if necessary, we have a horizontal annulus A_1 of $T \setminus \Sigma$ whose components of ∂A_1 are parallel to $\delta_1 = \gamma_1 \times \{t_1\}$.

Suppose that there exist no vertical annuli of $T \setminus \Sigma$. Then a horizontal annulus A_1 can only connect to a horizontal annulus A_2 with ∂A_2 running parallel to $f^{\pm 1}(\gamma_1) \times \{t_1\}$. But then T will be boundary parallel (peripheral) in N, contrary to our assumption.

From the above discussion, we may suppose that $T \setminus \Sigma$ contains a vertical annulus A. Let P denote the component of ∂A on one copy of $\Sigma \times \{t_1\}$ on ∂W_Σ. By the construction of W_Σ in (2.1), the boundary ∂A is disjoint from the level surface in W_Σ resulting from the identification of $\Sigma \times \{1\}$ to $\Sigma \times \{-1\}$ via $(x, 1) \sim (f(x), -1)$. The intersection of the annulus A with the later level surface is the curve resulting from $P \times \{1\} \sim f(P) \times \{-1\}$ under above identification of $\Sigma \times \{1\}$ to $\Sigma \times \{-1\}$. Thus the component of ∂A on the second copy of $\Sigma \times \{t_1\} \subset \partial W_\Sigma$ is $f(P)$. That is A runs from P to $f(P)$ in W_Σ.

We have two cases.

(1) $T \setminus \Sigma$ contains a horizontal annulus A_1, or
(2) all of the components of $T \setminus \Sigma$ are vertical annuli.

We first consider the case (1). As discussed earlier, without loss of generality, we may assume that ∂A_1 is formed by two curves parallel to $\gamma_1 \times \{t_1\}$.

Since the case that the horizontal annulus A_1 connects to another horizontal annulus was excluded earlier, we may now assume that A_1 connects a vertical annulus A. Now this vertical annulus A eventually connects to another horizontal annulus A' of type (A2) such that the solid torus bounded by $A' \cup \widehat{A'}$ contains a curve δ_j for some $j \in \{1, \dots, k\}$.

Assume that $j = 1$. Then the torus T must have a self-intersection in N, and this is a contradiction.

Next, we assume that $j \in \{2, \dots, k\}$. Then $\partial A'$ is formed by two curves parallel to $\gamma_j \times \{t_1\}$. Recall that the curves $P \times \{t_1\}$ and $f(P) \times \{t_1\}$, viewed on different copies of $\Sigma \times \{t_1\} \subset \partial W_\Sigma$, form the boundary of the vertical annulus A. This implies that on $\Sigma := \Sigma \times \{t_1\}$ we have $P, f(P) \in O_f(\gamma_j) \cap O_f(\gamma_1) \neq \emptyset$. This contradicts the assumption that the orbits of $\gamma_1, \dots, \gamma_k$ under f are distinct.

We turn to the case (2). To form the torus T from vertical annuli, we have $f^m(P) = P$ for some $m > 0$. Arguing as above, we conclude that the curves $P, f(P), \dots, f^{m-1}(P)$ on $\Sigma \times \{t_1\} \subset T_f$ lie on the torus T. Since T is embedded in N and all of the components of $T \setminus \Sigma$ are vertical annuli, we have $\gamma_i \cap f^j(P) = \emptyset$ for any $i = 1, \dots, k$ and $j = 1, \dots, m$. Equivalently we have $f^{-j}(\gamma_i) \cap P = \emptyset$ for any $i = 1, \dots, k$ and $j = 1, \dots, m$. For any $n \in \mathbb{Z}$, write $n = m\ell - j$ for some $\ell \in \mathbb{Z}$ and some $j = 1, \dots, m$. For any $i = 1, \dots, k$, we obtain

$$f^{m\ell}(f^{-j}(\gamma_i) \cap P) = f^{m\ell}(f^{-j}(\gamma_i)) \cap f^{m\ell}(P) = f^{m\ell - j}(\gamma_i) \cap P = f^n(\gamma_i) \cap P = \emptyset.$$

Thus for any $i = 1, \dots, k$, the curve P must be disjoint from $O_f(\gamma_i)$. However this contradicts our assumption that the orbits of $\gamma_1, \dots, \gamma_k$ under f fill Σ. This implies that N is atoroidal.

To finish the proof of Claim 1, it is enough to show that N contains no essential annuli. Suppose that there exists an essential annulus A in N. Then N must be a

Seifert manifold (see [9, Lemma 1.16]), and the components of ∂N consist of fibers of the Seifert fibration of N. In N, we can find a copy of the fiber Σ of T_f, say S, that is disjoint from the components $T_1, \dots, T_k$ of ∂N that are created by drilling out the curves $\delta_1, \dots, \delta_k$. Then S is a surface that is essential in the Seifert manifold N with non-empty boundary. Since we assumed that $3g-3+m>0$, S is not a torus or an annulus. Thus up to isotopy, we can make S horizontal which means that S must intersect all the fibers of the Seifert fibration of N transversely, see [9, Proposition 1.11]. Since S is disjoint from the components $T_1, \dots, T_k$ of ∂N, it cannot become horizontal. This contradiction implies that N contains no essential annuli. Thus by work of Thurston [20], the manifold N is hyperbolic. This completes the proof of Claim 1.

We now prove the claim (a). We denote by $N_{\boldsymbol{n}}$, the mapping torus $T_{f_{\boldsymbol{n}}}$ of $f_{\boldsymbol{n}} = \tau_{\gamma_k}^{n_k} \dots \tau_{\gamma_1}^{n_1} f$ for $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k$. We use the fact that $N_{\boldsymbol{n}}$ is obtained from N by the Dehn filling, where the boundary component $T_i \subset \partial N$ corresponding to δ_i is filled. Given $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k$, let s_i denote the Dehn filling slope on $T_i \subset \partial N$ to obtain $N_{\boldsymbol{n}}$ for $i = 1, \dots, k$. Since N is hyperbolic, each torus boundary component of N corresponds to a cusp of $T_f \setminus L_k$. Taking a maximal disjoint horoball neighborhood about the cusps, each torus T_i inherits a Euclidean structure, well-defined up to similarity. The slope s_i can then be given a geodesic representative. We define the length of s_i, denoted by $\ell(s_i)$, to be the length of this geodesic representative. (Note that when $k>1$, this definition of slope length depends on the choice of maximal horoball neighborhood. See [16].)

The length $\ell(s_i)$ of the slope s_i is an increasing function of $|n_i|$. Let $\lambda > 0$ denote the minimum length of the slopes, that is

$$\lambda = \min\{\ell(s_i) \mid i = 1, \dots, k\}.$$

By Thurston's hyperbolic Dehn surgery theorem [19], there exists $n \in \mathbb{N}$ such that for all $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k$ with $|n_i| > n$ for $i = 1, \dots, k$, the resulting manifold $N_{\boldsymbol{n}} (= T_{f_{\boldsymbol{n}}})$ obtained by filling N is hyperbolic, and hence $f_{\boldsymbol{n}}$ is pseudo-Anosov. Thus the claim (a) holds.

We turn to the claim (b). As $|n_i| \to \infty$ for all $i = 1, \dots, k$, the volumes of the filled manifolds $N_{\boldsymbol{n}}$'s approach the volume of the 3-manifold $T_f \setminus L_k$ from bellow. To make things more concrete, we use an effective form proved in [7, Theorem 1.1], which states that if $\lambda > 2\pi$, then $N_{\boldsymbol{n}}$ is hyperbolic and we have

$$\left(1 - \left(\frac{2\pi}{\lambda}\right)^2\right)^{3/2} \operatorname{vol}(T_f \setminus L_k) \leqslant \operatorname{vol}(N_{\boldsymbol{n}}) < \operatorname{vol}(T_f \setminus L_k). \tag{2.2}$$

Since $T_f \setminus L_k$ is a hyperbolic 3-manifold with at least k cusps, we have

$$k\ v_3 < \operatorname{vol}(T_f \setminus L_k),$$

where $v_3 = 1.01494\dots$ is the volume of the ideal regular tetrahedron, see [1, Theorem 7].

On the other hand, by taking $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k$ with all $|n_i|$ sufficiently larger than n, we can assure that

$$\frac{1}{2} < \left(1 - \left(\frac{2\pi}{\lambda}\right)^2\right)^{3/2}.$$

By (2.2), we obtain

$$\frac{1}{2}\, k < \frac{1}{2}\, kv_3 < \frac{1}{2}\, \mathrm{vol}(T_f \setminus L_k) < \mathrm{vol}(N_{\boldsymbol{n}}). \tag{2.3}$$

We set $\boldsymbol{n}_1 = \boldsymbol{n}$ with the above inequality $\frac{1}{2}\, k < \mathrm{vol}(N_{\boldsymbol{n}_1})$. Suppose that there exists a finite sequence $\{\boldsymbol{n}_\ell\}_{\ell=1}^m$ of the k-tuple of integers $\boldsymbol{n}_\ell \in \mathbb{Z}^k$ such that

$$\frac{1}{2}\, k < \mathrm{vol}(N_{\boldsymbol{n}_1}) < \cdots < \mathrm{vol}(N_{\boldsymbol{n}_m}) < \mathrm{vol}(T_f \setminus L_k).$$

Now we choose $\boldsymbol{n}_{m+1} = (n'_1, \dots, n'_k) \in \mathbb{Z}^k$ with all $|n'_i|$ sufficiently larger than n so that if we let $\lambda = \lambda_{\boldsymbol{n}_{m+1}}$ be the minimal length of the slopes corresponding to $\boldsymbol{n}_{m+1} \in \mathbb{Z}^k$, then we have

$$\lambda > \frac{2\pi}{\sqrt{1 - x_m^{2/3}}} > 2\pi, \quad \text{where} \quad x_m = \frac{\mathrm{vol}(N_{\boldsymbol{n}_m})}{\mathrm{vol}(T_f \setminus L_k)}.$$

Hence $\left(\frac{2\pi}{\lambda}\right)^2 < 1 - x_m^{2/3}$. This tells us that

$$\frac{\mathrm{vol}(N_{\boldsymbol{n}_m})}{\mathrm{vol}(T_f \setminus L_k)} = x_m < \left(1 - \left(\frac{2\pi}{\lambda}\right)^2\right)^{3/2}.$$

Thus

$$\mathrm{vol}(N_{\boldsymbol{n}_m}) < \left(1 - \left(\frac{2\pi}{\lambda}\right)^2\right)^{3/2} \mathrm{vol}(T_f \setminus L_k).$$

By (2.2), we have

$$\left(1 - \left(\frac{2\pi}{\lambda}\right)^2\right)^{3/2} \mathrm{vol}(T_f \setminus L_k) \le \mathrm{vol}(N_{\boldsymbol{n}_{m+1}}).$$

Putting them together, we obtain

$$\mathrm{vol}(N_{\boldsymbol{n}_m}) < \left(1 - \left(\frac{2\pi}{\lambda}\right)^2\right)^{3/2} \mathrm{vol}(T_f \setminus L_k) \le \mathrm{vol}(N_{\boldsymbol{n}_{m+1}}),$$

and the conclusion follows inductively. This completes the proof of Theorem 5. □

3. Curves on surfaces and open book decompositions

In this section, we quickly review curve graphs and open book decompositions of 3-manifolds. We prove a lemma that is needed for the proof of Theorem 2.

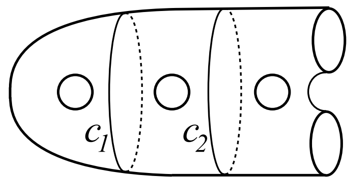

Figure 1. Case $(g, m) = (3.2)$. An example of simple closed curves $c_1, \dots, c_{g-1}$ in $\Sigma_{g,m}$.

3.1. **Curves on surfaces.** Suppose that $g \geq 2$. The *curve graph* $\mathcal{C}(\Sigma)$ for $\Sigma = \Sigma_{g,m}$ is defined as follows. The set of vertices $\mathcal{C}_0(\Sigma)$ is the set of isotopy classes of essential simple closed curves. Two vertices in $\mathcal{C}_0(\Sigma)$ are connected by an edge if they can be represented by disjoint essential simple closed curves. The space $\mathcal{C}(\Sigma)$ is a geodesic metric space with the path metric $d(\cdot, \cdot)$ that assigns length 1 to each edge of the graph. The mapping class group $\mathrm{MCG}(\Sigma)$ acts on $\mathcal{C}(\Sigma)$ as isometries.

Lemma 6. *Let $f \in \mathrm{MCG}(\Sigma)$ be a pseudo-Anosov mapping class defined on $\Sigma = \Sigma_{g,m}$, where $g \geq 2$. Then for any $1 \leq k \leq g$, there exist mutually disjoint, essential simple closed curves $\gamma_1, \dots, \gamma_k$ in Σ such that*

(a) *the orbits of $\gamma_1, \dots, \gamma_k$ under f are distinct and fill Σ, and*
(b) *the surface $\Sigma \setminus \{\gamma_1, \dots, \gamma_k\}$ obtained from Σ cutting along $\gamma_1 \cup \cdots \cup \gamma_k$ is connected.*

Remark 7. *The curve graph $\mathcal{C}(\Sigma)$ is locally infinite, i.e. for each vertex $v \in \mathcal{C}_0(\Sigma)$, there exist infinitely many vertices of $\mathcal{C}_0(\Sigma)$ that are at distance 1 from v. It is not hard to see that if $d(a, b) \geq 3$ for $a, b \in \mathcal{C}_0(\Sigma)$, then $\{a, b\}$ fills Σ. Furthermore, if $f \in \mathrm{MCG}(\Sigma)$ is pseudo-Anosov, then the distance $d(a, f^n(a))$ grows linearly with $|n|$ for any $a \in \mathcal{C}_0(\Sigma)$, see* [13, Proposition 4.6]. *We define the ball*

$$B_1(a) = \{b \in \mathcal{C}_0(\Sigma) \mid d(a, b) \leq 1\}.$$

Since $d(a, f^n(a)) \to \infty$ as $|n| \to \infty$, a single orbit $O_f(a)$ of $a \in \mathcal{C}_0(\Sigma)$ under f fills Σ. Take any $b \in \mathcal{C}_0(\Sigma)$. Then the cardinality of the set $B_1(a) \cap O_f(b)$ is finite, since $d(b, f^n(b)) \to \infty$ as $|n| \to \infty$. Moreover $B_1(a) \setminus O_f(b)$ is an infinite set, since $\mathcal{C}(\Sigma)$ is locally infinite. Hence one can pick an element of $B_1(a) \setminus O_f(b)$ at distance 1 from a.

Proof of Lemma 6. We first take mutually disjoint, essential simple closed curves $c_1, \dots, c_{g-1}$ in Σ so that the surface obtained from Σ by cutting along $c_1 \cup \cdots \cup c_{g-1}$ has g connected components $\Sigma^{(1)}, \dots, \Sigma^{(g)}$, each of which is a surface of genus 1 with nonempty boundary. See Figure 1.

For each $1 \leq k \leq g$, there exists an infinite family $\{a_i^{(k)}\}_{i \in \mathbb{N}}$ of $\mathcal{C}_0(\Sigma)$ such that $a_i^{(k)} \neq a_j^{(k)} \in \mathcal{C}_0(\Sigma)$ if $i \neq j$ and $a_i^{(k)}$ is represented by a non-separating simple closed curve in the surface $\Sigma^{(k)}$. Then

$$d(a_i^{(k)}, a_j^{(\ell)}) = 1 \text{ if } k \neq \ell \text{ and } i, j \in \mathbb{N}. \tag{3.1}$$

In the family $\{a_i^{(1)}\}_{i\in\mathbb{N}}$, take any $a_{i_1}^{(1)} = [\gamma_1]$. Then the orbit of γ_1 under f fills Σ by Remark 7. The statement of the lemma holds in the case $k = 1$.

We turn to the case $k = 2$. By (3.1), we have $\{a_i^{(2)}\}_{i\in\mathbb{N}} \subset B_1([\gamma_1]) = B_1(a_{i_1}^{(1)})$. By Remark 7, one sees that $\{a_i^{(2)}\}_{i\in\mathbb{N}} \cap O_f(\gamma_1)$ $\Big(\subset B_1([\gamma_1]) \cap O_f(\gamma_1)\Big)$ is finite. Hence one can pick an element $a_{i_2}^{(2)} = [\gamma_2] \in \{a_i^{(2)}\}_{i\in\mathbb{N}} \setminus O_f(\gamma_1)$. Then the orbits of γ_1 and γ_2 under f are distinct by the choice of γ_2. By (3.1), two curves γ_1 and γ_2 are disjoint. Moreover the orbits of γ_1 and γ_2 under f fill Σ, since a single orbit of γ_1 under f fills Σ. Since γ_1 (resp. γ_2) is non-separating in the surface $\Sigma^{(1)}$ (resp. $\Sigma^{(2)}$), one sees that $\Sigma \setminus \{\gamma_1, \gamma_2\}$ is connected. Thus the statement of the lemma holds in the case $k = 2$.

Similarly for $3 \le k \le g$, one can find the vertices $a_{i_3}^{(3)} = [\gamma_3], \dots, a_{i_k}^{(k)} = [\gamma_k]$ such that the orbits of $\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_k$ under f are distinct and fill Σ. By (3.1), $\gamma_1, \gamma_2, \gamma_3 \dots, \gamma_k$ are mutually disjoint. Each γ_i is non-separating in the surface $\Sigma^{(i)}$ for $i = 1, \dots, k$, and this implies that $\Sigma \setminus \{\gamma_1, \dots, \gamma_k\}$ is connected. This completes the proof. □

3.2. **Open book decompositions of closed 3-manifolds.** An *open book decomposition* of M is a pair (K, θ), where K is a link in M and $\theta : M \setminus K \to S^1$ is a fibration whose fiber is an interior of a Seifert surface of K. We call K the *binding* of the open book decomposition. We also call K the *fibered link* in M. An open book decomposition of M is determined by the closure $\Sigma = \overline{\theta^{-1}(t)} \subset M$ of a fiber $\theta^{-1}(t)$ $(t \in S^1)$ of the fibration θ together with the monodromy $h : \Sigma \to \Sigma$ with $h|_{\partial\Sigma} = \mathrm{id}$. Conversely, each pair (Σ, h) with $h|_{\partial\Sigma} = \mathrm{id}$ gives rise to an open book decomposition of some 3-manifold M as the *relative mapping torus* of h, i.e. M is homeomorphic to the quotient of the mapping torus T_h of h under the identification $(y, t) \sim (y, t')$ for all $y \in \partial\Sigma$ and $t, t' \in [-1, 1]$. We also call such a pair (Σ, h) the open book decomposition of a 3-manifold.

By the proof of [4, Theorem 1.1] by Colin-Honda, the following result holds. See also Detcherry-Kalfagianni [5, Propositions 4.9, 4.10].

Theorem 8. *Let M be a closed, connected, oriented* 3*-manifold containing a hyperbolic fibered knot of genus $g_0 \ge 2$. Then for any $g \ge g_0$ and $j \in \{1, 2\}$, the manifold M admits an open book decomposition $(\Sigma_{g,j}, h_{g,j})$, where $\partial\Sigma_{g,j}$ has j components and $h_{g,j}$ is isotopic to a pseudo-Anosov homeomorphism.*

4. Proof of Theorem 2

Theorem 2 in Section 1 follows from the following result.

Theorem 9. *Let M be a closed, connected, oriented* 3*-manifold containing a hyperbolic fibered knot of genus $g_0 \ge 2$. Then for any $g \ge g_0$, $j \in \{1, 2\}$ and $2 \le k \le g$, there exists $n \in \mathbb{N}$ which satisfies the following. For any $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k$ with $|n_i| \ge n$ for $i = 1, \dots, k$, there exists a hyperbolic* 3*-manifold $N_{\boldsymbol{n}}$ such that*

(a) *$N_{\boldsymbol{n}}$ is a Σ_{2g+j-1}-surface bundle over S^1,*
(b) *$N_{\boldsymbol{n}}$ is a* 2*-fold branched cover of M branched over a $2j$-component link, and*

(c) *there exists a sequence* $\{\boldsymbol{n}_\ell\}_{\ell\in\mathbb{N}}$ *of the* k*-tuple of integers* $\boldsymbol{n}_\ell = (n_{\ell_1}, \dots, n_{\ell_k}) \in \mathbb{Z}^k$ *with* $|n_{\ell_i}| \geq n$ *for* $i = 1, \dots, k$ *such that*

$$\frac{1}{2}\ k < \mathrm{vol}(N_{\boldsymbol{n}_\ell}) < \mathrm{vol}(N_{\boldsymbol{n}_{\ell+1}}) \quad \textit{for} \quad \ell \in \mathbb{N}.$$

Proof. By Theorem 8, for any $g \geq g_0$ and $j \in \{1, 2\}$, there exists an open book decomposition $(\Sigma_{g,j}, h_{g,j})$ of M, where $h_{g,j}$ is isotopic to a pseudo-Anosov homeomorphism. We set $F_{g,j} = \Sigma_{g,j}$. Then by Lemma 6, we have mutually disjoint, essential simple closed curves $\gamma_1, \dots, \gamma_k$ in $F_{g,j}$ such that the orbits of $\gamma_1, \dots, \gamma_k$ under $h_{g,j}$ are distinct and fill $F_{g,j}$. Moreover $F_{g,j} \setminus \{\gamma_1, \dots, \gamma_k\}$ is connected.

Let $B = \partial F_{g,1}$ when $j = 1$, and let B and B' be the components of $\partial F_{g,2}$ when $j = 2$. When $j = 1$, let $\beta_1, \dots, \beta_k$ be properly embedded, mutually disjoint arcs in $F_{g,1} \setminus \{\gamma_1, \dots, \gamma_k\}$ so that one of the endpoints of each β_i lies on γ_i and the other endpoint of β_i lies on $B = \partial F_{g,1}$. Since $F_{g,1} \setminus \{\gamma_1, \dots, \gamma_k\}$ is connected, one can choose those arcs $\beta_1, \dots, \beta_k \subset F_{g,1} \setminus \{\gamma_1, \dots, \gamma_k\}$ so that they are mutually disjoint. When $j = 2$, let $\beta_1, \dots, \beta_k$ be properly embedded, mutually disjoint arcs in $F_{g,2} \setminus \{\gamma_1, \dots, \gamma_k\}$ so that one of the endpoints of each β_i lies on γ_i and the other endpoint of β_i lies on B (resp. B') if $i = 1, \dots, k-1$ (resp. $i = k$). In both cases $j = 1, 2$, consider a small neighborhood $\mathcal{N} = \mathcal{N}(\gamma_i \cup \beta_i)$ in $F_{g,j}$. We set α_i to be a component of $\partial\mathcal{N} \setminus \partial F_{g,j}$ which is not parallel to γ_i. See Figures 2(1), 3(1). Then $\alpha_1, \dots, \alpha_k$ are mutually disjoint, essential arcs in $F_{g,j}$.

We claim that the orbits of $\alpha_1, \dots, \alpha_k$ under $h_{g,j}$ are distinct. Assume that $O_{h_{g,j}}(\alpha_i) = O_{h_{g,j}}(\alpha_{i'})$ for some $i, i' \in \{1, \dots, k\}$ with $i \neq i'$. This implies that $O_{h_{g,j}}(\gamma_i) = O_{h_{g,j}}(\gamma_{i'})$, since γ_i is obtained from each α_i by concatenating with an arc of B or B'. This contradicts the choice of $\gamma_1, \dots, \gamma_k$.

Let us consider the closed surface $\Sigma_{2g+j-1} = DF_{g,j}$ of genus $2g + j - 1$ that is obtained as the double $DF_{g,j}$ of $F_{g,j}$ along $\partial F_{g,j}$. There exists an involution

$$\iota : \Sigma_{2g+j-1} \to \Sigma_{2g+j-1}$$

that interchanges the two copies of $F_{g,j}$ and $\iota|_{\partial F_{g,j}} = \mathrm{id}$ holds. (Notice that ι is orientation reversing.) For the above essential arc α_i, there is a corresponding arc $\iota(\alpha_i)$ on the second copy of $F_{g,j}$ so that $\widehat{\gamma}_i = \alpha_i \cup \iota(\alpha_i)$ becomes an essential simple closed curve in Σ_{2g+j-1}. Since $\alpha_1, \dots, \alpha_k$ are mutually disjoint, $\widehat{\gamma}_1, \dots, \widehat{\gamma}_k$ are mutually disjoint, essential simple closed curves in Σ_{2g+j-1}. See Figures 2(2), 3(2).

Let

$$\widehat{h}_{g,j} = h_{g,j} \# h_{g,j}^{-1} : \Sigma_{2g+j-1} \to \Sigma_{2g+j-1}$$

be a homeomorphim induced by $h_{g,j}$. More precisely, $\widehat{h}_{g,j}(x) = h_{g,j}(x)$ if x is in one copy of $F_{g,j}$ and $\widehat{h}_{g,j}(\iota(x)) = \iota(h_{g,j}^{-1}(x))$ if $\iota(x)$ is in the second copy of $F_{g,j}$.

Note that $\widehat{h}_{g,j}$ is a reducible homeomorphism, since $\widehat{h}_{g,j}$ preserves the essential simple closed curves $\partial F_{g,j} \subset \Sigma_{2g+j-1}$.

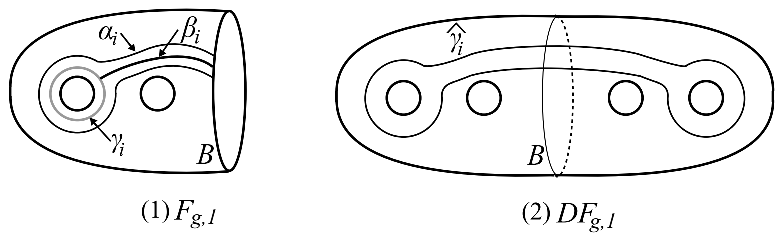

FIGURE 2. Case $(g, j) = (2, 1)$. (1) The arc α_i in $F_{g,1}(= \Sigma_{g,1})$. (2) The simple closed curve $\widehat{\gamma}_i$ in $\Sigma_{2g} = DF_{g,1}$.

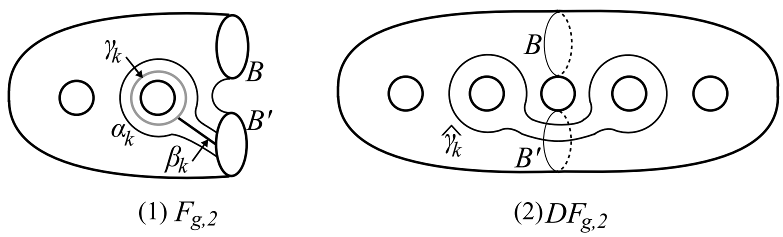

FIGURE 3. Case $(g, j) = (2, 2)$. (1) The arc α_k in $F_{g,2}(= \Sigma_{g,2})$. (2) The simple closed curve $\widehat{\gamma}_k$ in $\Sigma_{2g+1} = DF_{g,2}$.

Claim 1. For $j \in \{1, 2\}$, the orbits of $\widehat{\gamma}_1, \dots, \widehat{\gamma}_k$ under $\widehat{h}_{g,j}$ are distinct.

Proof of Claim 1. By the definition of $\widehat{h}_{g,j}$, we have $\widehat{h}_{g,j}|_{\partial F_{g,j}} = \mathrm{id}$, and the orbits of the arc $\alpha_i \subset \widehat{\gamma}_i$ under $\widehat{h}_{g,j}$ are contained in one copy of $F_{g,j}$ for $i = 1, \dots, k$. Claim 1 follows, since the orbits of $\alpha_1, \dots, \alpha_k$ under $h_{g,j}$ are distinct.

Claim 2. For $j \in \{1, 2\}$, the orbits of $\widehat{\gamma}_1, \dots, \widehat{\gamma}_g$ under $\widehat{h}_{g,j}$ fill Σ_{2g+j-1}.

Proof of Claim 2. We prove the claim when $j = 2$. (The proof for the case of $j = 1$ is similar.) By the proof of Lemma 6, a single orbit of γ_1 under $h_{g,2}$ fill $F_{g,2}$. This means that there exists an integer $n > 0$ such that each component of $F_{g,2} \setminus \{h^{\ell}_{g,2}(\gamma_1) \mid \ell \in \{0, \pm 1, \dots, \pm n\}\}$ is a disk or a once-holed disk. Let A (resp. A') be the annular component of $F_{g,2} \setminus \{h^{\ell}_{g,2}(\gamma_1) \mid \ell \in \{0, \pm 1, \dots, \pm n\}\}$ such that one of the boundary components of A (resp. A') coincides with $B \subset \partial F_{g,2}$ (resp. $B' \subset \partial F_{g,2}$). Then the annulus A is cut into disks by cutting $F_{g,2}$ along the arc α_1 (since $\partial \alpha_1$ lies on B). The other annulus A' is also cut into disks by cutting $F_{g,2}$ along the arc α_k (since $\partial \alpha_k$ lies on B'). Thus the surface obtained from the double $\Sigma_{2g+1} = DF_{g,2}$ by cutting along all $\widehat{h}^{\ell}_{g,2}(\widehat{\gamma}_1)$ and $\widehat{h}^{\ell}_{g,2}(\widehat{\gamma}_k)$ ($\ell \in \{0, \pm 1, \dots, \pm n\}$) is a disjoint union of disks. This means that the orbits of $\widehat{\gamma}_1, \widehat{\gamma}_k$ under $\widehat{h}_{g,2}$ fill Σ_{2g+1}. Thus the orbits of $\widehat{\gamma}_1, \dots, \widehat{\gamma}_k$ under $\widehat{h}_{g,2}$ fill Σ_{2g+1}. This completes the proof of Claim 2.

We build the mapping torus

$$T_{\widehat{h}_{g,j}} = \Sigma_{2g+j-1} \times [-1, 1]/_{(x,1) \sim (\widehat{h}_{g,j}(x), -1)}.$$

Claim 3. For $j \in \{1,2\}$, the mapping torus $T_{\widehat{h}_{g,j}}$ is a 2-fold branched cover of M branched over a $2j$-component link.

Proof of Claim 3. The statement of Claim 3 follows from the proof of [15, Lemma 1]. Here we prove the claim for completeness. Consider the involution $u : T_{\widehat{h}_{g,j}} \to T_{\widehat{h}_{g,j}}$ defined by

$$u(x,t) = (\iota(x), -t) \text{ for } (x,t) \in \Sigma_{2g+j-1} \times [-1,1],$$

where ι is the previous involution on Σ_{2g+j-1}. In the case $j = 1$, u fixes $2 = 2j$ curves $B \times \{1\} (= B \times \{-1\})$ and $B \times \{0\}$. In the case $j = 2$, u fixes $4 = 2j$ curves $B \times \{1\}$, $B \times \{0\}$ and $B' \times \{1\}$, $B' \times \{0\}$.

The quotient of $T_{\widehat{h}_{g,j}}$ by the action of u is the mapping torus $T_{h_{g,j}}$ of $h_{g,j}$ under the identification $(y,t) \sim (y,-t)$ for all $(y,t) \in \partial T_{h_{g,j}} = \partial F_{g,j} \times [-1,1]/_{(y,1)\sim(y,-1)}$. This is equivalent to identifying $\{y\} \times [-1,0]$ with $\{y\} \times [0,1]$ for all $y \in \partial F_{g,j}$. The resulting quotient is homeomorphic to the manifold obtained from $T_{h_{g,j}}$ by collapsing the set $\{y\} \times S^1$ to a point for all $y \in \partial F_{g,j}$. Thus the quotient of $T_{\widehat{h}_{g,j}}$ under u is the relative mapping torus of $h_{g,j}$ which is the open book decomposition $(F_{g,j}, h_{g,j})$ of M. In other words, the mapping torus $T_{\widehat{h}_{g,j}}$ of $\widehat{h}_{g,j}$ is a 2-fold branched cover of M branched cover the $2j$-component link that comes from the above $2j$ curves fixed by u.

This completes the proof of Claim 3.

By Claims 1 and 2, one can apply Theorem 5 to the orbits of $\widehat{\gamma}_1, \dots, \widehat{\gamma}_k$ under $\widehat{h}_{g,j}$. Then we have $n \in \mathbb{N}$ given by Theorem 5. For $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k$, we set

$$f_{\boldsymbol{n}} = \tau_{\widehat{\gamma}_k}^{n_k} \dots \tau_{\widehat{\gamma}_1}^{n_1} \widehat{h}_{g,j} \quad \text{and} \quad N_{\boldsymbol{n}} = T_{f_{\boldsymbol{n}}}.$$

By Theorem 5(a), if $|n_i| \geq n$ for $i = 1, \dots, k$, then $f_{\boldsymbol{n}}$ is pseudo-Anosov and $N_{\boldsymbol{n}}$ is a hyperbolic 3-manifold which is a Σ_{2g+j-1}-bundle over S^1. Thus $N_{\boldsymbol{n}}$ has a property of Theorem 9(a). By Theorem 5(b), $N_{\boldsymbol{n}}$ also has a property of Theorem 9(c).

Claim 4. For $j \in \{1,2\}$ and $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k$, the mapping torus $N_{\boldsymbol{n}}$ of $f_{\boldsymbol{n}} = \tau_{\widehat{\gamma}_k}^{n_k} \dots \tau_{\widehat{\gamma}_1}^{n_1} \widehat{h}_{g,j}$ is a 2-fold branched cover of M branched over a $2j$-component link.

Proof of Claim 4. We use Montesinos' trick [14, 15]. (cf. Auckly [2, Example 1].) See Figure 4, which illustrates Montesinos' trick. Since $\widehat{\gamma}_1, \dots, \widehat{\gamma}_k$ are mutually disjoint simple closed curves in Σ_{2g+j-1}, the curves $\delta_i^* = \widehat{\gamma}_i \times \{0\} \subset \Sigma_{2g+j-1} \times \{0\}$ for $i = 1, \dots, k$ are mutually disjoint. We consider the link $L_k^* = \delta_1^* \cup \dots \cup \delta_k^*$ in $T_{\widehat{h}_{g,j}}$. Then the 3-manifold $N_{\boldsymbol{n}}$ is obtained from the mapping torus $T_{\widehat{h}_{g,j}}$ of $\widehat{h}_{g,j}$ by the Dehn surgery along the link L_k^*.

Notice that each δ_i^* is invariant under the involution $u : T_{\widehat{h}_{g,j}} \to T_{\widehat{h}_{g,j}}$. Now we do Dehn surgery along L_k^*. For $i = 1, \dots, k$, we remove the interior of a neighborhood N_i of δ_i^*, and replace it with a new solid torus V_i. The involution $u_i := u|_{\partial N_i} : \partial N_i \to \partial N_i$ extends to an elliptic involution on the solid torus V_i added with Dehn filling. The effect of the Dehn surgery on the quotient by the elliptic involution on

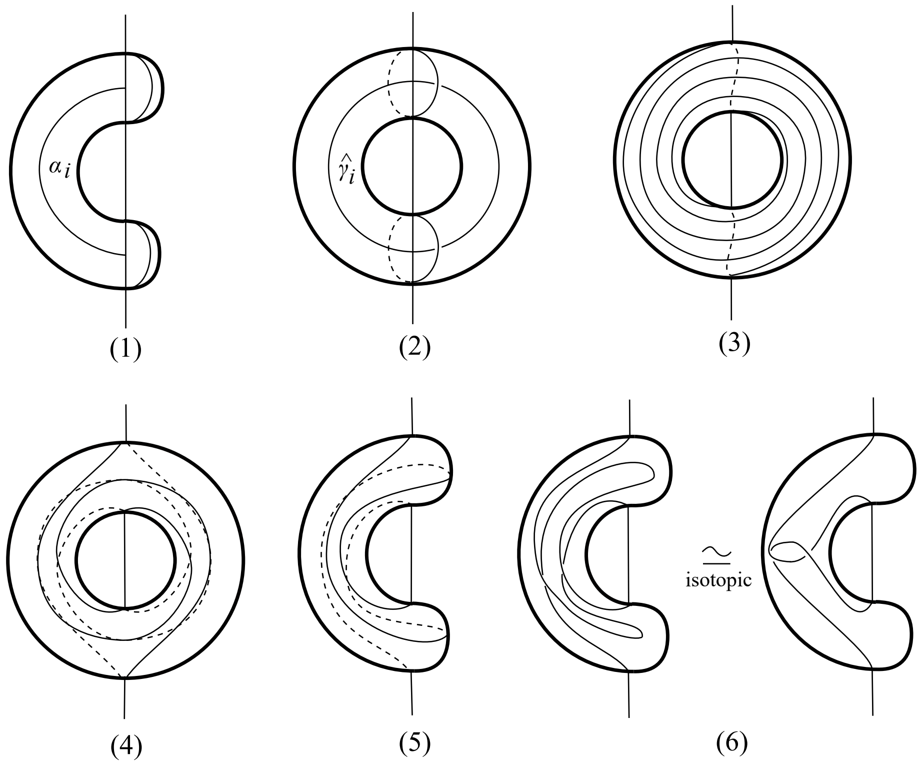

FIGURE 4. (1) The regular neighborhood of $\alpha_i \times \{0\}$ in M, which is a 3-ball. (2) The regular neighborhood of $\widehat{\gamma}_i \times \{0\}$ in $T_{\widehat{h}_{g,j}}$, which is a solid torus. (3) In order to obtain $T_{\tau^2_{\widehat{\gamma}_i}\widehat{h}_{g,j}}$, we remove the solid torus from $T_{\widehat{h}_{g,j}}$ and glue this solid torus again so that the boundary of its meridian is identified with one of the circles on the torus. (4) Isotope the circles to be invariant by the elliptic involution. (5) Take a quotient by the elliptic involution. (6) Push the arcs into the 3-ball, then we have the branched set of the new 2-fold branched cover of M after the Dehn surgery.

V_i is a modification of the 3-manifold M inside a collection of 3-balls that changes the branched set for the 2-fold branched cover $T_{\widehat{h}_{g,j}} \to M$, but not the ambient manifold M. Thus $N_{\boldsymbol{n}}$ is still a 2-fold branched covers of M branched over a link with $2j$ components. This completes the proof of Claim 3.

By Claim 4, the manifold $N_{\boldsymbol{n}}$ satisfies a property of Theorem 9(b), and we have finished the proof of Theorem 9. $\square$

5. Large volume vs. fixed genus

In this section we discuss conditions on 3-manifolds under which Question 4 has a positive answer.

Theorem 10. *Let M be a closed, connected, oriented 3-manifold containing a hyperbolic fibered knot of genus $g_0 \geq 2$. Suppose that for any $g \geq g_0$ and $j \in \{1, 2\}$, M contains a family $\{K_r^j(g)\}_{r \in \mathbb{N}}$ of hyperbolic fibered links of genus g with j components such that $\mathrm{vol}(M \setminus K_r^j(g)) \to \infty$ as $r \to \infty$. Then Question 4 has a positive answer for M.*

Proof. For notational simplicity we will assume that $j = 1$. The case $j = 2$ is analogous. Fix $g \geq g_0$, we consider the family of the hyperbolic knots $\{K_r := K_r^1(g)\}_{r \in \mathbb{N}}$ satisfying the assumption of Theorem 10, where $F_r = \Sigma_{g,1}$ denotes the fiber of K_r and $h_r := h_{g,1}$ the monodromy. As in the proof of Theorem 9, for any $r \in \mathbb{N}$, we build the 2-fold branched cover $T_{\widehat{h}_r}$ of M with fiber $\Sigma_{2g} = DF_r$ and monordomy $\widehat{h}_r = h_r \# h_r^{-1}$. We apply the process in the proof of Claims 1–4 of the proof of Theorem 9: For $1 < k \leq g$, say for $k = 2$, we take simple closed curves $\widehat{\gamma^r}_1, \widehat{\gamma^r}_2$ on Σ_{2g} so that Theorem 5 can be applied. By pushing $\widehat{\gamma^r}_1$, $\widehat{\gamma^r}_2$, we get a hyperbolic link L^r in $T_{\widehat{h}_r}$. Recall that for $\boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}^2$ with n_1, n_2 large enough, the mapping class $f_{\boldsymbol{n}}^r := \tau_{\widehat{\gamma^r}_2}^{n_2} \tau_{\widehat{\gamma^r}_1}^{n_1} \widehat{h}_r$ defined on Σ_{2g} is pseudo-Anosov and its mapping torus $N_{\boldsymbol{n}}^r$ is hyperbolic. Furthermore, $N_{\boldsymbol{n}}^r$ is obtained by Dehn filling of $\overline{T_{\widehat{h}_r} \setminus \mathcal{N}(L^r)}$, and we have

$$\frac{1}{2}\,\mathrm{vol}(T_{\widehat{h}_r} \setminus L^r) < \mathrm{vol}(N_{\boldsymbol{n}}^r), \tag{5.1}$$

where (5.1) follows from (2.3). To finish the proof of the theorem, we need to show that $\mathrm{vol}(N_{\boldsymbol{n}}^r) \to \infty$ as $r \to \infty$. This follows immediately from (5.1) and the following.

Claim 1. We have $\mathrm{vol}(T_{\widehat{h}_r} \setminus L^r) \to \infty$ as $r \to \infty$.

Proof of Claim 1. For any orientable 3-manifold X with ∂X empty or ∂X consisting only of tori, let $||X||$ denote the Gromov norm of X. See [19, Definition 6.1.2, the beginning of Section 6.5]. If X is closed and hyperbolic, or if ∂X consists only of tori and the interior of X is hyperbolic, then $v_3||X|| = \mathrm{vol}(X)$, where v_3 is the volume of the ideal regular tetrahedron. (See [19, Theorem 6.2, Lemma 6.5.4].) By construction, $M \setminus K_r \simeq M \setminus \mathcal{N}(K_r)$ is a submanifold of $T_{\widehat{h}_r}$ and $\partial(M \setminus K_r)$ is an incompressible torus in $T_{\widehat{h}_r}$. Indeed, we can think of $T_{\widehat{h}_r}$ as obtained by identifying two copies of $M \setminus K_r$ along their torus boundary. By [19, Theorem 6.5.5], we have

$$v_3\ ||T_{\widehat{h}_r}|| \geq \mathrm{vol}(M \setminus K_r),$$

which implies that $||T_{\widehat{h}_r}|| \to \infty$ as $r \to \infty$. Since $T_{\widehat{h}_r}$ is obtained from $T_{\widehat{h}_r} \setminus L^r$ by adding solid tori, by [19, Proposition 6.5.2] we obtain

$$\mathrm{vol}(T_{\widehat{h}_r} \setminus L^r) = v_3\ ||T_{\widehat{h}_r} \setminus L^r|| \geq v_3\ ||T_{\widehat{h}_r}||,$$

and $\mathrm{vol}(T_{\widehat{h}_r} \setminus L^r) \to \infty$ as $r \to \infty$. □

For $M = S^3$, families of knots satisfying the assumption of Theorem 10 are constructed in Futer-Purcell-Schleimer [8, Theorem 1]. Thus we have the following result.

Corollary 11. *For any $g \geq 2$, the set $\mathcal{D}_{2g}(S^3)$ contains an infinite family of pseudo-Anosov elements whose mapping tori have arbitrarily large volume.*

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