CR geometry and ACH-Einstein metrics

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Davis

conformal — Poincaré-Einstein

CR (Cauchy-Riemann) — ACH-Einstein
asymp. complex hyperbolic

Conformal / CR structures

M^n $[g]$ conformal structure $\quad g \sim e^{2u} g \quad u \in C^\infty(M)$

M^{2n+1} (H, J) CR structure (in this talk) if :

* $H \subset TM$ contact distribution, $H^\perp$ orientable

  (i.e., $\exists\, \theta$ 1-form, $H = \ker \theta$, $\theta \wedge (d\theta)^n$ nowhere vanishing)

* $J \in \Gamma(\mathrm{End}(H))$, $J^2 = -\mathrm{id}$,

  $h_\theta(X, Y) := d\theta(X, JY)$, $\quad X, Y \in H$

  is symmetric in X, Y and positive definite

  (Replace θ with $-\theta$ if necessary)

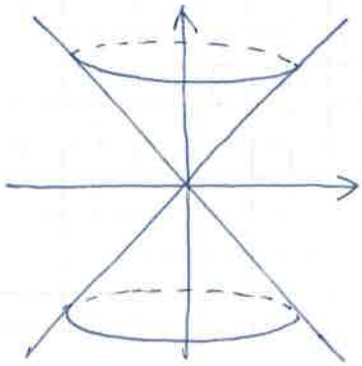
Rem, (1) Also called "strictly pseudoconvex partially integrable almost CR structure".

(H, J) integrable $\overset{\text{def}}{\Longleftrightarrow}$ $\Gamma(H^{1,0})$ closed under Lie bracket

(2) $\hat{\theta} = e^u \theta \rightsquigarrow h_{\hat{\theta}} = e^u h_\theta$

Standard models — spheres

* Conformal



$$C = \left\{ x \in \mathbb{R}^{n+2} \;\middle|\; x \neq 0,\ -(x^0)^2 + \sum_{i=1}^{n+1} (x^i)^2 = 0 \right\}$$

$$S^n = C / \mathbb{R}^*$$

$PO(n+1, 1)$ acts naturally

* CR

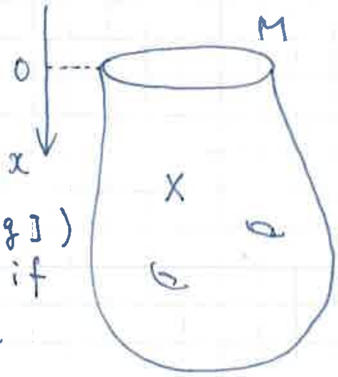
$$C = \left\{ z \in \mathbb{C}^{n+2} \;\middle|\; z \neq 0,\ -|z^0|^2 + \sum_{i=1}^{n+1} |z^i|^2 = 0 \right\}$$

$S^{2n+1} = C / \mathbb{C}^*$ $\quad$ $PU(n+1, 1)$ acts naturally

Conformal / CR manifolds as boundary at infinity

* Poincaré - Einstein metrics

$(M^n, [g])$ given.



g_+ is PE with conformal infinity $(M, [g])$ if

- $g_+ \sim \dfrac{dx^2 + g}{x^2}$ on $(0, \varepsilon) \times M^n$

  for some $g \in [g]$

- $\mathrm{Ric}(g_+) = -n g_+$

Model : $\mathbb{H}^{n+1}$. Conf. infinity $= S^n$

* ACH-Einstein metrics

(M^{2n+1}, H, J) given.

g_+ is <u>ACHE</u> with <u>conf. infinity</u> (M, H, J) if

- $g_+ \sim 4\dfrac{dx^2}{x^2} + \dfrac{\theta^2}{x^4} + \dfrac{h_\theta}{x^2}$ on $(0,\varepsilon) \times M^{2n+1}$

  for some contact form θ

- $\mathrm{Ric}(g_+) = -\dfrac{n+2}{2} g_+$

Model: $\mathbb{C}H^{n+1}$.

Existence & uniqueness issues

(1) Formal aspects

For a given conformal infinity,

* PE — Fefferman-Graham (1985)

$\exists\, g_+$ satisfying $\mathrm{Ric}(g_+) = \begin{cases} -n g_+ + O(x^\infty) & (n \text{ odd}) \\ -n g_+ + O(x^n) & (n \text{ even}) \end{cases}$

g_+ unique mod. diffeomorphisms and high-order terms

* ACHE — Fefferman (1976),
Biquard-Herzlich (2005), M. (2014)

(2) Global aspects

<u>Perturbative existence</u> * PE — Graham-Lee (1991)
* ACHE — Biquard (2000)
M. (preprint)

<u>Non-uniqueness</u> (PE) Hawking-Page (1983)
<u>Non-existence</u> (PE) Gursky-Han (2017)

$\Big]$ Not known for ACHE

Construction of conformal / CR-invariant diff. op's

* Conformal   Graham-Jenne-Mason-Sparling (1992)
Graham-Zworski (2003)

n even.

(X^{n+1}, g_+) approx. PE (of Fefferman-Graham)
$(M^n, [g])$ conf. infinity

Consider harmonic extension from boundary

$$f \in C^\infty(M) \rightsquigarrow \tilde f \in C^\infty(X), \quad \Delta_{g_+} \tilde f = 0,$$
$$\tilde f = f + F + x^n \log x \cdot G, \quad F, G \in C^\infty(\bar X), \quad F|_{\partial X} = 0$$

$$P_g : C^\infty(M) \to C^\infty(M) \qquad f \longmapsto c_n \cdot G|_{\partial X}$$

$$\rightsquigarrow \quad P_g = \Delta_g^{n/2} + \cdots$$
$$\hat g = e^{2u} g \qquad P_{\hat g} = e^{-nu} P_g$$

(GJMS op. of critical order)

<u>Ex.</u>
$$n=2 \qquad P_g = \Delta_g$$
$$n=4 \qquad P_g = \Delta_g^2 + \delta\left(\tfrac{2}{3} R \cdot \mathrm{id} - 2 \mathrm{Ric}\right) d \qquad \text{Paneitz op.}$$

* CR   Gover-Graham (2005), Hislop-Perry-Tang (2008), M. (2016)

(X^{2n+2}, g_+) approx. ACHE (M^{2n+1}, H, J)

$$f \in C^\infty(M) \rightsquigarrow \tilde f \in C^\infty(X), \quad \Delta_{g_+} \tilde f = 0,$$
$$\tilde f = f + F + x^{2n+2} \log x \cdot G, \quad F|_{\partial X} = 0$$
$$P_\theta : C^\infty(M) \to C^\infty(M) \qquad f \longmapsto c_n' \cdot G|_{\partial X}$$

CR-specific phenomena

(1) Obstruction tensor

* Conformal, n even — Fefferman-Graham (1985)
  $\mathcal{O} \in \Gamma(S^2 T^*M)$

* CR — M. (2014) $\mathcal{O} \in \Gamma(S^2_A H^*)$
  anti-hermitian
  $\mathcal{O}(JX, JY) = -\mathcal{O}(X, Y)$
  (or: $\mathcal{O} \in \Gamma(S^2(H^{1,0})^*)$)

Fact $\mathcal{O} \equiv 0$ for integrable CR structures.

(2) GJMS-type op. on 2-tensors (M. 2013, 2016)

* Conformal $P_g : \Gamma(S^2 T^*M) \to \Gamma(S^2 T^*M)$
* CR $P_\theta : \Gamma(S^2_A H^*) \to \Gamma(S^2_A H^*)$

Consider "harmonic extension" w.r.t. the linearized gauged Einstein operator.

Fact $P_\theta = B_\theta \circ N$, N linearized Nijenhuis op.

(3) P'-operator & Q'-curvature in CR geometry
Case-Yang (2014), Hirachi (2014)
Observation For integrable CR, P_θ annihilates CR-pluriharmonics.

"$P'_\theta := \lim_{N \to n} \frac{1}{N-n} \left(P_\theta \text{ of order } 2n+2 \text{ in dim } 2N+1 \right)$"

acting on CR-pluriharmonic functions. (locally Re(CR-fn.))

Associated "Q'-curvature" $Q'_\theta \in C^\infty(M)$ makes sense for pseudo-Einstein contact forms θ.