BRAIDS, ENTROPIES AND FIBERED 2-FOLD BRANCHED COVERS OF 3-MANIFOLDS

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Dedicated to the Memory of Toshie Takata

Abstract. It is proved by Brooks that any closed orientable 3-manifold with a Heegaard splitting of genus g admits a 2-fold branched cover that is a hyperbolic 3-manifold and a genus g surface bundle over the circle. This paper concerns entropy of pseudo-Anosov monodromies for hyperbolic fibered 3-manifolds. We prove that there exist infinitely many closed orientable 3-manifolds M such that the minimal entropy over all hyperbolic, genus g surface bundles over the circle as 2-fold branched covers of the 3-manifold M is comparable to $1/g$.

1. Introduction

Let M be a closed orientable 3-manifold which admits a genus g Heegaard splitting. Sakuma [Sak81] proved that there exists a 2-fold branched cover $\widetilde{M}$ of M such that $\widetilde{M}$ is a genus g surface bundle over the circle S^1. It is proved by Brooks [Bro85] that the branched cover $\widetilde{M}$ of M can be chosen to be hyperbolic if $g \geq 2$.

To state our results of this paper, let $\Sigma = \Sigma_{g,p}$ be an orientable, connected surface of genus g with p punctures, possibly $p = 0$. We set $\Sigma_g = \Sigma_{g,0}$ for a closed orientable surface of genus g. The mapping class group $\mathrm{MCG}(\Sigma)$ is the group of isotopy classes of orientation-preserving self-homeomorphisms on Σ which preserve the punctures setwise.

By the Nielsen-Thurston classification [Thu88, FM12], an element in $\mathrm{MCG}(\Sigma)$ is one of the following types: periodic, reducible, pseudo-Anosov. If an element in $\mathrm{MCG}(\Sigma)$ is neither periodic nor reducible, then it is pseudo-Anosov. For a mapping class $\phi = [f] \in \mathrm{MCG}(\Sigma)$, the *mapping torus* T_ϕ of ϕ is defined by

$$T_\phi = \Sigma \times \mathbb{R}/ \sim,$$

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where $(x,t) \sim (f(x), t+1)$ for $x \in \Sigma$ and $t \in \mathbb{R}$. We call Σ the *fiber* of T_ϕ. The 3-manifold T_ϕ is a Σ-bundle over S^1 with the monodromy ϕ. By Thurston [Thu98, Ota01], T_ϕ admits a hyperbolic structure of finite volume if and only if ϕ is pseudo-Anosov.

Thanks to the above result by Sakuma, one can define the non-empty subset $\mathcal{D}_g(M) \subset \mathrm{MCG}(\Sigma_g)$ consisting of elements $\phi \in \mathrm{MCG}(\Sigma_g)$ such that T_ϕ is a 2-fold branched cover of M branched over a link, i.e.,

$$\mathcal{D}_g(M) = \{\phi \in \mathrm{MCG}(\Sigma_g) \mid T_\phi \text{ is a 2-fold branched cover of } M\}.$$

By the above result of Brooks, there exists a pseudo-Anosov element in $\mathcal{D}_g(M)$ if $g \geq \max(2, g(M))$, where $g(M)$ is the Heegaard genus of M.

To each pseudo-Anosov mapping class $\phi \in \mathrm{MCG}(\Sigma)$ on the surface $\Sigma = \Sigma_{g,p}$, there exists an associated dilatation (stretch factor) $\lambda(\phi) > 1$ ([FM12]). The logarithm $\log(\lambda(\phi))$ of the dilatation is called the *entropy* of ϕ. We call

$$\mathrm{Ent}(\phi) = |\chi(\Sigma)| \log(\lambda(\phi)) \tag{1.1}$$

the *normalized entropy* of ϕ, where $\chi(\Sigma)$ is the Euler characteristic of Σ.

Consider the set

$$\mathrm{Spec}(\Sigma) = \{\log(\lambda(\phi)) \mid \phi \in \mathrm{MCG}(\Sigma) \text{ is pseudo-Anosov}\}.$$

For any subset of $\mathrm{Spec}(\Sigma)$, there exists a minimum. Then for any subset $G \subset \mathrm{MCG}(\Sigma)$ containing a pseudo-Anosov element, we set

$$\ell(G) = \min\{\log(\lambda(\phi)) \mid \phi \in G \text{ is pseudo-Anosov}\},$$

that is the minimal entropy of pseudo-Anosov elements of G. Clearly we have $\ell(G) \geq \ell(\mathrm{MCG}(\Sigma))$. Penner [Pen91] proved that $\ell(\mathrm{MCG}(\Sigma_g))$ is comparable to $1/g$. Here we say that for two real valued functions A and B of g, A is comparable to B and write $A \asymp B$ if there exists a constant $C > 0$ independent of g so that $B/C \leq A \leq CB$.

Asymptotic behaviors of minimal entropies of various subgroups (subsets) of mapping class groups have been studied by many authors ([FLM08, Tsa09, Val12, ALM16, HK17, Yaz18, HIKK22]). For the hyperelliptic mapping class group $\mathcal{H}(\Sigma_g)$ defined on Σ_g, the minimal entropy $\ell(\mathcal{H}(\Sigma_g))$ for $\mathcal{H}(\Sigma_g)$ is also comparable to $1/g$ (Hironaka-Kin [HK06]). In contrast, the minimal entropy $\ell(\mathcal{I}(\Sigma_g))$ for the Torelli group $\mathcal{I}(\Sigma_g)$ defined on Σ_g has a uniform lower bound (Farb-Leininger-Margalit [FLM08]).

Given a 3-manifold M, we consider the subset $\mathcal{D}_g(M) \subset \mathrm{MCG}(\Sigma_g)$ and we write

$$\ell_g(M) = \ell(\mathcal{D}_g(M)).$$

Then $\ell_g(M) \geq \ell(\mathrm{MCG}(\Sigma_g))$. The authors proved in [HK20b] that for the 3-sphere S^3, it holds $\ell_g(S^3) \asymp \frac{1}{g}$. In this paper, we prove that there exist infinitely many closed 3-manifolds with the same property as S^3. More precisely, we prove the following result.

Theorem 1.1. *There exist infinitely many closed orientable non-hyperbolic* 3*-manifolds* M *such that* $\ell_g(M) \asymp \dfrac{1}{g}$.

For a link L in S^3, let $M_L \to S^3$ be the 2-fold branched cover of S^3 branched over a link L. Every link L can be expressed by the closure $\mathrm{cl}(b)$ of some braid b. Along the way in the proof of Theorem 1.1, we prove in Theorem 4.2 that if b is a homogeneous braid with certain conditions, then we have $\ell_g(M_{\mathrm{cl}(b)}) \asymp \dfrac{1}{g}$. The 3-manifolds $M_{\mathrm{cl}(b)}$ with this property include the following examples.

- The lens space $L_{(2m,1)}$ of type $(2m, 1)$ with $m \neq 0$ (Corollary 4.5).
- The connected sum $\sharp_n S^2 \times S^1$ of n copies of $S^2 \times S^1$ for $n \geq 1$ (Theorem 4.6).
- Dehn fillings of the minimally twisted $2k$-chain link $\mathcal{C}_{2k}$ for $k \geq 3$.

Here a chain link is a link having the form of a circular chain. The *minimally twisted chain link* $\mathcal{C}_{2k}$ with $2k$ components is the chain link with every other link component lying flat in the plane of projection, and alternate link components to be perpendicular to the plane of projection. (See (3) of Figure 12 for $\mathcal{C}_6$.)

Since $\mathcal{C}_{2k}$ is a hyperbolic link, all Dehn fillings (with a finite exceptions) are hyperbolic. Moreover $\mathrm{vol}(S^3 \setminus \mathcal{C}_{2k}) \geq 2k\, v_3$, where $v_3 = 1.01494\ldots$ is the volume of the ideal regular tetrahedron. Hence we have the following result.

Theorem 1.2. *For any* $R \geq 0$*, there exists a closed orientable hyperbolic* 3*-manifold* M *with volume more than* R *such that* $\ell_g(M) \asymp \dfrac{1}{g}$.

Theorems 1.1 and 1.2 imply that there exist infinitely many links L in S^3 such that the minimal entropy $\ell_g(M_L)$ is comparable to $1/g$. Our conjecture is that every link in S^3 holds this property.

Conjecture 1.3. *For any link* L *in* S^3*, we have* $\ell_g(M_L) \asymp \dfrac{1}{g}$.

We ask the following question.

Question 1.4. *Is there a closed orientable* 3*-manifold* M *such that the minimal entropy* $\ell_g(M)$ *has a uniform lower bound?*

This paper is organized as follows. In Section 2 we review basic facts on braids groups, mapping class groups and pseudo-Anosov mapping classes. In Section 3 we introduce the notion of braids that are increasing in the middle. Then we combine some results in [HK20a, HK20b] into new claims that can be used for the study of pseudo-Anosov elements in the set $\mathcal{D}_g(M_L)$ for each link L in S^3. In Section 4 we prove Theorems 1.1 and 1.2 and give some applications.

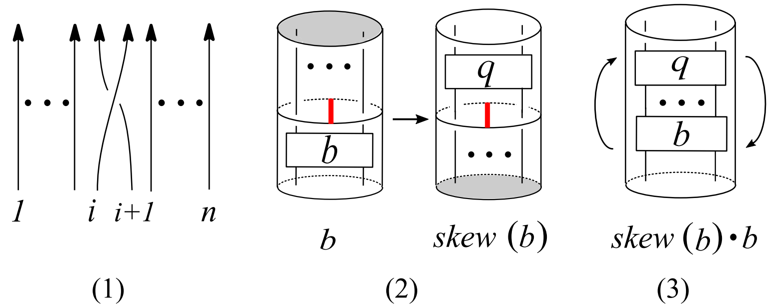

Figure 1. (1) $\sigma_i \in B_n$. (2) The involution $R : D^2 \times [0,1] \to D^2 \times [0,1]$ with the fixed point set $\{(\pm ri, \frac{1}{2}) \mid 0 \le r \le 1\} \subset D^2 \times \{\frac{1}{2}\}$. (3) The braid $\widetilde{b} = \mathrm{skew}(b) \cdot b$ that is invariant under the involution R.

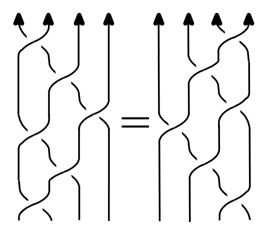

Figure 2. A half twist $\Delta_4 = \mathrm{skew}(\Delta_4) \in B_4$.

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2. Backgrounds and preliminaries

2.1. Homogeneous braids and skew-palindromic braids.

Let B_n be the (planar) braid group with n strands. Let $a_1, \dots, a_n$ be the bottom end points of an n-braid $b \in B_n$. We call a_i's the base points of b. We put indices $1, \dots, n$ to indicate the base points $a_1, \dots, a_n$ respectively. Let σ_i ($i = 1, \dots, n$) denote the Artin generator of B_n as in Figure 1(1).

A braid word written by $\sigma_i^{\pm 1}$ ($i = 1, \dots, n-1$) is said to be *homogeneous* if for each $i \in \{1, \dots, n-1\}$, the exponents of all occurrences of σ_i have the same sign. A braid b is said to be *homogeneous* if it can be represented by a homogeneous word. For example, the braid $\sigma_1\sigma_3\sigma_2^{-1}\sigma_3^2\sigma_2^{-3}$ is homogeneous.

Now, we define an involution

$$\begin{array}{rcl} \mathrm{skew} : B_n & \to & B_n \\ \sigma_{n_1}^{\epsilon_1}\sigma_{n_2}^{\epsilon_2}\ldots\sigma_{n_k}^{\epsilon_k} & \mapsto & \sigma_{n-n_k}^{\epsilon_k}\ldots\sigma_{n-n_2}^{\epsilon_2}\sigma_{n-n_1}^{\epsilon_1}, \qquad \epsilon_i = \pm 1. \end{array}$$

The map skew is an anti-homomorphism. A braid $b \in B_n$ is said to be *skew-palindromic* if $\mathrm{skew}(b) = b \in B_n$.

Note that $\mathrm{skew} : B_n \to B_n$ is induced by the involution R on the cylinder $D^2 \times [0,1]$:

$$\begin{array}{rcl} R : D^2 \times [0,1] & \to & D^2 \times [0,1], \\ (re^{i\theta}, t) & \mapsto & (re^{i(\pi-\theta)}, 1-t) \end{array}$$

see Figure 1(2). Here we identify the disk D^2 with the unit disk centered at the origin in the complex plane $\mathbb{C}$.

Notice that the product $\mathrm{skew}(b) \cdot b \in B_n$ is a skew-palindromic braid for any $b \in B_n$. We put

$$\widetilde{b} := \mathrm{skew}(b) \cdot b,$$

and we say that $\widetilde{b}$ is the *skew-palindromization* of b. See Figure 1(3).

Example 2.1. *For a braid $b = \sigma_3^2\sigma_4^{-2} \in B_5$, the skew-palindromization is*

$$\widetilde{b} = \mathrm{skew}(b) \cdot b = \sigma_1^{-2}\sigma_2^2\sigma_3^2\sigma_4^{-2},$$

that is a homogeneous braid.

Let $\Delta = \Delta_n \in B_n$ be a half twist defined by

$$\begin{array}{rcl} \Delta & = & (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2)\sigma_1 \\ & = & \sigma_{n-1}(\sigma_{n-2}\sigma_{n-1}) \cdots (\sigma_2 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-1}). \end{array}$$

See Figure 2. This means that $\Delta = \mathrm{skew}(\Delta)$, and hence $\Delta \in B_n$ is skew-palindromic for each n.

2.2. **Dilatations and normalized entropies of braids.** Let D_n be the n-punctured disk. We consider the mapping class group $\mathrm{MCG}(D_n)$, the group of isotopy classes of orientation preserving self-homeomorphisms on D_n preserving the boundary ∂D of the disk setwise. There exists a surjective homomorphism

$$\Gamma : B_n \to \mathrm{MCG}(D_n)$$

which sends each generator σ_i to the right-handed half twist h_i between the i-th and $(i+1)$-th punctures. Since the kernel of Γ is isomorphic to the center $Z(B_n) = \langle \Delta^2 \rangle$ generated by a full twist Δ^2, we have

$$B_n / \langle \Delta^2 \rangle \simeq \mathrm{MCG}(D_n).$$

Collapsing the boundary ∂D to a puncture in the sphere Σ_0, we have a homomorphism

$$\mathfrak{c} : \mathrm{MCG}(D_n) \to \mathrm{MCG}(\Sigma_{0,n+1}).$$

We say that $b \in B_n$ is *periodic* (resp. *reducible*, *pseudo-Anosov*) if the mapping class $\mathfrak{c}(\Gamma(b))$ is of the corresponding Nielsen-Thurston type.

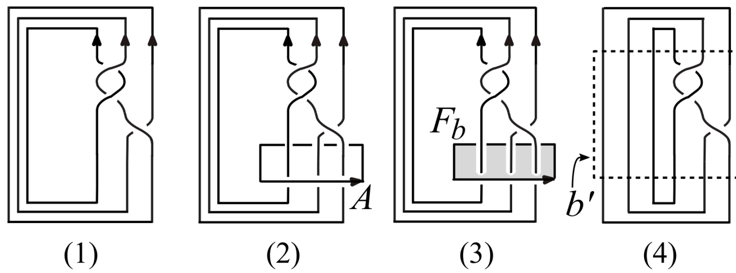


FIGURE 3. Case $b = \sigma_1^2\sigma_2^{-1} \in B_3$: (1) $\mathrm{cl}(b)$. (2) $\mathrm{br}(b)$. (3) F_b. (4) $C(b') = \mathrm{cl}(b)$, where $b' = \sigma_4^2\sigma_5^{-1} \in B_6$.

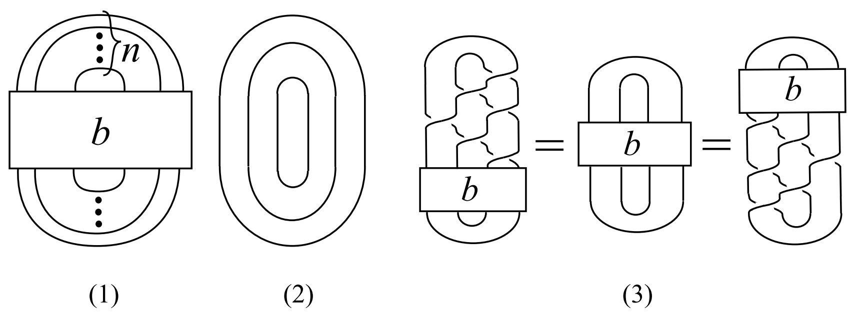


FIGURE 4. (1) $C(b)$ for $b \in B_{2n}$. (2) $E_3 = C(e_6)$. (3) $C(\Delta_4 b) = C(b) = C(b\Delta_4)$ for $b \in B_4$.

When $b \in B_n$ is pseudo-Anosov, we call $\lambda(b) := \lambda(\mathfrak{c}(\Gamma(b)))$ the *dilatation* of b, and call $\mathrm{Ent}(b) := \mathrm{Ent}(\mathfrak{c}(\Gamma(b)))$ the *normalized entropy* of b, see (1.1). By definition we have

$$\mathrm{Ent}(b) = |\chi(\Sigma_{0,n+1})| \log(\lambda((\mathfrak{c}(\Gamma(b))))) = (n-1)\log(\lambda((\mathfrak{c}(\Gamma(b))))).$$

2.3. Closures, braided links, and circular plat closures of braids.

In this section we introduce three kinds of links in S^3, closures, braided links and circular plat closures obtained from planar braids. Given a link L in a 3-manifold M, we denote by $\mathcal{N}(L)$, a regular neighborhood of L. We denote by $\mathcal{E}(L)$, the exterior $M \setminus \mathrm{int}(\mathcal{N}(L))$.

The *closure* $\mathrm{cl}(b)$ of b is an oriented knot or link in S^3 whose orientation is induced by those of the strands of b, see Figure 3(1). The *braided link*

$$\mathrm{br}(b) = A \cup \mathrm{cl}(b)$$

is a link in S^3 obtained from $\mathrm{cl}(b)$ with the braid axis A, see Figure 3(2). We think of $\mathrm{br}(b)$ as an oriented link in S^3 choosing an orientation of A arbitrarily. (In Section 2.7, we assign an orientation of A for *i-increasing* *braids.*) Let T_b denote the exterior of the link $\mathrm{br}(b)$:

$$T_b = \mathcal{E}(\mathrm{br}(b)) = S^3 \setminus \mathrm{int}(\mathcal{N}(\mathrm{br}(b))).$$

We define an $(n+1)$-holed sphere $F_b \subset T_b$ by

$$F_b = D_A \setminus \mathrm{int}(\mathcal{N}(\mathrm{cl}(b))),$$

where D_A is the disk bounded by the longitude of the regular neighborhood $\mathcal{N}(A)$ of the braid axis A of b. See Figure 3(3). We give an orientation of F_b which induces the orientation of A. The surface F_b is a fiber of the fibration $T_b \to S^1$ and the braid b determines the monodromy $\phi_b : F_b \to F_b$ (up to conjugation).

The *circular plat closure* $C(b)$ of $b \in B_{2n}$ with even strands is an unoriented knot or link in S^3 as in Figure 4(1). For example, the n-component trivial link E_n is of the form $E_n = C(e_{2n})$, where $e_{2n} \in B_{2n}$ is the identity element, see Figure 4(2). It is not hard to see that the links $C(\Delta b)$, $C(b)$ and $C(b\Delta)$ are ambient isotopic to each other:

$$C(\Delta b) = C(b) = C(b\Delta) \tag{2.1}$$

as links in S^3. See Figure 4(3).

Remark 2.2. *Any link L in S^3 can be represented by the circular plat closure $C(b')$ for some braid b' with even strands. To see this, we recall the fact that any link L can be expressed by the closure $\mathrm{cl}(b)$ for some $b \in B_n$ $(n \geq 1)$. The desired braid b' with $2n$ strands can be obtained from the n-braid b by adding n straight strands: $b' = e_n \cup b \in B_{2n}$. Then we have $C(b') = \mathrm{cl}(b) = L$ as links in S^3. See Figure 3(4).*

2.4. **A criterion to be pseudo-Anosov braids.** In this section, we give a criterion for deciding planar braids to be pseudo-Anosov.

Given an oriented link $L = K_1 \cup \cdots \cup K_m$ with m components in S^3, we denote by $\mathrm{lk}(K_i, K_j)$, the linking number between the two components K_i and K_j. See [Kaw96] for the definition of the linking number.

Let

$$\pi : B_n \to \mathfrak{S}_n$$

be the surjective homomorphism from the n-braid group B_n to the permutation group $\mathfrak{S}_n$ of degree n which sends σ_j to the transposition $(j, j+1)$. A braid $b \in B_n$ is *pure* if $\pi(b)$ is the identity element of $\mathfrak{S}_n$.

For example, a 3-braid $\beta = \sigma_1^4 \sigma_2^{-2}$ is pure. Let $\mathrm{cl}(\beta) = \ell_1 \cup \ell_2 \cup \ell_3$ be the closure of β, where ℓ_i denotes the closure $\mathrm{cl}(\beta(i))$ of the i-th strand $\beta(i)$ with the base point a_i for $i = 1, 2, 3$. Then $\mathrm{lk}(\ell_1, \ell_2) = 2$, $\mathrm{lk}(\ell_2, \ell_3) = -1$ and $\mathrm{lk}(\ell_3, \ell_1) = 0$.

Proposition 2.3 (Kobayashi-Umeda [KU10])**.** *Let $\beta \in B_n$ be a pure braid for $n \geq 3$. Let $\mathrm{cl}(\beta) = \ell_1 \cup \cdots \cup \ell_n$ be the closure of β, where ℓ_i denotes the closure $\mathrm{cl}(\beta(i))$ of the i-th strand $\beta(i)$ with the base point a_i for $i = 1, \ldots, n$.*

1. *Suppose that β is periodic. Then there exists an integer n_0 such that $\mathrm{lk}(\ell_i, \ell_j) = n_0$ for all i, j with $i \neq j$.*

(2) *Suppose that β is reducible. Let c be an inner most component of the system of the reducing curves for the mapping class $\Gamma(\beta) \in \mathrm{MCG}(D_n)$, and let D_c be the disk bounded by c. Suppose that a_s and a_t are distinct base points in D_c. Then for each base point $a_j \notin D_c$, the equality $\mathrm{lk}(\ell_j, \ell_s) = \mathrm{lk}(\ell_j, \ell_t)$ holds.*

The proof of the claim (1) (resp. the claim (2)) in Proposition 2.3 can be found in [KU10, Proposition 1] (resp. [KU10, Proposition 2]). For the definition of *the system of the reducing curves* in the claim (2), see [KU10], [FM12, Chapter 13.2.2].

Lemma 2.4. *Let $\beta \in B_n$ be a pure braid for $n \geq 3$. Let ℓ_i ($i = 1, \ldots, n$) be as in Proposition 2.3. Suppose that for any proper subset*

$$\mathcal{I} = \{i_1, \ldots, i_k\} \subsetneq \mathcal{J} := \{1, 2, \ldots, n\}$$

consisting of k distinct elements with $2 \leq k < n$, there exist three elements $j \in \mathcal{J} \setminus \mathcal{I}$ and $i_s, i_t \in \mathcal{I}$ such that $\mathrm{lk}(\ell_j, \ell_{i_s}) \neq \mathrm{lk}(\ell_j, \ell_{i_t})$. Then β is pseudo-Anosov.

Proof. By Proposition 2.3(1), the braid β with the assumption of Lemma 2.4 can not be periodic. Assume that β is reducible. Let c be an inner most component of the system of reducing curves for the mapping class $\Gamma(\beta) \in \mathrm{MCG}(D_n)$, and let D_c be the disk bounded by c. Let $a_{i_1}, \ldots, a_{i_k}$ be the set of all base points of β contained in D_c. Then $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\} \neq \emptyset$. By the assumption of Lemma 2.4, there exist three elements $j \in \{1, 2, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$ and $i_s, i_t \in \{i_1, \ldots, i_k\}$ such that $\mathrm{lk}(\ell_j, \ell_{i_s}) \neq \mathrm{lk}(\ell_j, \ell_{i_t})$. By the choice of j, we have $a_j \notin D_c$ for the base point a_j of the strand $\beta(j)$ and $a_{i_s}, a_{i_t} \in D_c$. By Proposition 2.3(2), it must hold that $\mathrm{lk}(\ell_j, \ell_{i_s}) = \mathrm{lk}(\ell_j, \ell_{i_t})$. This is a contradiction, and hence β is not reducible. Since β is neither periodic nor reducible, we conclude that β is pseudo-Anosov. $\square$

Lemma 2.5. *Let $b \in B_n$ be a pure braid for $n \geq 4$ of the form*

$$b = \sigma_{j_1}^{2m_1} \sigma_{j_2}^{2m_2} \ldots \sigma_{j_k}^{2m_k},$$

where $m_1, \ldots, m_k$ are non-zero integers and $j_1, \ldots, j_k \in \{1, \ldots, n-1\}$. Suppose that b is homogeneous, and each σ_i for $i = 1, \ldots, n-1$ appears in b at least once, i.e., $\{j_1, \ldots, j_k\} = \{1, \ldots, n-1\}$. Then b is pseudo-Anosov. In particular, if $b = \sigma_1^{2m_1} \sigma_2^{2m_2} \ldots \sigma_{n-1}^{2m_{n-1}} \in B_n$, then b is pseudo-Anosov.

Proof. Let $\ell_i = \mathrm{cl}(b(i))$ ($i = 1, \ldots, n$) be the component of $\mathrm{cl}(b)$ as in Proposition 2.3. The assumption of Lemma 2.5 implies that $\mathrm{lk}(\ell_i, \ell_j) \neq 0$ if and only if $|i - j| = 1$. It is sufficient to prove the following: For any proper subset $\mathcal{I} = \{i_1, \ldots, i_k\} \subsetneq \mathcal{J} = \{1, 2, \ldots, n\}$ with $2 \leq k < n$, there exist three elements $j \in \mathcal{J} \setminus \mathcal{I}$ and $i_s, i_t \in \mathcal{I}$ such that

$$|j - i_s| = 1 \ \text{ and } \ |j - i_t| > 1, \tag{2.2}$$

i.e., $\mathrm{lk}(\ell_j, \ell_{i_s}) \neq 0$ and $\mathrm{lk}(\ell_j, \ell_{i_t}) = 0$. Then $\mathrm{lk}(\ell_j, \ell_{i_s}) \neq \mathrm{lk}(\ell_j, \ell_{i_t})$, and Lemma 2.4 tells us that b is pseudo-Anosov.

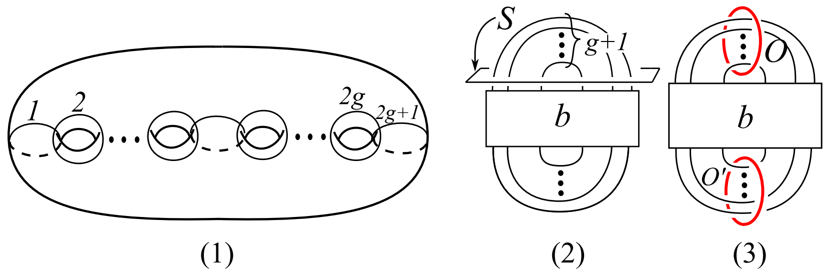


FIGURE 5. (1) Simple closed curves labeled $1,\dots,2g+1$ in Σ_g. (2) A $(g+1)$-bridge sphere S of $C(b)$ and (3) the link $C(b)\cup W$ for $b\in B_{2g+2}$, where $W=O\cup O'$.

Since $\mathcal{I}$ is a proper subset of $\mathcal{J}$, there are $i_u\in\mathcal{I}$ and $h\in\mathcal{J}\setminus\mathcal{I}$ such that $|i_u-h|=1$. Moreover we can take an element $i_v\in\mathcal{I}$ such that $i_v\neq i_u$. It is possible to take such $i_v\in\mathcal{I}$ because $|\mathcal{I}|\geq 2$, where $|S|$ denotes the cardinality of the finite set S. In case where $|i_v-h|>1$, the three elements $j:=h$, $i_s:=i_u$ and $i_t:=i_v$ satisfy (2.2). In case where $|i_v-h|=1$, the three elements i_u,h,i_v are consecutive integers. Without loss of generality, we may assume that $i_u<h<i_v$. Since $n\geq 4$, the following cases occur: (1) $1=i_u<h<i_v<n$, (2) $1<i_u<h<i_v<n$, (3) $1<i_u<h<i_v=n$. In cases (1) and (2), we have $i_v+1\in\mathcal{J}$. If $i_v+1\in\mathcal{I}$, then $j:=h$, $i_s:=i_u$ and $i_t:=i_v+1$ satisfy (2.2). If $i_v+1\notin\mathcal{I}$, then $j:=i_v+1$, $i_s:=i_v$ and $i_t:=i_u$ satisfy (2.2). In case (3), we can choose three elements j, i_s and i_t that satisfy (2.2) in the same way as above. This completes the proof. □

Example 2.6. *By Lemma 2.5, the braid* $\widetilde{b}=\sigma_1^{-2}\sigma_2^2\sigma_3^2\sigma_4^{-2}\in B_5$ *given in Example 2.1 is pseudo-Anosov*

2.5. Branched virtual fibering theorem.

We recall the branched virtual fibering theorem due to Sakuma [Sak81]. See also Koda-Sakuma [KS22, Theorem 9.1].

Theorem 2.7. *Let M be a closed orientable 3-manifold. Suppose that M admits a genus g Heegaard splitting. Then there exists a 2-fold branched cover $\widetilde{M}$ of M which is a Σ_g-bundle over the circle.*

In [HK20b], the authors gave an alternative construction of surface bundles over the circle in Sakuma's result when closed 3-manifolds are 2-fold branched covers of S^3 branched over links. We recall our construction in this section.

Let τ_i denote the right-handed Dehn twist about the simple closed curve labeled i in Figure 5. We have the Birman-Hilden homomorphism $\mathfrak{t}$ from the braid group B_{2g+2} to the mapping class group $\mathrm{MCG}(\Sigma_g)$:

$$\mathfrak{t}: B_{2g+2}\to \mathrm{MCG}(\Sigma_g)$$

which sends σ_i to τ_i for $i = 1, \dots, 2g+1$. Notice that its image $\mathfrak{t}(B_{2g+2})$ is the *hyperelliptic mapping class group* $\mathcal{H}(\Sigma_g)$. This is the subgroup of $\mathrm{MCG}(\Sigma_g)$ consisting of elements with representative homeomorphisms that commute with some fixed hyperelliptic involution on Σ_g.

Let L be a link in S^3. By Remark 2.2 we may suppose that L is of the form $L = C(b)$ for some $b \in B_{2g+2}$. Let

$$q = q_L : M_L \to S^3$$

denote the 2-fold branched covering map of S^3 branched over L. We have a $(g+1)$-bridge sphere S for $L = C(b)$ as in Figure 5(2). The 3-manifold M_L admits a genus g Heegaard splitting with the Heegaard surface $q^{-1}(S)$. Consider the trivial link $W = O \cup O'$ with 2 components and the link $C(b) \cup W$ in S^3 as shown in Figure 5(3). Then we have the following result.

Theorem 2.8 (Theorem B in [HK20b]). *Let $q : M_L \to S^3$ be the 2-fold branched covering map of S^3 branched over a link $L = C(b)$ for a braid $b \in B_{2g+2}$. Consider the skew-palindromization $\widetilde{b}$ of b and the mapping class $\mathfrak{t}(\widetilde{b}) \in \mathcal{H}(\Sigma_g) \subset \mathrm{MCG}(\Sigma_g)$. Then $T_{\mathfrak{t}(\widetilde{b})} \to M_L$ is a 2-fold branched cover of M_L branched over the link $q^{-1}(W)$. In particular $\mathfrak{t}(\widetilde{b}) \in \mathcal{D}_g(M_L)$.*

Sketch of Proof. We regard $\widetilde{M_L}$ as the $\mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$-cover of S^3 branched over the link $C(b) \cup W$ associated with the epimorphism

$$H_1(S^3 \setminus (C(b) \cup W)) \to \mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$$

which maps the meridians of $C(b)$ to $(1,0)$ and the meridians of W to $(0,1)$. Let $q_W : M_W \to S^3$ be the 2-fold branched covering map of S^3 branched over the link W. Note that $M_W = S^2 \times S^1$. Then $\widetilde{M_L}$ is the 2-fold branched cover of $S^2 \times S^1$ branched over the link $q_W^{-1}(C(b)) = \mathrm{cl}(\widetilde{b})$ associated with the epimorphism

$$H_1(S^2 \times S^1 \setminus \mathrm{cl}(\widetilde{b})) \to \mathbb{Z}/2\mathbb{Z}$$

which maps the meridians of $\mathrm{cl}(\widetilde{b})$ to 1 and $\{\mathrm{pt}\} \times S^1$ to 0. Therefore, $\widetilde{M_L}$ is homeomorphic to the mapping torus $T_{\mathfrak{t}(\widetilde{b})}$ of $\mathfrak{t}(\widetilde{b})$. This completes the proof. $\square$

We are interested in the case where the mapping class $\mathfrak{t}(\widetilde{b}) \in \mathrm{MCG}(\Sigma_g)$ given in Theorem 2.8 is pseudo-Anosov. The following lemma will be used in the later section.

Lemma 2.9 (Lemma 5 in [HK20b]). *Let $\beta \in B_{2g+2}$ be a pseudo-Anosov braid and let $\Phi_\beta : D_{2g+2} \to D_{2g+2}$ be a pseudo-Anosov homeomorphism which represents $\Gamma(\beta) \in \mathrm{MCG}(D_{2g+2})$. Suppose that the pseudo-Anosov braid β possesses the following condition:*

> $\diamondsuit$ *The stable foliation $\mathcal{F}$ for Φ_β defined on D_{2g+2} is not 1-pronged at the boundary ∂D of the disk.*

Then $\mathfrak{t}(\beta) \in \mathrm{MCG}(\Sigma_g)$ is pseudo-Anosov, and the equality $\lambda(\mathfrak{t}(\beta)) = \lambda(\beta)$ holds.

The basic facts on (un)stable foliations for pseudo-Anosov homeomorphisms can be found in Chapter 11.2 and Chapter 13 in [FM12].

2.6. **Thurston norm.** Let M be a 3-manifold with boundary (possibly $\partial M = \emptyset$). When M is a hyperbolic 3-manifold, there exists a norm $\|\cdot\|$ on $H_2(M, \partial M; \mathbb{R})$, that is called the Thurston norm [Thu86]. The norm $\|\cdot\|$ has the property such that for any integral class $a \in H_2(M, \partial M; \mathbb{R})$, we have

$$\|a\| = \min_S\{-\chi(S)\},$$

where the minimum is taken over all oriented surface S embedded in M with $a = [S]$ and with no components of non-negative Euler characteristic. The following result by Thurston describes a relation between the norm $\|\cdot\|$ and fibrations on M.

Theorem 2.10 (Thurston [Thu86])**.** *The norm $\|\cdot\|$ on $H_2(M, \partial M; \mathbb{R})$ has the following properties.*

(1) *There exist a set of maximal open cones $\mathscr{C}_1, \dots, \mathscr{C}_k$ in $H_2(M, \partial M; \mathbb{R})$ and a bijection between the set of isotopy classes of connected fibers of fibrations $M \to S^1$ and the set of primitive integral classes in $\mathscr{C}_1 \cup \cdots \cup \mathscr{C}_k$.*
(2) *The restriction of $\|\cdot\|$ to $\mathscr{C}_j$ is linear for each $j = 1, \dots, k$.*
(3) *For a fiber F_a of the fibration $M \to S^1$ associated with a primitive integral class $a \in \mathscr{C}_j$ for $j = 1, \dots, k$, we have $\|a\| = -\chi(F_a)$.*

We call the open cones $\mathscr{C}_j$ the *fibered cones* of M.

Theorem 2.11 (Fried [Fri82])**.** *For a fibered cone $\mathscr{C}$ of a hyperbolic 3-manifold M, there exists a continuous function $\mathrm{ent} : \mathscr{C} \to \mathbb{R}$ with the following properties.*

(1) *For the monodromy $\phi_a : F_a \to F_a$ of the fibration $M \to S^1$ associated with a primitive integral class $a \in \mathscr{C}$, we have $\mathrm{ent}(a) = \log(\lambda(\phi_a))$, i.e., $\mathrm{ent}(a)$ equals the entropy of the pseudo-Anosov monodromy ϕ_a.*
(2) $\mathrm{Ent} = \|\cdot\|\mathrm{ent} : \mathscr{C} \to \mathbb{R}$ *is a continuous function which becomes constant on each ray through the origin.*

We call $\mathrm{ent}(a)$ and $\mathrm{Ent}(a)$ the *entropy* and *normalized entropy* of the class $a \in \mathscr{C}$. By Theorem 2.10(3) and Theorem 2.11(1), if $a \in \mathscr{C}$ is a primitive integral class, then

$$\mathrm{Ent}(a) = \|a\|\mathrm{ent}(a) = |\chi(F_a)| \log(\lambda(\phi_a))(= \mathrm{Ent}(\phi_a)).$$

2.7. **i-increasing braids.**

In [HK20a], the authors introduced *i-increasing braids.* In this section, we review some properties of i-increasing braids that are needed in the later section.

Recall the surjective homomorphism $\pi : B_n \to \mathfrak{S}_n$ introduced in Section 2.4. We denote by π_b, the permutation $\pi(b) \in \mathfrak{S}_n$ for $b \in B_n$. Suppose that

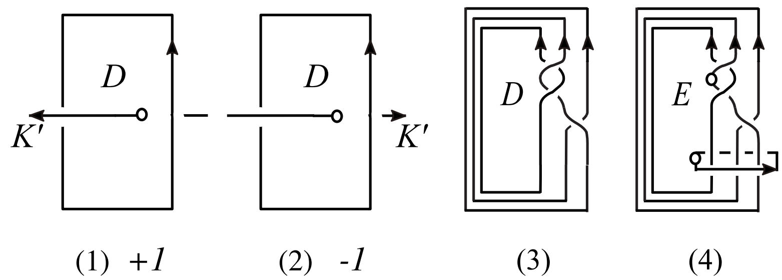


FIGURE 6. The sign of the intersection point: (1) $+1$ and (2) -1. (3) The associated disk $D = D_{(b,1)}$ and (4) the surface $E = E_{(b,1)}$ for a 1-increasing braid $b = \sigma_1^2\sigma_2^{-1} \in B_3$. ($E_{(b,1)}$ is a twice punctured disk in this case.)

$b \in B_n$ is a braid with $\pi_b(i) = i$, i.e., the permutation π_b fixes the index i. The closure $\mathrm{cl}(b(i))$ of the i-th strand $b(i)$ is a component of the closure $\mathrm{cl}(b)$ of b. We consider an oriented disk $D = D_{(b,i)}$ bounded by the longitude ℓ_i of a regular neighborhood $\mathcal{N}(\mathrm{cl}(b(i)))$ of $\mathrm{cl}(b(i))$. Such a disk D is unique up to isotopy on $\mathcal{E}(\mathrm{cl}(b(i)))$. Let $b - b(i) \in B_{n-1}$ be a braid with $n-1$ strands obtained from b by removing the i-th strand $b(i)$. The braid b is said to be *i-increasing* (resp. *i-decreasing*) if there exists a disk $D = D_{(b,i)}$ as above with the following conditions (D1) and (D2).

(D1) There exists at least one component K' of $\mathrm{cl}(b - b(i))$ such that $D \cap K' \neq \emptyset$.
(D2) Each component of $\mathrm{cl}(b - b(i))$ and D intersect with each other transversally, and every intersection point has the same sign $+1$ (resp. -1), see Figure 6(1)(2).

We call $D = D_{(b,i)}$ the *associated disk* of the pair (b, i). Then we set

$$I(b, i) = D \cap \mathrm{cl}(b - b(i)).$$

By (D1) we have $I(b, i) \neq \emptyset$. Let $u(b, i) \geq 1$ be the cardinality $|I(b, i)|$ of $I(b, i)$. We call $u(b, i)$ the *intersection number* of the pair (b, i).

Example 2.12.

(1) *A braid $b = \sigma_1^2\sigma_2^{-1} \in B_3$ is 1-increasing with $u(b, 1) = 1$. See Figure 6(3).*
(2) *A pure braid $b = \sigma_1^4\sigma_2^{-2} \in B_3$ is 1-increasing with $u(b, 1) = 2$ and 3-decreasing with $u(b, 3) = 1$.*

Properties of i-increasing braids are given in the next two lemmas. The same properties hold for i-decreasing braids.

Lemma 2.13. *If b and b' are i-increasing braids with the same number of strands, then the product bb' is also i-increasing such that $u(bb', i) = u(b, i) + u(b', i)$.*

Proof. Let D^2 be the disk with radius 1. For $k = 0, 1, 2$, let $(D^2 \times [0, 1])_k$ be cylinders in S^3 such that $(D^2 \times [0, 1])_1 \cap \mathrm{cl}(b) = b$, $(D^2 \times [0, 1])_0 \cap \mathrm{cl}(b') = b'$ and $(D^2 \times [0, 1])_2 \cap \mathrm{cl}(bb') = bb'$. We set $R_\theta := \{re^{i\theta} \mid 0 \le r \le 1\} \subset D^2$. We denote by D (resp. D'), an associated disk of the pair (b, i) (resp. (b', i)). By an ambient isotopy, we may assume that there exists $\theta_0 \in [0, 2\pi)$ such that $D \cap (D^2 \times [0, 1])_1 = R_{\theta_0} \times [0, 1]$ (resp. $D' \cap (D^2 \times [0, 1])_0 = R_{\theta_0} \times [0, 1]$) and $D \cap \mathrm{cl}(b) = R_{\theta_0} \times [0, 1] \cap \mathrm{cl}(b)$ (resp. $D' \cap \mathrm{cl}(b') = R_{\theta_0} \times [0, 1] \cap \mathrm{cl}(b')$). We stuck the cylinder $(D^2 \times [0, 1])_1$ over the cylinder $(D^2 \times [0, 1])_0$ so that $(D^2 \times \{0\})_1$ is attached to $(D^2 \times \{1\})_0$, and we identify the result with $D^2 \times [0, 2]$. Let

$$F : D^2 \times [0, 2] \to (D^2 \times [0, 1])_2$$

be the homeomorphism defined by $F(x, t) = (x, t/2)$. Then $F(b \cup b') = bb'$ by the definition of the product of braids. The image of the union of $R_{\theta_0} \times [0, 1]$ in $(D^2 \times [0, 1])_1$ and $R_{\theta_0} \times [0, 1]$ in $(D^2 \times [0, 1])_0$ under the homeomorphism F is $R_{\theta_0} \times [0, 1] \subset (D^2 \times [0, 1])_2$, which intersects with the closure $\mathrm{cl}(bb')$ of the product bb' positively at $(u(b, i) + u(b', i))$ points. Therefore, bb' is i-increasing and the equality $u(bb', i) = u(b, i) + u(b', i)$ holds. □

Lemma 2.14. *Suppose that $b \in B_n$ is an i-increasing braid. Then* $\mathrm{skew}(b) \in B_n$ *is an $(n-i+1)$-increasing braid such that $u(\mathrm{skew}(b), n-i+1) = u(b, i)$.*

Proof. The assertion follows from that fact that $\mathrm{skew} : B_n \to B_n$ is induced by the involution R on the cylinder $D^2 \times [0, 1]$ given in Section 2.1. □

Example 2.15. *Let $b = \sigma_3^2\sigma_4^{-2}$ be the 5-braid as in Example 2.1. Then b is 3-increasing with $u(b, 3) = 1$. By Lemma 2.14, the braid* $\mathrm{skew}(b) = \sigma_1^{-2}\sigma_2^2$ *is 3-increasing with* $u(\mathrm{skew}(b), 3) = 1$. *Then by Lemma 2.13, the braid* $\widetilde{b} = \mathrm{skew}(b) \cdot b = \sigma_1^{-2}\sigma_2^2\sigma_3^2\sigma_4^{-2}$ *is also 3-increasing with* $u(\widetilde{b}, 3) = u(\mathrm{skew}(b), 3) + u(b, 3) = 2$.

Recall that $T_b = \mathcal{E}(\mathrm{br}(b))$ is the exterior of the braided link $\mathrm{br}(b)$ and the surface F_b is a genus 0 fiber of the fibration $T_b \to S^1$. See Section 2.3. We shall define the 2-dimensional subcone $\mathrm{C}_{(b,i)}$ of $H_2(T_b, \partial T_b; \mathbb{R})$ for an i-increasing braid b. To do this, we first consider the braided link $\mathrm{br}(b) = \mathrm{cl}(b) \cup A$. The associated disk $D = D_{(b,i)}$ has a unique point of intersection with A, and the cardinality of $I(b, i) \cup (D \cap A)$ is $u(b, i) + 1$. To deal with $\mathrm{br}(b) = \mathrm{cl}(b) \cup A$ as an oriented link, we consider an orientation of $\mathrm{cl}(b)$ as we described in Section 2.3, and assign an orientation of the braid axis A of b so that the sign of the intersection between D and A is $+1$ as in Figure 6(1). See Figure 3(2) for the orientation of A of the 3-braid $\sigma_1^2\sigma_2^{-1}$ that is 1-increasing.

Next, we define an oriented surface $E_{(b,i)}$ of genus 0 embedded in T_b. Consider small $u(b, i) + 1$ disks in the associated disk $D = D_{(b,i)}$ whose centers are points of $I(b, i) \cup (D \cap A)$. Then $E_{(b,i)}$ is a surface of genus 0 with $u(b, i) + 2$ boundary components obtained from D by removing the interiors of those small disks. We choose the orientation of $E_{(b,i)}$ so that it agrees with the orientation of D. See Figure 6(4).

Lastly, we define the 2-dimensional subcone $\mathrm{C}_{(b,i)}$ of $H_2(T_b, \partial T_b; \mathbb{R})$ spanned by the two integral classes $[F_b]$ and $[E_{(b,i)}]$ as follows.

$$\mathrm{C}_{(b,i)} = \{x[F_b] + y[E_{(b,i)}] \mid x > 0,\ y > 0\}. \tag{2.3}$$

We write $(x, y) = x[F_b] + y[E_{(b,i)}] \in \mathrm{C}_{(b,i)}$. The Thurston norm of (x, y) is denoted by $\|(x, y)\|$.

Theorem 2.16 ([HK20a]). *Let b be an i-increasing braid. Suppose that b is pseudo-Anosov. Let $\mathscr{C}$ be the fibered cone of the 3-manifold T_b containing $[F_b] = (1, 0) \in \mathrm{C}_{(b,i)}$. Then we have the following.*

1. $\mathrm{C}_{(b,i)} \subset \mathscr{C}$.
2. *The fiber $F_{(x,y)}$ for each primitive integral class $(x, y) \in \mathrm{C}_{(b,i)}$ has genus* 0.
3. *Let $\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}$ denote the monodromy of the fibration $T_b \to S^1$ associated with a primitive integral class $(x, y) \in \mathrm{C}_{(b,i)}$. Then there exists a j-increasing braid $b_{(x,y)} \in B_{\|(x,y)\|+1}$ for some index $j = j_{(x,y)}$ which gives the monodromy $\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}$.*

The proof of Theorem 2.16(1)(2) can be found in Theorem 3.2(1)(2) in [HK20a]. The statement of Theorem 2.16(3) follows from the argument in the proof of Theorem 3.2(3) in [HK20a].

3. Braids increasing in the middle

Let b be a braid with $2n+1$ strands. Then the notion 'i-increasing braid' makes sense for $i = 1, \dots, 2n + 1$. (See Section 2.7.) In this section, we restrict our attention to the case $i = n + 1$: Suppose that b is an $(n + 1)$-increasing braid. In this case we say that b is *increasing in the middle*. We write $\mathrm{C}_b := \mathrm{C}_{(b,n+1)}$ for the subcone of $H_2(T_b, \partial T_b; \mathbb{R})$. (See (2.3) for the definition of the subcone.) Then $b^\bullet \in B_{2n}$ denotes the braid obtained from $b \in B_{2n+1}$ by removing the strand of the middle index $n + 1$.

Example 3.1. *(cf. Example 2.15) Suppose that $b \in B_{2n+1}$ is a braid increasing in the middle. By Lemma 2.14, the braid* $\mathrm{skew}(b)$ *is increasing in the middle. By Lemma 2.13 the braid $\widetilde{b} = \mathrm{skew}(b) \cdot b$ is also increasing in the middle with the intersection number*

$$u(\widetilde{b}, n + 1) = u(\mathrm{skew}(b), n + 1) + u(b, n + 1) = 2u(b, n + 1).$$

The skew-palindromization $\widetilde{(b^\bullet)}$ of $b^\bullet$ satisfies

$$\widetilde{(b^\bullet)} = \mathrm{skew}(b^\bullet)b^\bullet = (\widetilde{b})^\bullet,$$

i.e., $\widetilde{(b^\bullet)} \in B_{2n}$ is obtained from the skew-palindromization $\widetilde{b}$ of b by removing the strand of the middle index $n + 1$. Hereafter we simply denote the braid $\widetilde{(b^\bullet)}$ by $\widetilde{b^\bullet}$. Applying Theorem 2.8 to the circular plat closure $L = C(b^\bullet)$ of $b^\bullet \in B_{2n}$, we have $\mathfrak{t}(\widetilde{b^\bullet}) \in \mathcal{D}_{n-1}(M_{C(b^\bullet)})$.

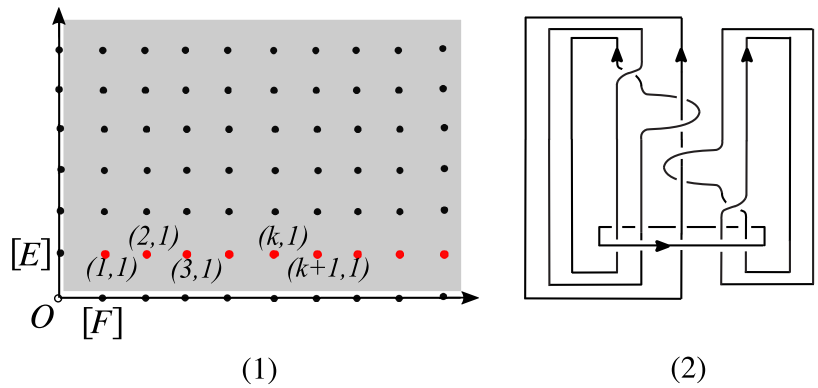


FIGURE 7. (1) The subcone $\mathrm{C}_{\widetilde{\beta}} = \mathrm{C}_{(\widetilde{\beta},n+1)}$ spanned by $[F]$ and $[E]$, where $F := F_{\widetilde{\beta}}$ and $E := E_{(\widetilde{\beta},n+1)}$. (Primitive integral classes $(1,1),(2,1),\dots,(k,1),\dots$ in $\mathrm{C}_{\widetilde{\beta}}$ are indicated.) (2) The braided link $\mathrm{br}(\widetilde{\beta})$ of $\widetilde{\beta} = \sigma_1\sigma_2^2\sigma_3^2\sigma_4 \in B_5$.

Example 3.2. *The half twist $\Delta = \Delta_{2n+1} \in B_{2n+1}$ is a braid increasing in the middle with $u(\Delta, n+1) = n$. By Lemma 2.13 the positive power Δ^p for $p \geq 1$ is a braid increasing in the middle, and it holds*

$$u(\Delta^p, n+1) = p \cdot u(\Delta, n+1) = pn.$$

The braid $\Delta^\bullet \in B_{2n}$ satisfies $\Delta^\bullet = \Delta_{2n} \in B_{2n}$, i.e., $\Delta^\bullet$ is equal to the half twist Δ_{2n} with $2n$ strands.

Given a braid $\beta \in B_{2n+1}$ increasing in the middle, we suppose that the skew-palindromization $\widetilde{\beta}$ of β is pseudo-Anosov. Consider the subcone $\mathrm{C}_{\widetilde{\beta}} = \mathrm{C}_{(\widetilde{\beta},n+1)}$ of $H_2(T_{\widetilde{\beta}}, \partial T_{\widetilde{\beta}}; \mathbb{R})$ for the hyperbolic 3-manifold $T_{\widetilde{\beta}}$ (Figure 7(1)). We now apply Theorem 2.16 for the skew-palindromization $\widetilde{\beta}$. For the class $(1,0) = [F_{\widetilde{\beta}}] \in \mathrm{C}_{\widetilde{\beta}}$, the monodromy $\phi_{(1,0)} : F_{(1,0)} \to F_{(1,0)}$ defined on the fiber $F_{(1,0)} = F_{\widetilde{\beta}}$ is given by the braid $\widetilde{\beta}$ that is increasing in the middle. Theorem 3.4 below tells us that this property is inherited for all primitive classes $(x,y) \in \mathrm{C}_{\widetilde{\beta}} \subset \mathscr{C}$, where $\mathscr{C}$ is the fibered cone of $T_{\widetilde{\beta}}$ containing $[F_{\widetilde{\beta}}] \in \mathrm{C}_{\widetilde{\beta}}$.

Example 3.3. *Consider the braid $\beta = \sigma_3^2\sigma_4 \in B_5$. The braid β is increasing in the middle. The skew-palindromization $\widetilde{\beta} = \sigma_1\sigma_2^2\sigma_3^2\sigma_4 \in B_5$ is pseudo-Anosov, see the proof of Step 1 in* [HK20a, Proof of Theorem D]. *See Figure 7(2) for the braided link.*

Theorem 3.4. *Let $\beta \in B_{2n+1}$ be a braid increasing in the middle. Suppose that the skew-palindromization $\widetilde{\beta}$ of β is pseudo-Anosov. Let $(x,y) \in \mathrm{C}_{\widetilde{\beta}}$ be a primitive integral class. Then we have the following.*

(1) *There exists a braid* $\alpha_{(x,y)} \in B_{\|(x,y)\|+1}$ *increasing in the middle such that the monodromy* $\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}$ *of the fibration* $T_{\widetilde{\beta}} \to S^1$ *associated with* (x,y) *is given by the skew-palindromization* $\widetilde{\alpha_{(x,y)}}$ *of* $\alpha_{(x,y)}$. *(In particular* $\widetilde{\alpha_{(x,y)}}$ *is a pseudo-Anosov braid.)*

(2) *Let* $\alpha^{\bullet}_{(x,y)} \in B_{\|(x,y)\|}$ *be the braid obtained from* $\alpha_{(x,y)}$ *by removing the strand of the middle index. Then* $C(\beta^{\bullet}) = C(\alpha^{\bullet}_{(x,y)})$, *and hence* $\mathfrak{t}(\widetilde{\alpha^{\bullet}_{(x,y)}}) \in \mathcal{D}_{\frac{\|(x,y)\|}{2}-1}(M_{C(\beta^{\bullet})})$.

For the proof of Theorem 3.4, we need some preparations from [HK20a]. Let L be a link in S^3. Suppose that an unknot K is a component of L. Then the exterior $\mathcal{E}(K)$ is a solid torus (resp. the boundary of the exterior $\partial\mathcal{E}(K)$ is a torus). We take a disk D bounded by the longitude of a tubular neighborhood $\mathcal{N}(K)$ of K. We define a mapping class t_D defined on $\mathcal{E}(K)$ as follows. We cut $\mathcal{E}(K)$ along D. We have resulting two sides obtained from D, and reglue two sides by twisting 360 degrees so that the mapping class defined on the torus $\partial\mathcal{E}(K)$ is the right-handed Dehn twist about ∂D. We call such a mapping class t_D on $\mathcal{E}(K)$ the *disk twist about* D. For simplicity, we also call a representative of the mapping class t_D the *disk twist about* D, and denote it by the same notation

$$t_D : \mathcal{E}(K) \to \mathcal{E}(K).$$

For any integer ℓ, consider the homeomorphism

$$t_D^{\ell} : \mathcal{E}(K) \to \mathcal{E}(K).$$

Observe that t_D^{ℓ} converts the link L into a link $K \cup t_D^{\ell}(L-K)$ so that $S^3 \setminus L$ is homeomorphic to $S^3 \setminus (K \cup t_D^{\ell}(L-K))$. Then t_D^{ℓ} induces a homeomorphism $h_{D,\ell}$ between the exteriors of links L and $K \cup t_D^{\ell}(L-K)$:

$$h_{D,\ell} : \mathcal{E}(L) \to \mathcal{E}(K \cup t_D^{\ell}(L-K)).$$

Consider the braided link $L = \mathrm{br}(b) = A \cup \mathrm{cl}(b)$ for a braid b with the braid axis A. We consider the k-th power of the disk twist about the disk D_A bounded by the longitude of $\mathcal{N}(A)$:

$$t_{D_A}^{k} : \mathcal{E}(A) \to \mathcal{E}(A).$$

Note that $A \cup t_{D_A}^{k}(\mathrm{cl}(b)) = A \cup \mathrm{cl}(b\Delta^{2k}) = \mathrm{br}(b\Delta^{2k})$. Hence $h_{D_A,k}$ sends $\mathcal{E}(\mathrm{br}(b)) = \mathcal{E}(A \cup \mathrm{cl}(b))$ to $\mathcal{E}(\mathrm{br}(b\Delta^{2k})) = \mathcal{E}(A \cup \mathrm{cl}(b\Delta^{2k}))$.

Following [HK20a, Section 4.1], we next introduce a sequence of braided links $\{\mathrm{br}(b_p)\}_{p=1}^{\infty}$ obtained from an i-increasing braid $b \in B_n$ such that $T_{b_p} \simeq T_b$, i.e., the mapping tori T_{b_p} and T_b are homeomorphic to each other for each $p \geq 1$. We set $u = u(b,i)$ that is the intersection number of the pair (b,i). Let D be an associated disk of the pair (b,i). We take a disk twist

$$t_D : \mathcal{E}(\mathrm{cl}(b(i))) \to \mathcal{E}(\mathrm{cl}(b(i)))$$

so that the point of intersection $D \cap A$ becomes the center of the twisting about D, i.e., $t_D(D \cap A) = D \cap A$. It follows that

$$t_D(\mathrm{br}(b - b(i))) \cup \mathrm{cl}(b(i))$$

is a braided link of a j-increasing braid for some index j with $(n+u)$ strands. (cf. Figures 11 and 12 in [HK20a].) We define b_1 to be such a braid with $(n+u)$ strands. The trivial knot $t_D(A)(= A)$ becomes a braid axis of b_1. By definition of the disk twist, we have $T_{b_1} \simeq T_b$. We remark that there is some ambiguity in defining b_1. However the braid b_1 is well defined up to conjugate, see [HK20a, Section 4.1]. The conjugacy class of b_1 is denoted by $\langle b_1 \rangle$.

To define the braid b_p obtained from the above b for $p \geq 1$, we consider the p-th power

$$t_D^p : \mathcal{E}(\mathrm{cl}(b(i))) \to \mathcal{E}(\mathrm{cl}(b(i)))$$

using the above disk twist t_D. As in the case of $p = 1$,

$$t_D^p(\mathrm{br}(b - b(i))) \cup \mathrm{cl}(b(i))$$

is a braided link of an increasing braid for some index with $(n+pu)$ strands. We define $b_p \in B_{n+pu}$ to be such a braid with $n+pu$ strands. Then $T_{b_p} \simeq T_b$. As in the case of $p = 1$, the braid b_p is well defined up to conjugate. We denote by $\langle b_p \rangle$, the conjugacy class of such a braid b_p. We say that $\langle b_p \rangle$ (or a representative b_p) is *obtained from b by the disk twist t_D (p times).*

Now we suppose that a braid b is of the form $b = \widetilde{\beta}$, where $\beta \in B_{2n+1}$ is a braid increasing in the middle. Then $\widetilde{\beta}$ is also increasing in the middle. The following lemma describes a property of a representative of the conjugacy class $\langle (\widetilde{\beta})_p \rangle$ obtained from $\widetilde{\beta}$ by the disk twist p times.

Lemma 3.5. *Let $\beta \in B_{2n+1}$ be a braid increasing in the middle with the intersection number $u = u(\beta, n+1)$. We consider the skew-palindromization $\widetilde{\beta}$ (that is increasing in the middle). Let $\langle (\widetilde{\beta})_p \rangle$ be the conjugacy class of a braid obtained from $\widetilde{\beta}$ by the disk twist p times for $p \geq 1$. We have the following.*

1. *There exists a braid $\alpha = \alpha(p) \in B_{2n+2pu+1}$ increasing in the middle such that $\widetilde{\alpha} = \widetilde{\alpha(p)} \in \langle (\widetilde{\beta})_p \rangle$, i.e., $\widetilde{\alpha} = \widetilde{\alpha(p)}$ represents the conjugacy class $\langle (\widetilde{\beta})_p \rangle$.*
2. *$C(\beta^\bullet) = C(\alpha^\bullet)$, where $\alpha^\bullet = \alpha(p)^\bullet \in B_{2n+2pu}$.*
3. *$\mathfrak{t}(\widetilde{\alpha^\bullet}) \in \mathcal{D}_g(M_{C(\beta^\bullet)})$, where $g = n + pu - 1$.*

Proof. (1) Figure 8 illustrates the procedure of the proof. (See also Example 3.3.) We put $\mathrm{cl}(\widetilde{\beta})$ in $D^2 \times S^1 = D^2 \times ([0,1]/0 \sim 1)$ so that $\mathrm{cl}(\widetilde{\beta}) \cap D^2 \times [0, 1/2] = \beta$, $\mathrm{cl}(\widetilde{\beta}) \cap D^2 \times [1, 1/2] = \mathrm{skew}(\beta)$ and $\mathrm{cl}(\widetilde{\beta}(n+1))$ corresponds to (the center of D^2) $\times S^1$, as shown in (1) of Figure 8. Let

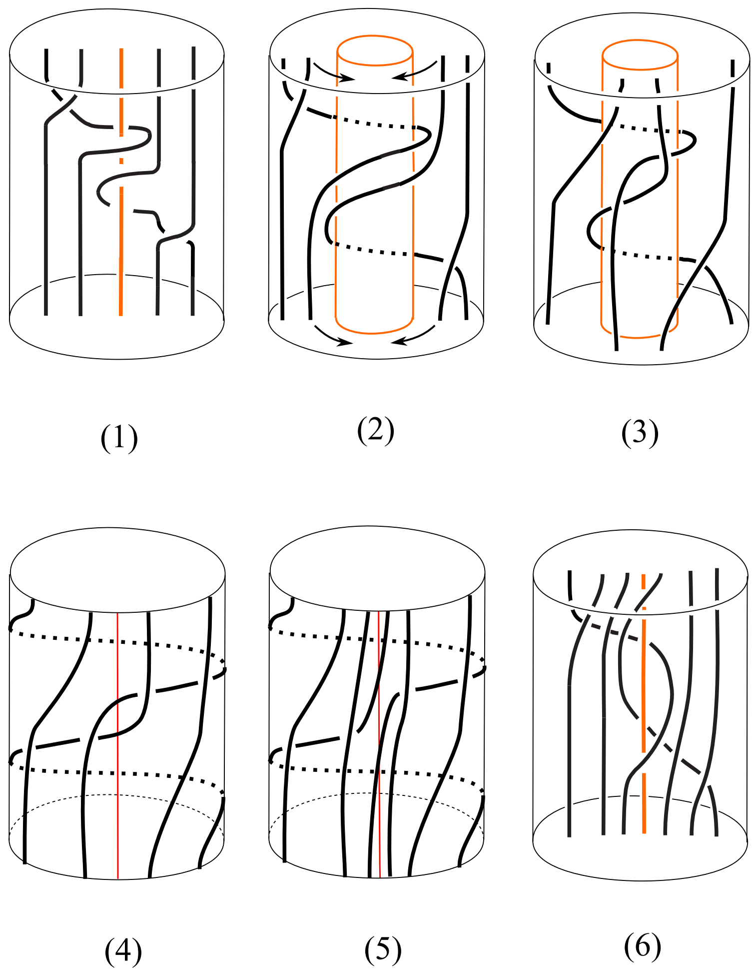


FIGURE 8. (1) The skew-palindromization $\widetilde{\beta}$ of $\beta = \sigma_3^2\sigma_4 \in B_5$. (2) The regular neighborhood of the middle strand (third strand) is removed. (3) The base points are moved into the circle $\{\frac{1}{2}e^{i\theta} \,|\, 0 \leq \theta \leq 2\pi\} \subset D^2$. (4) Project $\mathrm{cl}(\widetilde{\beta^\bullet})$ to the torus. (5) Dehn twist about the circle $c = \{-\frac{\pi}{2}\} \times [0,1]$ on the torus. (6) The skew-palindromization $\widetilde{\alpha(1)}$ of $\alpha(1) = \sigma_3\sigma_4\sigma_5\sigma_6\sigma_3 \in B_7$. (cf. Figure 7 in [HK20b].)

$R = R_{D^2 \times S^1} : D^2 \times S^1 \to D^2 \times S^1$ be the involution induced by the involution $R(re^{i\theta}, t) = (re^{i(\pi-\theta)}, 1-t)$ on $D^2 \times [0,1]$. Let $\mathcal{N}$ be the tubular neighborhood of (the center of D^2)$\times S^1$ such that $\mathrm{cl}(\widetilde{\beta}) \cap (D^2 \times S^1 \setminus \mathrm{int}\,\mathcal{N}) = \mathrm{cl}(\widetilde{\beta^\bullet})$, as shown in (2) of Figure 8. We identify $D^2 \times S^1 \setminus \mathrm{int}\,\mathcal{N}$ with $[0,1] \times S^1 \times S^1 = [0,1] \times ([0,2\pi]/0 \sim 2\pi) \times ([0,1]/0 \sim 1)$ so that $\partial\mathcal{N} = \{0\} \times S^1 \times S^1$, $\{\frac{1}{2}e^{i\theta} \,|\, 0 \leq \theta \leq 2\pi\} \times \{\mathrm{pt}\} = \{\frac{1}{2}\} \times S^1 \times \{\mathrm{pt}\}$, and for the disk D associated to the pair $(\widetilde{\beta}, n+1)$, we have $(D^2 \times S^1 \setminus \mathrm{int}\,\mathcal{N}) \cap D = [0,1] \times \{-\frac{\pi}{2}\} \times S^1$.

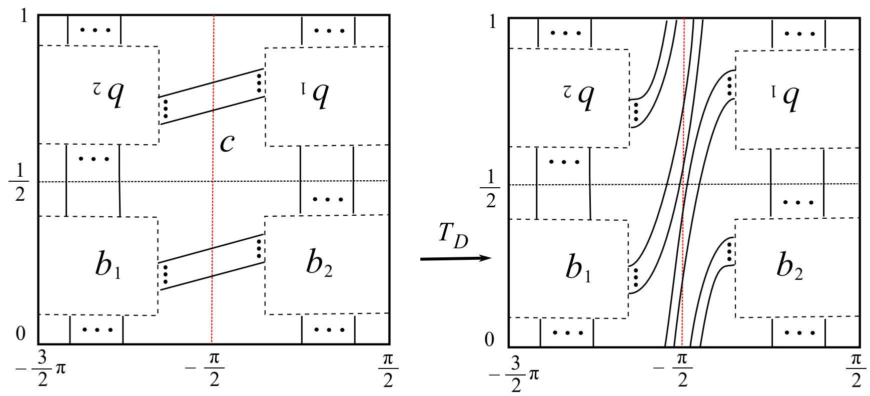


FIGURE 9. The projection $\mathcal{D}(\mathrm{cl}(\widetilde{\beta^\bullet}))$ consists of b_1, b_2 and slanted parallel arcs between them. The disk twist t_D induces a Dehn twist about $c = \{-\frac{\pi}{2}\} \times [0,1]$ on the torus. Compare Figure 8(4)(5) with Figure 9.

We deform the intersection of $\mathrm{cl}(\widetilde{\beta^\bullet})$ and $D^2 \times \{0\} = D^2 \times \{1\}$ together with the strings (along the arrows in (2) of Figure 8) by an isotopy commuting with the involution R so as to intersect $D^2 \times \{0\}$ in $2n$ points $\{\frac{1}{2}e^{i\theta} | \theta = -\frac{1}{2n+2}\pi, -\frac{2}{2n+2}\pi, \ldots, -\frac{n}{2n+2}\pi, -\frac{n+2}{2n+2}\pi, -\frac{n+3}{2n+2}\pi, \ldots, -\frac{2n+1}{2n+2}\pi\}$ and not to occur new intersections with $[0,1] \times \{-\frac{\pi}{2}\} \times S^1$. See also (3) of Figure 8. We make a projection $\mathcal{D}(\mathrm{cl}(\widetilde{\beta^\bullet}))$ of $\mathrm{cl}(\widetilde{\beta^\bullet})$ onto $\{\frac{1}{2}\} \times S^1 \times S^1$ with at most double points, and each double point indicates which is over pass or under pass in the same way as a knot diagram, as shown in (4) of Figure 8. For short, we drop $\{\frac{1}{2}\}$ from $\{\frac{1}{2}\} \times S^1 \times S^1$ and identify it with a torus $S^1 \times S^1 = ([-\frac{3}{2}\pi, \frac{1}{2}\pi]/-\frac{3}{2}\pi \sim \frac{1}{2}\pi) \times ([0,1]/0 \sim 1)$. With this identification, the restriction of R on $\{\frac{1}{2}\} \times S^1 \times S^1$ is an involution which maps (θ, t) to $(-\pi-\theta, 1-t)$, i.e., a π-rotation about $(-\frac{\pi}{2}, \frac{1}{2})$. Since β is an increasing braid in the middle, $\mathcal{D}(\mathrm{cl}(\widetilde{\beta^\bullet}))$ intersects $-\frac{\pi}{2} \times [\frac{1}{2}, 1]$ in $u = u(\beta, n+1)$ points and $-\frac{\pi}{2} \times [0, \frac{1}{2}]$ in $u = u(\beta, n+1)$ points as shown in Figure 9. The disk twist t_D induces the Dehn twist about the circle $c = \{-\frac{\pi}{2}\} \times [0,1]$ in $S^1 \times S^1$ which commutes with $R|_{S^1 \times S^1}$. This Dehn twist changes $\mathcal{D}(\mathrm{cl}(\widetilde{\beta^\bullet}))$ as shown in the right of Figure 9 and (5) of Figure 8. Now we append (the center of $D^2) \times S^1$ to the above result, that is, append an under-going string $\{-\frac{\pi}{2}\} \times [0,1]$ in the right of Figure 9 and regard the new diagram as the closure of the planar braid, as shown in (6) of Figure 8. Let $\alpha(p)$ be the braid indicated by the lower part of the result. Then the upper part corresponds to $\mathrm{skew}(\alpha(p))$, and hence $\widetilde{\alpha(p)}$ is a representative of $\langle \widetilde{\beta_p} \rangle$. The proof of (1) is done.

(2) We set $b = \beta^\bullet \in B_{2g+2}$ and $f = (t_c)^p : S^1 \times S^1 \to S^1 \times S^1$, following the notations in [HK20b, p.1811]. Let $\Phi_f = f \times \mathrm{id}_{[-1,1]}$ then, by (1), $\Phi_f(\mathrm{cl}(\widetilde{b})) = \Phi_f(\mathrm{cl}(\widetilde{\beta^\bullet})) = \mathrm{cl}(\widetilde{\alpha(p)^\bullet})$. Therefore, $\gamma = \widetilde{\alpha(p)^\bullet}$ and $b_f = \alpha(p)^\bullet = \alpha^\bullet$ satisfy

the assumptions $(*)$ and $(**)$ in [HK20b, p.1811]. By Lemma 4 of [HK20b], we conclude that $C(\beta^\bullet)$ and $C(\alpha^\bullet)$ are ambient isotopic. This completes the proof of (2).

(3) The claim (3) holds from the claim (2) and Theorem 2.8. This completes the proof of Lemma 3.5. $\square$

Remark 3.6. *The proof of Lemma 3.5 tells us that one can describe the braid $\alpha(p)$ in Lemma 3.5 concretely. In fact it is possible to read off $\alpha(p)$ from Figure 9. For example, in the case $\beta = \sigma_3^2\sigma_4 \in B_5$ as in Figure 8(1), the representative $\widetilde{\alpha(p)} \in \langle(\widetilde{\beta})_p\rangle$ is the skew-palindromization of the braid $\alpha(p) = \sigma_3\sigma_4 \dots \sigma_{4+2p}\sigma_{2+p} \in B_{5+2p}$ that is increasing in the middle.*

Let $\beta \in B_{2n+1}$ be the braid increasing in the middle with the intersection number $u = u(\beta, n+1)$ as before. By Lemma 2.13 and Example 3.2, the product $\beta\Delta^k \in B_{2n+1}$ of β and Δ^k for $k \geq 1$ is also increasing in the middle such that

$$u(\beta\Delta^k, n+1) = u(\beta, n+1) + u(\Delta^k, n+1) = u + kn.$$

Consider the skew-palindromization $\widetilde{\beta\Delta^k}$ of $\beta\Delta^k$. Recall that $\mathrm{skew}(\Delta) = \Delta$ (see Section 2.1), and hence $\mathrm{skew}(\Delta^k) = \Delta^k$. Thus it follows that

$$\widetilde{\beta\Delta^k} = \mathrm{skew}(\beta\Delta^k)\beta\Delta^k = \mathrm{skew}(\Delta^k)\mathrm{skew}(\beta)\beta\Delta^k = \Delta^k\widetilde{\beta}\Delta^k.$$

By Examples 3.1 and 3.2, $\widetilde{\beta\Delta^k}$ is a braid increasing in the middle with $u(\widetilde{\beta\Delta^k}, n+1) = 2u(\beta\Delta^k, n+1) = 2u + 2kn$. Since $\Delta^k\widetilde{\beta}\Delta^k$ is conjugate to $\widetilde{\beta}\Delta^{2k}$ in B_{2n+1}, we have

$$\mathrm{br}(\widetilde{\beta\Delta^k}) = \mathrm{br}(\Delta^k\widetilde{\beta}\Delta^k) = \mathrm{br}(\widetilde{\beta}\Delta^{2k}).$$

Lemma 3.7. *Let $\beta \in B_{2n+1}$ be a braid increasing in the middle with the intersection number $u = u(\beta, n+1)$. For $k \geq 1$, we consider $\beta\Delta^k \in B_{2n+1}$ and its skew-palindromization $\widetilde{\beta\Delta^k}$. Let $\langle(\widetilde{\beta\Delta^k})_p\rangle$ be the conjugacy class of a braid obtained from $\widetilde{\beta\Delta^k}$ by the disk twist p times for $p \geq 1$. We have the following.*

1. *There exists a braid $\gamma = \gamma(k,p) \in B_{2n+1+p(2u+2kn)}$ increasing in the middle such that $\widetilde{\gamma} = \widetilde{\gamma(k,p)} \in \langle(\widetilde{\beta\Delta^k})_p\rangle$.*
2. *$C(\beta^\bullet) = C(\gamma^\bullet)$, where $\gamma^\bullet = \gamma(k,p)^\bullet \in B_{2n+p(2u+2kn)}$.*
3. *$\mathfrak{t}(\widetilde{\gamma^\bullet}) \in \mathcal{D}_g(M_{C(\beta^\bullet)})$ for $g = n + pu + pkn - 1 \equiv pu - 1 \pmod{n}$.*

Proof. Recall that for each $k \geq 1$, $\beta\Delta^k \in B_{2n+1}$ is a braid increasing in the middle with the intersection number $u(\beta\Delta^k, n+1) = u + kn$. The claim (1) follows immediately from Lemma 3.5(1). For the proof of the claim (2), note that

$$(\beta\Delta^k)^\bullet = \beta^\bullet(\Delta^k)^\bullet = \beta^\bullet\Delta_{2n}^k \in B_{2n},$$

see Example 3.2. Thus $C((\beta\Delta^k)^\bullet) = C(\beta^\bullet\Delta_{2n}^k) = C(\beta^\bullet)$, see (2.1) in Section 2.3 for the second equality. By Lemma 3.5(2), we have $C((\beta\Delta^k)^\bullet) = C(\gamma^\bullet)$.

Putting them together, we obtain $C(\beta^\bullet) = C(\gamma^\bullet)$. The proof of (2) is done. The claim (3) holds from the claim (2) and Theorem 2.8. This completes the proof. □

We are now ready to prove Theorem 3.4.

Proof of Theorem 3.4. By the assumption of Theorem 3.4, $\widetilde{\beta}$ is pseudo-Anosov, and it is increasing in the middle. By Theorem 2.16(1)(2), the subcone $\mathrm{C}_{\widetilde{\beta}} = \mathrm{C}_{(\widetilde{\beta},n+1)}$ is a subset of the fibered cone $\mathscr{C}$ containing $[F_{\widetilde{\beta}}]$. Moreover the fiber $F_{(x,y)}$ for each primitive integral class $(x,y) \in \mathrm{C}_{\widetilde{\beta}}$ has genus 0.

Let $D = D_{(\widetilde{\beta},n+1)}$ be the associated disk of the braid $\widetilde{\beta}$ increasing in the middle. We consider two types of the disk twists. One is $t^k_{D_A} : \mathcal{E}(A) \to \mathcal{E}(A)$ for the braid axis A of $\widetilde{\beta}$, and the other is $t^p_D : \mathcal{E}(\mathrm{cl}(\widetilde{\beta}(n+1))) \to \mathcal{E}(\mathrm{cl}(\widetilde{\beta}(n+1)))$, where $\widetilde{\beta}(n+1)$ is the middle strand of the $(2n+1)$-braid $\widetilde{\beta}$. Consider the homeomorphisms

$$\begin{aligned} h_{D_A,k} &: \mathcal{E}(\mathrm{br}(\widetilde{\beta})) \to \mathcal{E}(\mathrm{br}(\widetilde{\beta}\Delta^{2k})) = \mathcal{E}(\mathrm{br}(\widetilde{\beta\Delta^k})), \\ h_{D,p} &: \mathcal{E}(\mathrm{br}(\widetilde{\beta})) \to \mathcal{E}(\mathrm{br}((\widetilde{\beta})_p)) \simeq \mathcal{E}(\mathrm{br}(\widetilde{\alpha(p)})), \end{aligned}$$

where $\widetilde{\alpha(p)}$ is the braid obtained from Lemma 3.5(1). We obtain the skew-palindromization $\widetilde{\beta\Delta^k} = \Delta^k\widetilde{\beta}\Delta^k$ (that is increasing in the middle) from the former homeomorphism $h_{D_A,k}$. We also obtain the skew-palindromization $\widetilde{\alpha(p)}$ (that is increasing in the middle) from the latter homeomorphism $h_{D,p}$. Both braids are pseudo-Anosov, since the exteriors of the links $\mathrm{br}(\widetilde{\beta})$, $\mathrm{br}(\widetilde{\beta\Delta^k})$ and $\mathrm{br}(\widetilde{\alpha(p)})$ are homeomorphic to each other. Hence one can further apply two types of the disk twists for each of the two braids $\widetilde{\beta\Delta^k}$ and $\widetilde{\alpha(p)}$. Then the resulting braids are again the skew-palindromization of some braids that are increasing in the middle by Lemmas 3.5(1) and 3.7(1). Choosing two types of the disk twists alternatively, one obtains a family of skew-paindromizations (of some braids) that are increasing in the middle. By the proof of Theorem 3.2(3) in [HK20a], the monodromy $\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}$ of the fibration $T_{\widetilde{\beta}} \to S^1$ associated with any primitive integral class $(x,y) \in \mathrm{C}_{\widetilde{\beta}}$ is given by a braid, say the skew-palindromization $\widetilde{\alpha_{(x,y)}}$ of some braid $\alpha_{(x,y)}$ in the family. The planar braid $\alpha_{(x,y)}$ is the desired braid. Let $2N+1$ be the number of the strands of $\alpha_{(x,y)}$ that is increasing in the middle. Since the Thurston norm $\|(x,y)\|$ of the class (x,y) is the negative Euler characteristic of the $(2N+1)$-punctured disk that is equal to $2N$. Thus $2N+1 = \|(x,y)\|+1$ and hence $\alpha_{(x,y)} \in B_{\|(x,y)\|+1}$. This completes the proof of (1).

The claim (2) follows from Lemma 3.5(2)(3) and Lemma 3.7(2)(3) together with the above argument in the proof of (1). □

4. Applications

For the proofs of Theorems 1.1 and 1.2, we first prove the following result.

Proposition 4.1. *Let L be a link in S^3. Let $\beta_{(1)}, \dots, \beta_{(n)} \in B_{2n+1}$ be increasing in the middle for some $n \geq 2$. Suppose that $\beta_{(1)}, \dots, \beta_{(n)}$ satisfy the following conditions (1)–(3): For each $j = 1, \dots, n$,*

1. *$u(\beta_{(j)}, n+1) \equiv j \pmod{n}$, where $u(\beta_{(j)}, n+1)$ is the intersection number of the pair $(\beta_{(j)}, n+1)$.*
2. *$L = C(\beta^{\bullet}_{(j)})$, where $\beta^{\bullet}_{(j)} \in B_{2n}$.*
3. *The skew-palindromization $\widetilde{\beta_{(j)}}$ of $\beta_{(j)}$ is pseudo-Anosov.*

Then we have $\ell_g(M_L) \asymp \dfrac{1}{g}$.

Proof. We fix $j \in \{1, \dots, n\}$ for a moment, and apply Lemma 3.7 to $\beta_{(j)}$, $k \geq 1$ and $p = 1$. Let $\gamma(k,1)$ be a braid increasing in the middle given by Lemma 3.7(1). By [HK20a, Theorem 5.2], a representative of $\langle (\widetilde{\beta_{(j)}\Delta^k})_1 \rangle$ gives the monodromy $\phi_{(k+1,1)} : F_{(k+1,1)} \to F_{(k+1,1)}$ corresponding to the primitive integral class $(k+1, 1) \in C_{\widetilde{\beta_{(j)}}}$ of the fibered 3-manifold $T_{\widetilde{\beta_{(j)}}}$. See Figure 7(1) for the class $(k+1,1)$. In particular the representative $\widetilde{\gamma(k,1)} \in \langle (\widetilde{\beta_{(j)}\Delta^k})_1 \rangle$ gives the monodromy $\phi_{(k+1,1)} : F_{(k+1,1)} \to F_{(k+1,1)}$ of the fibration $T_{\widetilde{\beta_{(j)}}} \to S^1$. Hence we can say that $\widetilde{\gamma(k,1)}$ is a braid with $\|(k+1,1)\| + 1$ strands. (Recall that $\|(x,y)\|$ is the Thurston norm of the class (x,y).)

Note that the ray of the class $(k+1,1) = (k+1)(1, \frac{1}{k+1})$ through the origin converges to the ray of $(1,0)$ as $k \to \infty$. This together with Theorem 2.11(2) implies that

$$\mathrm{Ent}(\widetilde{\gamma(k,1)}) = \mathrm{Ent}((k+1,1)) = \mathrm{Ent}((1, \tfrac{1}{k+1})) \to \mathrm{Ent}((1,0)) \text{ as } k \to \infty.$$

Since the monodromy on the fiber $F_{(1,0)} = F_{\widetilde{\beta_{(j)}}}$ is given by $\widetilde{\beta_{(j)}}$,

$$\mathrm{Ent}(\widetilde{\gamma(k,1)}) = \mathrm{Ent}((k+1,1)) \to \mathrm{Ent}((1,0)) = \mathrm{Ent}(\widetilde{\beta_{(j)}}) \text{ as } k \to \infty. \tag{4.1}$$

By [HK20a, Lemma 6.3], for k large, $\widetilde{\gamma(k,1)}^{\bullet} \in B_{\|(k+1,1)\|}$ is pseudo-Anosov with the same dilatation as $\widetilde{\gamma(k,1)}$. By the arguments in the proof of [HK20a, Lemma 6.3], one sees that for k large, the pseudo-Anosov braid $\widetilde{\gamma(k,1)}^{\bullet}$ satisfies the condition $\diamondsuit$ in Lemma 2.9. Therefore, for k large, $\mathfrak{t}(\widetilde{\gamma(k,1)}^{\bullet})$ is still pseudo-Anosov with the same dilatation as $\widetilde{\gamma(k,1)}^{\bullet}$. Then by Theorem 2.8, it holds $\mathfrak{t}(\widetilde{\gamma(k,1)}^{\bullet}) \in \mathcal{D}_{\frac{\|(k+1,1)\|}{2}-1}(M_L)$, where $L = C(\beta^{\bullet}_{(j)}) = C(\gamma(k,1)^{\bullet})$ by Lemma 3.7(2). Putting them together, we have

$$\lambda(\mathfrak{t}(\widetilde{\gamma(k,1)}^{\bullet})) = \lambda(\widetilde{\gamma(k,1)}^{\bullet}) = \lambda(\widetilde{\gamma(k,1)}) = \lambda((k+1,1)),$$

where $\lambda((x,y))$ denotes the dilatation of the class (x,y), i.e., $\log(\lambda((x,y))) = \mathrm{ent}((x,y))$. (See Theorem 2.11(1).) Since $\mathfrak{t}(\widetilde{\gamma(k,1)^\bullet})$ is the mapping class on the closed surface of genus $\frac{\|(k+1,1)\|}{2} - 1$, we have

$$
\begin{aligned}
\mathrm{Ent}(\mathfrak{t}(\widetilde{\gamma(k,1)^\bullet})) &= (\|(k+1,1)\| - 4)\log(\lambda(\mathfrak{t}(\widetilde{\gamma(k,1)^\bullet}))) \\
&= (\|(k+1,1)\| - 4)\mathrm{ent}((k+1,1)).
\end{aligned}
\tag{4.2}
$$

Claim 1. $\mathrm{ent}((k+1,1)) \to 0$ as $k \to \infty$.

Proof of Claim 1. By (4.1), $\mathrm{Ent}((k+1,1))(= \|(k+1,1)\|\mathrm{ent}((k+1,1))) \to \mathrm{Ent}((1,0))$ as $k \to \infty$. This implies that there exists a constant $P > 0$ independent of k such that

$$0 < \|(k+1,1)\|\mathrm{ent}((k+1,1)) < P$$

for all $k \geq 1$. Since $\|(k+1,1)\| \to \infty$ as $k \to \infty$, we obtain

$$\mathrm{ent}((k+1,1)) < \frac{P}{\|(k+1,1)\|} \to 0 \ \text{ as } k \to \infty.$$

This completes the proof of Claim 1.

By (4.2), Claim 1 and (4.1), one has

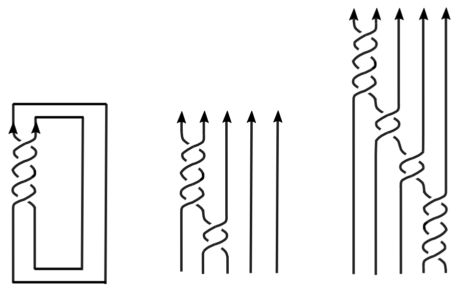
$$
\begin{aligned}
\lim_{k\to\infty} \mathrm{Ent}(\mathfrak{t}(\widetilde{\gamma(k,1)^\bullet})) &= \lim_{k\to\infty} (\|(k+1,1)\| - 4)\mathrm{ent}((k+1,1)) \\
&= \lim_{k\to\infty} \|(k+1,1)\|\mathrm{ent}((k+1,1)) \\
&= \lim_{k\to\infty} \mathrm{Ent}((k+1,1)) \ \ (\because \text{definition of } \mathrm{Ent}(\cdot)) \\
&= \mathrm{Ent}(\widetilde{\beta_{(j)}}).
\end{aligned}
\tag{4.3}
$$

On the other hand, Lemma 3.7(3) tells us that $\mathfrak{t}(\widetilde{\gamma(k,1)^\bullet}) \in \mathcal{D}_g(M_L)$, where

$$g = n + u(\beta_{(j)}, n+1) + kn - 1 \equiv u(\beta_{(j)}, n+1) - 1 \equiv j - 1 \pmod{n}. \tag{4.4}$$

(See the condition (1) of Proposition 4.1.) For $k \geq 1$, consider the set of all pseudo-Anosov mapping classes $\mathfrak{t}(\widetilde{\gamma(k,1)^\bullet})$ obtained from $\beta_{(j)}$ over all $j = 1, \ldots, n$. Then by (4.4) together with the condition (1) of Proposition 4.1, one can find a sequence of pseudo-Anosov elements $\phi_g \in \mathcal{D}_g(M_L)$ for all $g \gg 0$ in this set. In fact when $g \equiv j-1 \pmod{n}$, one can put $\phi_g = \mathfrak{t}(\widetilde{\gamma(k,1)^\bullet})$ obtained from the braid $\beta_{(j)}$, where k satisfies the equality (4.4). Since each of braids $\widetilde{\beta_{(1)}}, \ldots, \widetilde{\beta_{(n)}}$ satisfies (4.3), there exists a constant $C' > 0$ independent of g so that $\ell_g(M_L) \leq \log(\lambda(\phi_g)) \leq \frac{C'}{g}$.

The result $\ell(\mathrm{MCG}(\Sigma_g)) \asymp \frac{1}{g}$ by Penner [Pen91] tells us that there exists a constant $C > 0$ independent of g so that $\frac{1}{Cg} \leq \ell_g(M_L)$. We conclude that $\ell_g(M_L) \asymp \frac{1}{g}$. This completes the proof. $\square$



(1) (2) (3)

FIGURE 10. Case $b = \sigma_1^4 \in B_2$. (1) $\mathrm{cl}(b)$ for $b \in B_2$. (2) $\eta_{(b,1)} = \bar{b}\sigma_2^2 = \sigma_1^4\sigma_2^2 \in B_5$. (3) $\eta_{(b,1)} \cdot \mathrm{skew}(\eta_{(b,1)}) = \sigma_1^4\sigma_2^2\sigma_3^2\sigma_4^4 \in B_5$.

Theorem 4.2. *Let $b \in B_n$ be a pure braid for $n \geq 2$ of the form*

$$b = \sigma_{j_1}^{2m_1}\sigma_{j_2}^{2m_2}\dots\sigma_{j_k}^{2m_k},$$

where $m_1, \dots, m_k$ are non-zero integers and $j_1, \dots, j_k \in \{1, \dots, n-1\}$. Suppose that b is homogeneous, and each σ_i for $i = 1, \dots, n-1$ appears in b at least once, i.e., $\{j_1, \dots, j_k\} = \{1, \dots, n-1\}$. Then for the 2-fold branched cover $M_{\mathrm{cl}(b)}$ of S^3 branched over the closure $\mathrm{cl}(b)$ of b, we have $\ell_g(M_{\mathrm{cl}(b)}) \asymp \dfrac{1}{g}$. In particular, if a pure braid $b \in B_n$ is of the form

$$b = \sigma_1^{2m_1}\sigma_2^{2m_2}\cdots\sigma_{n-1}^{2m_{n-1}},$$

then we have $\ell_g(M_{\mathrm{cl}(b)}) \asymp \dfrac{1}{g}$.

To prove Theorem 4.2, we need the following lemma.

Lemma 4.3. *Let $b = \sigma_{j_1}^{2m_1}\sigma_{j_2}^{2m_2}\dots\sigma_{j_k}^{2m_k} \in B_n$ be a pure braid for $n \geq 2$ with the same assumption as in Theorem 4.2. Let $\bar{b}$ be a $(2n+1)$-braid with the same braid word as b. We take a braid*

$$\eta_{(j)} = \eta_{(b,j)} := \bar{b}\sigma_n^{2j} = (\sigma_{j_1}^{2m_1}\sigma_{j_2}^{2m_2}\cdots\sigma_{j_k}^{2m_k})\sigma_n^{2j} \in B_{2n+1}$$

for a positive integer j. Then $\eta_{(j)}$ is increasing in the middle with the intersection number $u(\eta_{(j)}, n+1) = j$, and the braid

$$\eta_{(j)} \cdot \mathrm{skew}(\eta_{(j)}) = (\sigma_{j_1}^{2m_1}\cdots\sigma_{j_k}^{2m_k})\sigma_n^{2j} \cdot \sigma_{n+1}^{2j}(\sigma_{2n+1-j_k}^{2m_k}\cdots\sigma_{2n+1-j_1}^{2m_1}) \in B_{2n+1}$$

is pseudo-Anosov.

Proof. By the definition of $\eta_{(j)}$, it is easy to check that $\eta_{(j)}$ is increasing in the middle with $u(\eta_{(j)}, n+1) = j$. The braid $\eta_{(j)} \cdot \mathrm{skew}(\eta_{(j)}) \in B_{2n+1}$ satisfies the assumption of Lemma 2.5, and hence it is pseudo-Anosov. $\square$

Example 4.4. *If $b = \sigma_1^4 \in B_2$, then $\eta_{(1)} = \eta_{(b,1)} = \bar{b}\sigma_2^2 = \sigma_1^4\sigma_2^2 \in B_5$. By Lemma 4.3, $\eta_{(1)} \cdot \mathrm{skew}(\eta_{(1)}) = \sigma_1^4\sigma_2^2\sigma_3^2\sigma_4^4 \in B_5$ is pseudo-Anosov. See Figure 10.*

Let us turn to the proof of Theorem 4.2.

Proof of Theorem 4.2. We consider the braid $\eta_{(j)} = \eta_{(b,j)} \in B_{2n+1}$ as in Lemma 4.3 for each $j = 1, \dots, n$. By Lemma 2.14, $\mathrm{skew}(\eta_{(j)}) \in B_{2n+1}$ is a braid increasing in the middle, and $u(\eta_{(j)}, n+1) = u(\mathrm{skew}(\eta_{(j)}), n+1) = j$. Note that

$$\eta_{(j)}^\bullet = \sigma_{j_1}^{2m_1}\sigma_{j_2}^{2m_2} \dots \sigma_{j_k}^{2m_k} \in B_{2n}$$

with the same word as the braid $b \in B_n$ for $j = 1, \dots, n$. Then we have

$$C(\Big(\mathrm{skew}(\eta_{(j)})\Big)^\bullet) = C(\eta_{(j)}^\bullet) = \mathrm{cl}(b)$$

as links in S^3. By Lemma 4.3, $\eta_{(j)} \cdot \mathrm{skew}(\eta_{(j)}) \in B_{2n+1}$ is pseudo-Anosov for $j = 1, \dots, n$. Notice that $\eta_{(j)} \cdot \mathrm{skew}(\eta_{(j)})$ is the skew-palindromization of $\mathrm{skew}(\eta_{(j)})$ (since $\mathrm{skew} : B_n \to B_n$ is an involution). Hence the braids $\mathrm{skew}(\eta_{(1)}), \dots, \mathrm{skew}(\eta_{(n)}) \in B_{2n+1}$ satisfy the conditions (1)–(3) of Proposition 4.1, where $L = C(\Big(\mathrm{skew}(\eta_{(j)})\Big)^\bullet) = \mathrm{cl}(b)$. Thus $\ell_g(M_{\mathrm{cl}(b)}) \asymp \dfrac{1}{g}$. This completes the proof. $\square$

Theorem 1.1 follows from the following result.

Corollary 4.5. *For the lens space $L_{(2m,1)}$ of type $(2m, 1)$ with $m \neq 0$, we have $\ell_g(L_{(2m,1)}) \asymp \dfrac{1}{g}$.*

Proof. The closure $\mathrm{cl}(\sigma_1^{2m})$ of the 2-braid σ_1^{2m} with $m \neq 0$ is the $(2m, 2)$-torus link T. (See Figure 10(1) when $m = 2$.) The 2-fold branched cover $M_T = M_{\mathrm{cl}(\sigma_1^{2m})}$ of S^3 branched over T is the lens space $L_{(2m,1)}$ of the type $(2m, 1)$. See [Rol90, p. 302] for example. This together with Theorem 4.2 completes the proof. $\square$

Theorem 4.6. *Let $\sharp_n S^2 \times S^1$ denote the connected sum of n copies of $S^2 \times S^1$. For each $n \geq 1$, we have $\ell_g(\sharp_n S^2 \times S^1) \asymp \dfrac{1}{g}$.*

Proof. For the n-component trivial link E_n, we have $M_{E_n} = \sharp_{n-1} S^2 \times S^1$. See [Rol90, p. 300] for example. Recall that $E_n = C(e_{2n})$ for the identity element $e_{2n} \in B_{2n}$. To prove Theorem 4.6, we check that $\ell_g(M_{C(e_{2n})})) \asymp \frac{1}{g}$.

For $n \geq 2$, we take a braid $\beta \in B_{2n+1}$ increasing in the middle with the following properties.

$$u(\beta, n+1) = 2n \equiv 0 \pmod{n} \text{ and } \beta^\bullet = e_{2n} \in B_{2n}.$$

One can choose such a braid β as follows.

$$\beta = \sigma_{n+1}\sigma_{n+2} \cdots \sigma_{2n-1}\sigma_{2n}^4\sigma_{2n-1}^3 \cdots \sigma_{n+2}^3\sigma_{n+1}^3 \in B_{2n+1}.$$

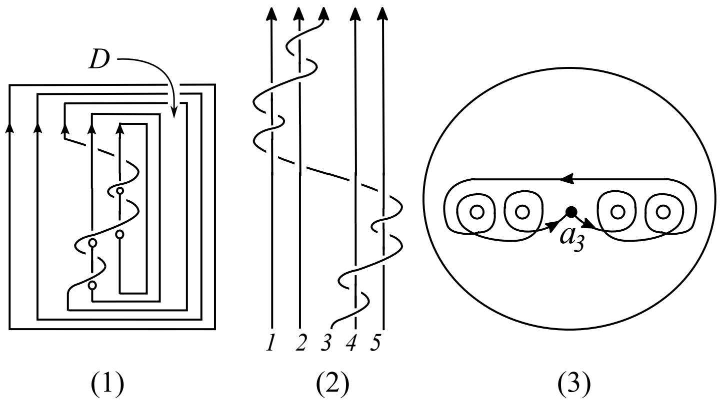


FIGURE 11. (1) The closure $\mathrm{cl}(\beta)$ for $\beta = \sigma_3\sigma_4^4\sigma_3^3 \in B_5$ increasing in the middle with $u(\beta, 3) = 4$. (D is an associated disk of the pair $(\beta, 3)$.) (2) The skew-palindromization $\widetilde{\beta}$ of β. (3) The closed curve $\gamma_{\widetilde{\beta}}$ (based at a_3) in $D_4 = D^2 \setminus \{a_1, a_2, a_4, a_5\}$.

See Figure 11(1) for the braid $\beta = \sigma_3\sigma_4^4\sigma_3^3 \in B_5$ when $n = 2$. Consider the skew-palindromization $\widetilde{\beta}$ of β, see Figure 11(2). Since $\beta^\bullet = e_{2n} \in B_{2n}$, it holds $\widetilde{\beta}^\bullet = e_{2n} \cdot e_{2n} = e_{2n} \in B_{2n}$.

Let $a_1, \dots, a_{2n+1}$ be base points of $\widetilde{\beta}$. The projection of the $(n+1)$-th strand $\widetilde{\beta}(n+1) \subset D^2 \times [0, 1]$ onto the first factor D^2 gives an oriented closed curve $\gamma_{\widetilde{\beta}}$ on the $2n$-punctured disk $D_{2n} = D^2 \setminus \{a_1, \dots, a_n, a_{n+2}, \dots, a_{2n+1}\}$, see Figure 11(2)(3). Note that the initial point of the closed curve $\gamma_{\widetilde{\beta}}$ corresponds to the base point a_{n+1} of $\widetilde{\beta}$. If we choose a braid β increasing in the middle as above, then one can check that $\gamma_{\widetilde{\beta}}$ fills D_{2n}. Here a closed curve $\gamma \subset D_N$ in an N-punctured disk D_N *fills* D_N if every loop that is freely homotopic to γ intersects every essential simple closed curve in D_N. By Kra's criterion [Kra81, Theorem 2'], $\widetilde{\beta} \in B_{2n+1}$ is pseudo-Anosov.

For each $j = 1, \dots, n$, we define a braid $\beta_{(j)} \in B_{2n+1}$ as follows: If $j = n$, then $\beta_{(n)} := \beta$. If $j = 1, \dots, n-1$, then $\beta_{(j)} := \beta\sigma_{n+1}^{2j}$. Notice that both braids $\beta \in B_{2n+1}$ and $\sigma_{n+1}^{2j} \in B_{2n+1}$ are increasing in the middles with the intersection numbers $2n$ and j respectively. Lemma 2.13 tells us that $\beta_{(j)} = \beta\sigma_{n+1}^{2j}$ is increasing in the middle with the intersection number $u(\beta_{(j)}, n+1) = 2n+j \equiv j \pmod n$. Moreover $\beta_{(j)}^\bullet = \beta^\bullet(\sigma_{n+1}^{2j})^\bullet = e_{2n} \cdot e_{2n} = e_{2n} \in B_{2n}$. Hence $E_n = C(\beta_{(j)}^\bullet)$. The closed curve $\gamma_{\widetilde{\beta_{(j)}}}$ still fills D_{2n}. Hence $\widetilde{\beta_{(j)}} \in B_{2n+1}$ for $j = 1, \dots, n$ is pseudo-Anosov by the same criterion of Kra. By Proposition 4.1, we obtain $\ell_g(M_{E_n}) \asymp \frac{1}{g}$. This completes the proof. $\square$

Finally we prove Theorem 1.2.

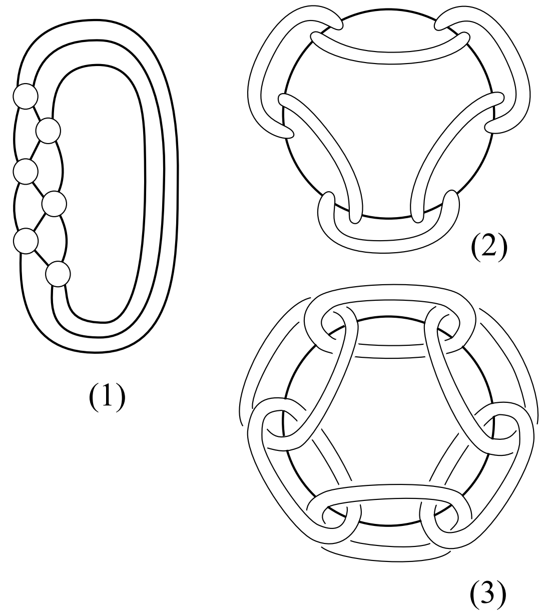


FIGURE 12. This figure illustrates the case where $k = 3$. (1) Let P be the complement of the $2k$ 3-balls containing twists. (2) We isotope (1) in order to have $2\pi/3$ symmetry. (3) The 2-fold branched cover of P branched over $P \cap \mathrm{cl}(b_{\boldsymbol{m}})$ (indicated by the thick circle) is homeomorphic to $S^3 - \mathcal{N}(\mathcal{C}_6)$, where $\mathcal{C}_6$ is the minimally twisted 6-chain link.

Proof of Theorem 1.2. Let $b_{\boldsymbol{m}} \in B_3$ be a pure braid of the form

$$b_{\boldsymbol{m}} = \sigma_1^{2m_1}\sigma_2^{2m_2}\sigma_1^{2m_3}\sigma_2^{2m_4}\dots\sigma_1^{2m_{2k-1}}\sigma_2^{2m_{2k}},$$

where $k \geq 3$ and $\boldsymbol{m} = (m_1, m_2, m_3, m_4, \dots, m_{2k}) \in (\mathbb{Z}_{>0})^{2k}$ is a $2k$-tuple of positive integers. Notice that $b_{\boldsymbol{m}}$ is homogeneous, and both σ_1 and σ_2 appear in $b_{\boldsymbol{m}}$. As shown in Figure 12, $M_{\mathrm{cl}(b_{\boldsymbol{m}})}$ is a closed 3-manifold obtained from the 3-sphere S^3 by Dehn surgery about the minimally twisted $2k$-chain link $\mathcal{C}_{2k}$. Let s_i $(i = 1, \dots, 2k)$ be the slope of this Dehn surgery. It is shown by Thurston [Thu79, Example 6.8.7] that $S^3 - \mathcal{C}_{2k}$ has a complete hyperbolic structure with $2k$ cusps. (See also [Pur11, Yos97].) Hence, we have $\mathrm{vol}(S^3 - \mathcal{C}_{2k}) > 2kv_3$, where $v_3 = 1.01494\dots$ is the volume of the ideal regular tetrahedron. See [Ada88, Theorem 7]. We consider the Euclidean structure on the torus boundary of $\mathcal{N}(\mathcal{C}_{2k})$ induced by a maximal disjoint horoball neighborhood about the cusps. Let λ be the minimum of the Euclidean lengths of the solpes s_i. By the 2π-theorem of Gromov-Thurston [BH96, Theorem 9], if $\lambda > 2\pi$, then $M_{\mathrm{cl}(b_{\boldsymbol{m}})}$ is hyperbolic. Furthermore, from the result by Futer-Kalfagianni-Purcell [FKP08, Theorem 1.1], we have

$$\left(1-\left(\frac{2\pi}{\lambda}\right)^2\right)^{3/2}\mathrm{vol}(S^3-\mathcal{C}_{2k}) \leq \mathrm{vol}(M_{\mathrm{cl}(b_{\boldsymbol{m}})}) < \mathrm{vol}(S^3-\mathcal{C}_{2k}).$$

For any $R>0$, if we choose k so that $2kv_3>R$ and each coordinate of $\boldsymbol{m}$ sufficiently large, then we have $\mathrm{vol}(M_{\mathrm{cl}(b_{\boldsymbol{m}})})>R$. Since the braid $b_{\boldsymbol{m}}$ satisfies the assumption in Theorem 4.2, we see $\ell_g(M_{\mathrm{cl}(b_{\boldsymbol{m}})}) \asymp \frac{1}{g}$. $\square$

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