Asymptotically complex hyperbolic Einstein metrics and CR geometry

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CHAPTER 1

Introduction

1. Overview

1.1. Background. An approach to geometry of smoothly bounded strictly pseudoconvex domains is studying its relation to that of the Cauchy–Riemann (CR) structure of the boundary. There is a clear-cut theorem, which is now classical, that describes an aspect of this correspondence. Let Ω_1, $\Omega_2 \subset \mathbb{C}^{n+1}$, $n \geq 1$, be smoothly bounded strictly pseudoconvex domains. Using the Hartogs–Bochner Theorem [**Bo**], or more precisely a version of this theorem proved by Kohn–Rossi [**KR**], one can easily show (see [**BSW**]) that if there is a CR-diffeomorphism $f\colon \partial\Omega_1 \longrightarrow \partial\Omega_2$ between the boundaries, then it necessarily extends to a diffeomorphism $F\colon \overline{\Omega}_1 \longrightarrow \overline{\Omega}_2$ which is biholomorphic in Ω_1. Conversely, a celebrated work of Fefferman [**F1**] shows that, if $F\colon \Omega_1 \longrightarrow \Omega_2$ is biholomorphic, it extends to a diffeomorphism $\overline{\Omega}_1 \longrightarrow \overline{\Omega}_2$ and thus induces a CR-diffeomorphism between the boundaries.

Since Fefferman's proof of the latter direction is based on the analysis of the boundary behavior of the Bergman metric, one is naturally lead to a more detailed research of the Bergman kernel. It was Fefferman's idea (see [**F3, BFG**]) that one can study the asymptotic expansion of the Bergman kernel as an analogue of the heat kernel expansion in Riemannian geometry. The actual work of expressing the singular parts of the expansion in terms of boundary CR invariants was carried out by Fefferman himself [**F3**], Bailey–Eastwood–Graham [**BEG**], and Hirachi [**Hi**]. This direction keeps being studied as the "Fefferman's program," and nowadays it is enlarged for a wider class of parabolic geometries (see [**ČSl**] and the forthcoming second volume).

The Laplacian of the Bergman metric g is undoubtedly one of the most interesting objects associated to g. A basic study of this operator is given by Epstein–Melrose–Mendoza [**EMM**], who showed the meromorphicity of the resolvent of the Bergman Laplacian. In their work, the importance of the Bergman metric g lives in the fact that the singularity of g at the boundary is in some sense controlled and that the leading part of the singularity recovers the CR structure naturally given to the boundary—the higher-order asymptotics is not crucial. We say, after Biquard [**Bi**], that a metric with a similar property is *asymptotically complex hyperbolic* (ACH). Epstein–Melrose–Mendoza took a geometric approach to state what we call the ACH condition[1]: they defined the notion of Θ-structures on manifolds-with-boundary, and formulate the ACH condition as a condition on fiber metrics of the associated Θ-tangent bundle.

The advantage of considering ACH metrics is, apart from the fact that it is a natural setting for the study of the Laplacian, that the class of CR structures at the boundary can now be broaden to partially integrable structures. Let M be a $(2n+1)$-dimensional C^∞-manifold, and suppose that $T^{1,0}M$ is a subbundle of the complexified tangent bundle $\mathbb{C}TM$ of rank n that gives an almost CR structure, i.e., such that $T^{1,0}M \cap \overline{T^{1,0}M} = 0$. Then it is *partially integrable* if and only if

$$[C^\infty(M, T^{1,0}M), C^\infty(M, T^{1,0}M)] \subset C^\infty(M, T^{1,0}M \oplus \overline{T^{1,0}M}). \tag{1.1}$$

Partially integrable CR structures are somewhat mysterious objects, in the sense that there are no known situations in which CR structures that are only partially integrable are induced naturally on real hypersurfaces (unless they sit at the boundaries at infinity as in our case). However, partially integrable CR structures are at the same time orthodox geometric objects because they are natural generalizations of usual integrable CR structures that fall in the class of parabolic geometries, and the existence of a canonical Cartan connection is established in [**ČSc**]. Moreover, a lot of examples of partially integrable CR structures that are not integrable are easily constructed by modifying integrable CR structures (see Subsection 3.7). Our claim behind the work presented in this thesis is that the space of partially integrable CR structures has a significance at least as the place where integrable CR structures are modified.

We study the Einstein equation for ACH metrics in this thesis to generate CR invariants on the boundary. The idea of using the Einstein equation for such a purpose is again originally due to Fefferman [**F2**]. He considered the zero boundary value problem of the complex Monge–Ampère equation on strictly pseudoconvex domains, which is the equation for the Kähler-Einstein potential function. Fefferman's finding was that the lower terms of the asymptotic expansion of the solution can be easily related to the geometry of the boundary. The existence of the unique exact solution is proved by Cheng–Yau [**CY**], and Lee–Melrose [**LM**] proved that it admits an asymptotic expansion containing logarithmic terms. Graham [**G1**] completely identified the locally-determined and undetermined terms, and this enabled to produce more CR invariants. Actually, the invariants built based on these works are the ones that are used in [**F3, BEG, Hi**] to describe the singularity of the Bergman kernel.

[1]Precise definitions of the ACH condition depend on authors. What they mean are essentially the same, but one must be careful to the technicality of each definition. For example, Biquard's definition in [**Bi**] is well understood if compared to our Corollary 4.13.

We also note that, as for ACH metrics, a perturbation result on the existence of solutions of the Einstein equation is obtained by Biquard [**Bi**], and the existence of asymptotic expansions is studied by Biquard–Herzlich [**BH2**].

1.2. Brief outline. In this thesis, we study asymptotic ACH solutions of the Einstein equation. As an application, two CR invariants are constructed: the CR obstruction tensor $\boldsymbol{\mathcal{O}}_{\alpha\beta}$, which is a local one, and the total CR Q-curvature $\overline{Q}$, which is global. Finally, $\boldsymbol{\mathcal{O}}_{\alpha\beta}$ is characterized as the variation of $\overline{Q}$ with respect to modifications of partially integrable CR structure. Here we explain some more idea about what is discussed, along with the relations to other works. For precise statements of the main theorems, see the next section.

In Chapter 2, basic materials and preparatory discussions on partially integrable CR structures, Θ-structures and ACH metrics, and Bergman-type metrics in the sense of [**EMM**] are presented.

In Chapter 3, we first construct an approximate solution to the Einstein equation that should be compared to Fefferman's approximate solution to the Monge–Ampère equation. Since there is no notion of potential functions for ACH metrics, our approach to the equation is direct. In order to compensate for the shortage of unknown functions due to the diffeomorphism invariance of the Einstein equation, we need a technical use of contracted Bianchi identity in Riemannian geometry. An obstruction $\mathcal{O}_{\alpha\beta}$ for nonexistence of logarithmic singularities occurs in the order a little bit lower than that of the Monge–Ampère obstruction. As it is discussed in the next section and proved in Chapter 3, this obstruction, which we call the CR obstruction tensor, is a new CR invariant that occurs only in the nonintegrable case. On the other hand, the counterpart for the Monge–Ampère obstruction does not appear in the ACH-Einstein expansion. This is not contradictory because an ACH metric may not be regarded as a Kähler metric even if it induces an integrable CR structure on the boundary.

After that, we discuss what happens if we introduce logarithmic terms into ACH metrics. We will formulate a system of partial differential equations on the boundary that should be solved to construct an expansion that solves the Einstein equation. We do not know if this system can be solved globally, or even locally, but the formal solvability at a given point is guaranteed by Cauchy–Kovalevskaya Theorem. All the locally undetermined terms are specified. A technical difficulty here is that the form of induction changes compared to the construction of non-logarithmic approximate solutions. We have to determine certain different-order terms at the same induction step.

There are similar results to the ones explained so far in even-dimensional conformal geometry. In the conformal case, one considers the Einstein equation for asymptotically hyperbolic (AH) metrics[2]. Then, as briefly explained in [**FG1**] and detailed in [**FG3**] by Fefferman–Graham, an obstruction $\mathcal{O}_{ij}$ appears and an AH-Einstein expansion can be constructed if a certain system of PDE can be solved. The obstruction tensors in conformal and CR geometries has a similarity that the both are symmetric 2-tensors. The CR obstruction tensor moreover has a property that it takes values in $\mathrm{Sym}^2(T^{1,0}M)^*$, which makes sense only in CR geometry, and it is preferable when one considers deformations of partially integrable CR structures as described below.

We also remark that, in the 4-dimensional ACH case (hence with 3-dimensional CR boundary), the asymptotics is investigated by Biquard–Herzlich [**BH1**] to some extent for the purpose of obtaining a Burns–Epstein type formula.

In Chapter 4, we discuss the CR Q-curvature, which integrates to a global CR-invariant real number called the total CR Q-curvature. This is constructed by using the terms of ACH-Einstein expansions appearing before the CR obstruction tensor occurs. Our construction is the ACH version of a work of Graham–Zworski [**GZ**] for AH metrics. Namely, we consider eigenvalue problems for the Laplacian of the non-logarithmic approximate ACH-Einstein metric. Depending on the spectral parameter, the form of asymptotic expansions of possible solutions is strongly controlled, and we can define a Dirichlet-to-Neumann-like operator P_k for each $k \in \mathbb{Z}_+$, which is a differential operator of order $2k$. It turns out that P_k is a CR-invariant operator for $k \le n+1$. Among them, P_{n+1} has a special property that $P_{n+1}1 = 0$, i.e., there is no zeroth-order term in P_{n+1}. This is why the CR Q-curvature appears.

Section 9 has some overlap with a work of Guillarmou–Sà Barreto [**GS**]. And also, the case of strictly pseudoconvex domains is due to Hislop–Perry–Tang [**HPT**].

The final several pages of Chapter 4 is devoted to the first variational formula of the total CR Q-curvature. Here deformations of partially integrable CR structure are considered, which are infinitesimally represented by symmetric 2-tensors of the type that allows taking the pairing with the CR obstruction tensor $\mathcal{O}_{\alpha\beta}$. It is shown that the derivative of the total CR Q-curvature is given by this pairing. Since $\mathcal{O}_{\alpha\beta}$ vanishes for integrable CR structures, one concludes that the total CR Q-curvature takes critical values at integrable structures. The proof is parallel to the case of even-dimensional conformal structures by Graham–Hirachi [**GH**]. In conformal geometry, conformally flat and conformally Einstein structures are large classes with vanishing Fefferman–Graham obstruction tensor, and one can say that integrable CR structures resemble to such conformal structures from the viewpoint of the total Q-curvature.

[2]They are special class of conformally compact metrics, and a conformally compact Einstein metric is necessarily AH. The term “Poincaré metrics” is also used for these metrics, but the usage of this term is not fixed.

2. Main theorems

Here we summarize important results that are shown in Chapters 3 and 4. Basic notions and facts that will be described in the next chapter are freely used.

2.1. Smooth approximate Einstein metrics. A large portion of Chapter 3 is devoted to a discussion on the existence of an approximate solution of the Einstein equation in the ACH category. By Proposition 4.4, the Levi-Civita connection of a Θ-metric is a Θ-connection, so the Riemann curvature tensor and the Ricci tensor are naturally regarded as Θ-tensors.

THEOREM 2.1. *Let $(\overline{X}, [\Theta])$ be a $(2n+2)$-dimensional Θ-manifold and $T^{1,0}M$ a compatible partially integrable CR structure on the boundary $M = \partial X$. Let $\rho \in C^\infty(\overline{X})$ be any boundary defining function. Then there exists a C^∞-smooth ACH metric g that induces $T^{1,0}M$ on the boundary whose Ricci tensor satisfies, as a Θ-tensor,*

$$\mathrm{Ric} = -\frac{1}{2}(n+2)g + O(\rho^{2n+2}). \tag{2.1}$$

Up to the action of boundary-fixing Θ-diffeomorphisms on $\overline{X}$, such an ACH metric g is unique modulo $O(\rho^{2n+2})$ Θ-tensors.

There is beauty in the simplicity of this statement, and it is also remarkable that the condition (2.1) is sufficient when we later define the CR obstruction tensor. On the other hand, for some purposes we will need a more improved metric. Note that (2.1) means that $\mathrm{Ric} + \frac{1}{2}(n+2)g$ vanishes to $(2n+2)^{\mathrm{nd}}$ order at ∂X as a 2-Θ-tensor, not as a usual 2-tensor. It is automatic from (2.1) that the scalar curvature satisfies $\mathrm{Scal} = -(n+1)(n+2) + O(\rho^{2n+2})$, but actually, (2.2) is possible.

THEOREM 2.2. *Under the assumption of Theorem 2.1, there exists a C^∞-smooth ACH metric g that induces $T^{1,0}M$ on the boundary for which* (2.1) *and*

$$\mathrm{Scal} = -(n+1)(n+2) + O(\rho^{2n+3}) \tag{2.2}$$

are satisfied. Up to the action of boundary-fixing Θ-diffeomorphisms on $\overline{X}$, such an ACH metric g is unique modulo $O(\rho^{2n+2})$ Θ-tensors with $O(\rho^{2n+3})$ trace.

Theorems 2.1 and 2.2 reduce to Theorem 6.1 by using the normalization of ACH metrics, and it is finally proved in Subsection 6.6. If we introduce the evenness condition in Subsection 6.7, then the solution automatically improves a bit more.

THEOREM 2.3. *Under the assumption of Theorem 2.1, there exists an even C^∞-smooth ACH metric g that induces $T^{1,0}M$ on the boundary for which* (2.1) *and*

$$\mathrm{Scal} = -(n+1)(n+2) + O(\rho^{2n+4}) \tag{2.3}$$

are satisfied. Up to the action of boundary-fixing Θ-diffeomorphisms on $\overline{X}$, such an even ACH metric g is unique modulo $O(\rho^{2n+2})$ even Θ-tensors with $O(\rho^{2n+4})$ trace.

Although it is not strictly necessary, a systematic use of the evenness simplifies various arguments. Chapter 4 can be seen as an example.

We introduce the following notions for the subsequent description.

DEFINITION 2.4. Let $(\overline{X}, [\Theta])$ be a $(2n+2)$-dimensional Θ-manifold and $T^{1,0}M$ a compatible partially integrable CR structure on the boundary. An ACH metric g with properties described in Theorem 2.1 is called a *smooth approximate ACH-Einstein metric.* An even ACH metric g with properties in Theorem 2.3 is called a *smooth approximate even ACH-Einstein metric.*

2.2. CR obstruction tensor. Construction of better approximate solutions is obstructed in general. From the proof of Theorem 2.1, one simultaneously observes the following. For a smooth approximate ACH-Einstein metric g, we set

$$\mathrm{Ric} = -\frac{1}{2}(n+2)g + \rho^{2n+2}S. \tag{2.4}$$

Let ${}^\Theta T\overline{X}|_{\partial X} = R \oplus K_2 \oplus L$ be the orthogonal decomposition (4.2) and $\lambda_\rho \colon H \longrightarrow L$ the isomorphism given in (4.5) associated to a choice of a boundary defining function ρ, where H is the underlying contact distribution for $T^{1,0}M$. Consider $\lambda_\rho^* S$, which is a C^∞-smooth section of $\mathrm{Sym}^2 \mathbb{C}H^*$. The decomposition $\mathbb{C}H = T^{1,0}M \oplus \overline{T^{1,0}M}$ induces

$$\mathrm{Sym}^2 \mathbb{C}H^* = \mathrm{Sym}^2(T^{1,0}M)^* \oplus \Big((T^{1,0}M)^* \odot (\overline{T^{1,0}M})^*\Big) \oplus \mathrm{Sym}^2(\overline{T^{1,0}M})^*, \tag{2.5}$$

where $\odot$ denotes the symmetric product. Using the abstract index notation (see Subsection 3.3), the components of $\lambda_\rho^* S$ with respect to (2.5) are written as

$$(\lambda_\rho^* S)_{\alpha\beta}, \qquad (\lambda_\rho^* S)_{\alpha\overline{\beta}}, \qquad \text{and} \qquad (\lambda_\rho^* S)_{\overline{\alpha}\overline{\beta}}.$$

Since $\lambda_\rho^* S$ is a real tensor, the first and the third ones are related: $(\lambda_\rho^* S)_{\overline{\alpha}\overline{\beta}} = \overline{(\lambda_\rho^* S)_{\alpha\beta}}$. So there are only two independent components. Our claim is that $(\lambda_\rho^* S)_{\alpha\beta}$ is determined only by $(M, T^{1,0}M)$. The following result is proved in Subsection 7.1. The notion of admissible boundary defining functions is given in Subsection 4.2.

THEOREM 2.5. *Let $(\overline{X}, [\Theta])$ be a $(2n+2)$-dimensional Θ-manifold, $T^{1,0}M$ a compatible partially integrable CR structure on $M = \partial X$, and θ any fixed contact form on M. Let g be a smooth approximate ACH-Einstein metric. Take an admissible boundary defining function $\rho \in \mathcal{F}_\theta$ and define S by (2.4). Then, the tensor*

$$\mathcal{O}_{\alpha\beta} := (\lambda_\rho^* S)_{\alpha\beta} \tag{2.6}$$

does not depend on the choice of g. Moreover, there exists a universal expression of $\mathcal{O}_{\alpha\beta}$ as a local pseudohermitian invariant.

The last statement of Theorem 2.5 means that there is a polynomial representing $\mathcal{O}_{\alpha\beta}$, which depends only on the dimension, of (the components of) the Levi form, its dual, the Nijenhuis tensor, the pseudohermitian torsion tensor, the pseudohermitian curvature tensor, and their first and higher-order covariant derivatives. Actually more is true—instead of the full curvature tensor, we only need the Ricci tensor.

In particular, $\mathcal{O}_{\alpha\beta}$ is determined only by the local geometry of $(M, T^{1,0}M, \theta)$. So we define as follows.

DEFINITION 2.6. The tensor $\mathcal{O}_{\alpha\beta} \in \mathcal{E}_{(\alpha\beta)} = C^\infty(M, \mathrm{Sym}^2(T^{1,0}M)^*)$ is called the *CR obstruction tensor* of $(M, T^{1,0}M, \theta)$.

Returning back to Theorem 2.1, one notices that the condition on g has nothing to do with choosing contact forms. So it is natural to consider another scale $\hat\theta = e^{2\Upsilon}\theta$ for (2.6) keeping g fixed. Then, since $\mathcal{F}_{\hat\theta} = e^{\tilde\Upsilon}\mathcal{F}_\theta$ (where $\tilde\Upsilon$ is an arbitrary smooth extension of Υ), it is immediate that the CR obstruction tensor $\hat{\mathcal{O}}_{\alpha\beta}$ for $\hat\theta$ is given by

$$\hat{\mathcal{O}}_{\alpha\beta} = e^{-2n\Upsilon}\mathcal{O}_{\alpha\beta}. \tag{2.7}$$

In other words, if we define the density-weighted version $\boldsymbol{\mathcal{O}}_{\alpha\beta}$ by

$$\boldsymbol{\mathcal{O}}_{\alpha\beta} := \mathcal{O}_{\alpha\beta} \otimes \theta^n \in \mathcal{E}_{(\alpha\beta)}(-n,-n), \tag{2.8}$$

then this is a CR-invariant tensor. Here θ is regarded as a density in $\mathcal{E}(-1,-1)$ by (3.23).

DEFINITION 2.7. The density-weighted tensor $\boldsymbol{\mathcal{O}}_{\alpha\beta} \in \mathcal{E}_{(\alpha\beta)}(-n,-n)$ is called the *CR obstruction tensor* of $(M, T^{1,0}M)$.

The CR obstruction tensor actually is a nontrivial local invariant as we will see in Subsection 7.2. Nevertheless we can prove the following remarkable fact here, which shows that it is essential for our study to broaden our scope to partially integrable CR structures.

THEOREM 2.8. *The CR obstruction tensor of a nondegenerate integrable CR manifold always vanishes.*

PROOF. Since $\mathcal{O}_{\alpha\beta}$ admits a universal expression as a local pseudohermitian invariant, the value of $\mathcal{O}_{\alpha\beta}$ at a point $p \in M$ depends only on some finite jets of $T^{1,0}M$ and θ. It is known that any nondegenerate integrable CR structure $T^{1,0}M$ can be formally embedded at a given point p (see [**K**]), in the sense that, for any given $N \in \mathbb{N}$, one can take a C^∞-embedding (not a CR embedding) of a neighborhood $U \subset M$ of p into $\mathbb{C}^{n+1}$ so that the N-jet of the induced CR structure is the same as that of $T^{1,0}M$. Therefore the claim reduces to the case where the CR structure is induced by some embedding into a domain of $\mathbb{C}^{n+1}$. For such type of integrable CR structures, by Proposition 5.6, one can always construct a smooth approximate ACH-Einstein metric such that $E = O(\rho^{2n+4})$. Hence $\mathcal{O}_{\alpha\beta} = 0$. $\square$

REMARK 2.9. When $n = 1$, the integrability condition for almost CR structures is automatically satisfied, so the obstruction tensor does not appear in this dimension. This fact was observed by Biquard–Herzlich [**BH1**, Corollary 5.4].

There is also an interesting property of the CR obstruction tensor that should be compared to the fact in conformal geometry that the Fefferman–Graham obstruction tensor is divergence-free [**FG3**]. Let $\boldsymbol{D}^{\alpha\beta} \colon \mathcal{E}_{(\alpha\beta)}(-n,-n) \longrightarrow \mathcal{E}(-n-2,-n-2)$ be a differential operator defined as follows. For any choice of a contact form θ, the trivialization $D^{\alpha\beta} \colon \mathcal{E}_{(\alpha\beta)} \longrightarrow \mathcal{E}$ is given by the following formula in terms of the Tanaka–Webster connection:

$$D^{\alpha\beta} := \nabla^\alpha\nabla^\beta - iA^{\alpha\beta} - N^{\gamma\alpha\beta}\nabla_\gamma - (\nabla_\gamma N^{\gamma\alpha\beta}).$$

One can check that this gives a well-defined operator by using the transformation formulae in Subsection 3.5.

THEOREM 2.10. *The imaginary part of* $\boldsymbol{D}^{\alpha\beta}\boldsymbol{\mathcal{O}}_{\alpha\beta}$ *always vanishes.*

This is, as becomes apparent in Subsection 7.1, a consequence of the so-called contracted Bianchi identity in Riemannian geometry.

2.3. Einstein metrics with logarithmic singularities. We shall also investigate how well the solution can be improved if we introduce logarithmic terms. A continuous Θ-tensor S on $\overline{X}$ that is C^∞-smooth in X is said to have *logarithmic singularity* if it admits an asymptotic expansion of the form

$$S \sim \sum_{q=0}^{\infty} S^{(q)}(\log\rho)^q, \qquad \text{where } S^{(q)} \text{ are } C^\infty\text{-smooth in } \overline{X}. \tag{2.9}$$

By this we mean that, for any $m \geq 0$, it holds that for sufficiently large N,

$$r_N := S - \sum_{q=0}^{N} S^{(q)}(\log\rho)^q \in C^m(\overline{X}) \qquad \text{and} \qquad r_N = O(\rho^m).$$

The set of Θ-tensors with logarithmic singularities is denoted by $\mathcal{A}(\overline{X})$ (suppressing the type of the tensors). If $S \in \mathcal{A}(\overline{X})$, then $S^{(q)}$ is unique modulo $O(\rho^\infty)$.

If g is a Θ-metric with logarithmic singularity, then its Ricci tensor also has only logarithmic singularity.

THEOREM 2.11. *Let* $(X, [\Theta])$ *and* $T^{1,0}M$ *be as in Theorem 2.1, and* $p \in \partial X$. *Then there exists an ACH metric* g *with logarithmic singularity that induces* $T^{1,0}M$ *for which* $E := \mathrm{Ric} + \frac{1}{2}(n+2)g$ *formally vanishes at* p, *i.e., the Taylor expansion of each coefficients* $E^{(q)}$ *vanishes at* p.

Theorem 2.11 is a consequence of Theorem 8.1 and its proof. On the other hand, even if we introduce logarithmic singularities, there might be a global obstruction to the existence of asymptotic solution to the Einstein equation. This point is also discussed in Theorem 2.11, and it should be compared to the asymptotically hyperbolic case.

In the language of logarithmic singularity, the CR obstruction tensor is "the first logarithmic term," or "the first obstruction to the existence of non-logarithmic solutions." It turns out that this is the first and last obstruction in the local sense. The following result is shown in Subsection 8.5.

THEOREM 2.12. *Let* $(X, [\Theta])$ *and* $T^{1,0}M$ *be as in Theorem 2.1, and* $p \in \partial X$. *If* $\boldsymbol{\mathcal{O}}_{\alpha\beta}(p) = 0$, *then there exists a smooth ACH metric that induces* $T^{1,0}M$ *for which* $E := \operatorname{Ric} + \frac{1}{2}(n+2)g$ *formally vanishes at* p.

Recall from [**G1**] that, in the case of the zero boundary value problem for the complex Monge–Ampère equation on bounded strictly pseudoconvex domains, there is an obstruction to the existence, which is one real scalar-valued function on $\partial\Omega$. Our result says that in the ACH category, at any given point on $\partial\Omega$, we can always erase the logarithmic terms.

2.4. Dirichlet-like problems. Next we consider the following Dirichlet-like problem on a C^∞-smooth ACH manifold $(\overline{X}, [\Theta], g)$. Let k be a positive integer, and suppose a real-valued function $f \in C^\infty(M)$ is given. We want to find a solution u to

$$\left(\Delta_g - \frac{1}{4}((n+1)^2 - k^2)\right) u = 0$$

in the form

$$u = \rho^{n+1-k} F, \qquad F \in C^\infty(\overline{X}), \qquad F|_{\partial X} = f.$$

Here Δ_g is the Laplacian with positive spectrum.

This problem is not solvable in general, even if we consider in formal level. So, to construct a formal solution we introduce logarithmic terms. For simplicity we restrict to the case of even metrics, and the situation is summarized in the following theorem, which is proved in Subsection 9.2.

THEOREM 2.13. *Let* $(\overline{X}, [\Theta])$ *be a* Θ*-manifold and* g *an even* C^∞*-smooth ACH metric. Suppose* $T^{1,0}M$ *is the induced partially integrable CR structure,* θ *is a fixed contact form, and* $\rho \in \mathcal{F}_\theta$ *is an admissible boundary defining function. Then, for any real-valued function* $f \in C^\infty(M)$, *there exists a function* $u \in C^\infty(\overline{X})$ *of the form*

$$u = \rho^{n+1-k} F + \rho^{n+1+k} \log\rho \cdot G, \qquad F,\, G \in C^\infty(\overline{X}), \qquad F|_{\partial X} = f \tag{2.10}$$

that solves

$$\left(\Delta_g - \frac{1}{4}((n+1)^2 - k^2)\right) u = O(\rho^\infty). \tag{2.11}$$

The function F is uniquely determined modulo $O(\rho^{2k})$, and G is unique modulo $O(\rho^\infty)$. Moreover, there exists a differential operator P_k determined by g and θ such that

$$G|_M = -2c_k P_k f, \qquad c_k = \frac{(-1)^k}{k!(k-1)!}. \tag{2.12}$$

The principal symbol $\sigma(P_k)$ of P_k is equal to $\sigma(\Delta_b^k)$, where Δ_b is the sublaplacian.

The condition (2.10) itself is independent of the boundary defining function ρ. So if we take another contact form $\hat\theta = e^{2\Upsilon}\theta$, then the function u is considered as a solution associated to the Dirichlet datum $\hat f = e^{-(n+1-k)\Upsilon} f$. Therefore we obtain

$$\hat P_k(e^{-(n+1-k)\Upsilon} f) = e^{-(n+1+k)\Upsilon} P_k f.$$

This means that, even if we do not specify θ, P_k is well-defined as an operator between appropriate densities.

The operator P_k actually extends to a family $P_{k,s}$ of differential operators for which $P_{k,(n+1+k)/2} = P_k$. The family $P_{k,s}$ appears when we consider a similar problem for equation (2.14). In Subsection 9.2, we also prove the following theorem. Note that, although s is not allowed to be $(n+1+k)/2$ in this theorem, if we formally put $s = (n+1+k)/2$ into (2.14), then we get equation (2.11).

Theorem 2.14. *Let $s \notin (n+1+\mathbb{Z}_+)/2$ be a real number. Suppose $(\overline{X}, [\Theta])$ and g are as in Theorem 2.13. Then, for any real-valued function $f \in C^\infty(M)$, there is a function $u \in C^\infty(X)$ of the form*

$$u = \rho^{2(n+1-s)} F, \qquad F \in C^\infty(\overline{X}), \qquad F|_{\partial X} = f \tag{2.13}$$

that solves

$$(\Delta_g - s(n+1-s))u \in C^\infty(X). \tag{2.14}$$

The function F is uniquely determined modulo $O(\rho^\infty)$.

The proof of Theorem 2.14 is based on the normalization of the ACH metric g. One concludes that the function F in (2.13) is given as

$$F \sim f + \rho^2 c_{1,s} P_{1,s} + \rho^4 c_{2,s} P_{2,s} + \cdots, \tag{2.15}$$

where $P_{l,s}$s are some differential operator determined by $(M, T^{1,0}M, \theta)$ and

$$c_{l,s} := (-1)^l \prod_{i=1}^{l} \frac{1}{2s-n-1-i}. \tag{2.16}$$

The construction shows that $P_{l,s}$ is actually polynomial in s, so the operator $P_{l,s}$ itself makes sense for any $s \in \mathbb{R}$. The fact is that

$$P_k = P_{k,(n+1+k)/2}. \tag{2.17}$$

There is a special case where the problem of Theorem 2.13 becomes trivial. Let $k = n+1$, and take $f \equiv 1$ as the boundary datum. Then, since (2.11) reduces to $\Delta_g u = O(\rho^\infty)$, the constant function $u \equiv 1$ solves (2.11). Hence we conclude:

$$P_{n+1} 1 = 0. \tag{2.18}$$

Then equation (2.17) implies that the zeroth-order term of $P_{n+1,s}$ has a factor $s - n - 1$. We define

$$Q := \left. \left(\frac{1}{n+1-s} P_{n+1,s} 1 \right) \right|_{s=n+1}. \tag{2.19}$$

Then Q satisfies the following remarkable transformation law.

Theorem 2.15. *Let g be an even ACH metric on $(\overline{X}, [\Theta])$. If θ and $\hat{\theta} = e^{2\Upsilon}\theta$ are two contact forms on the infinity $(M, T^{1,0}M)$, then Q and $\hat{Q}$ are related by*

$$e^{2(n+1)\Upsilon} \hat{Q} = Q + P_{n+1}\Upsilon. \tag{2.20}$$

As a consequence of this transformation law, if M is compact, the integral

$$\overline{Q} := \int_M Q\, \theta \wedge (d\theta)^n \tag{2.21}$$

does not depend on θ. This follows from (2.18) and the self-adjointness of P_{n+1} with respect to the volume form $\theta \wedge (d\theta)^n$. The proof of the self-adjointness can be given by the idea of Graham–Zworski [**GZ**] or the one of Fefferman–Graham [**FG2**].

2.5. Invariance and first variation of total CR Q-curvature. In the previous subsection, we discussed general even ACH metrics. Now it is time to introduce the Einstein condition.

Theorem 2.16. *Let g be a smooth even approximate ACH-Einstein metric on a Θ-manifold $(\overline{X}, [\Theta])$ that induces $T^{1,0}M$ on the boundary. Then the operator P_k does not depend on the ambiguity that lives in g if $k \le n+1$, and Q is also independent of the ambiguity. There are universal expressions of P_ks and Q in terms of local pseudohermitian invariants, and in the case of P_ks, covariant differentiations by the Tanaka–Webster connection.*

The proof of Theorem 2.16 given in Subsection 10.1 is based on a careful observation of the Laplacian Δ_g. This theorem allows us to give the following definition.

Definition 2.17. The operator P_k is called the k^{th} *CR-invariant power of the sublaplacian* of $(M, T^{1,0}M)$. The function Q is called the *CR Q-curvature* of $(M, T^{1,0}M, \theta)$. If M is compact, then the integral $\overline{Q}$ given by (2.21) is the *total CR Q-curvature*, which is a CR-invariant real number.

In the case of integrable CR structures, there are other constructions of the CR-invariant powers of the sublaplacian and the CR Q-curvature. In Subsection 5.3, we discuss the fact that our P_k and Q turn into the known ones for integrable

CR structures. To prove this, we need to show that our Q vanishes for embedded CR structures associated with an invariant contact form, and this will be given as Proposition 10.2.

As the final theorem in this thesis, we prove the first variational formula of the total CR Q-curvature in Subsection 10.2.

THEOREM 2.18. *Let $(M, T^{1,0}M)$ be a compact nondegenerate partially integrable CR manifold of dimension $2n+1$. Let $\psi_{\alpha\beta} \in \mathcal{E}_{(\alpha\beta)}(1,1)$, and $\hat{T}_t^{1,0}$ a smooth 1-parameter family of partially integrable CR structures based at $T^{1,0}M$, with fixed underlying contact structure, that is tangent to $\psi_{\alpha\beta}$. Let $\overline{Q}_t$ be the total CR Q-curvature of $(M, \hat{T}_t^{1,0})$. Then, the derivative of $\overline{Q}_t$ at $t = 0$ is given by*

$$\left(\frac{d}{dt}\overline{Q}_t\right)\bigg|_{t=0} = \frac{8 \cdot (-1)^n \cdot n!(n+1)!}{n+2} \int_M \mathrm{Re}(\boldsymbol{\mathcal{O}}^{\alpha\beta}\psi_{\alpha\beta}). \tag{2.22}$$

Here, the indices are raised by the weighted Levi form $\boldsymbol{h}_{\alpha\beta}$ of $(M, T^{1,0}M)$, and $\boldsymbol{\mathcal{O}}_{\alpha\beta}$ is the obstruction tensor of $(M, T^{1,0}M)$.

If this is combined with Theorem 2.8, then one can conclude that any nondegenerate integrable CR structure is a critical point of the total Q-curvature.

CHAPTER 2

Preparations

3. Partially integrable CR structures

3.1. Partially integrable CR structures. Nijenhuis tensor. Recall that an almost CR structure $T^{1,0}M$ on an odd-dimensional smooth manifold M with $\dim M = 2n+1$, $n \geq 1$, is a rank n subbundle of the complexified tangent bundle $\mathbb{C}TM$ such that $T^{1,0}M \cap \overline{T^{1,0}M} = 0$. If the space of smooth sections $C^\infty(M, T^{1,0}M)$ is closed under the Lie bracket, then the almost CR structure $T^{1,0}M$ is *integrable.* Even when $T^{1,0}M$ is not necessarily integrable, we say that it is *partially integrable* if (1.1) is satisfied.

It is customary to say that $T^{1,0}M$ is a CR structure when it is an integrable almost CR structure. However, as we have two different integrability conditions, we refrain from using this term in rigorous statements. Instead, we will say *integrable CR structures* (resp. *partially integrable CR structures*) to call almost CR structures that are integrable (resp. partially integrable).

If $T^{1,0}M$ is an almost CR structure, then the direct sum $T^{1,0}M \oplus \overline{T^{1,0}M}$ is invariant under conjugation. Let H be the hyperplane distribution given by

$$H := \operatorname{Re}(T^{1,0}M \oplus \overline{T^{1,0}M}) \subset TM.$$

Then $T^{1,0}M \oplus \overline{T^{1,0}M}$ is equal to the complexification of H. We define a real endomorphism $J \in \operatorname{End}(H)$ satisfying $J^2 = -\mathrm{id}_H$ by

$$J|_{T^{1,0}M} = i\mathrm{id}_{T^{1,0}M}, \qquad J|_{\overline{T^{1,0}M}} = -i\mathrm{id}_{\overline{T^{1,0}M}}.$$

Conversely, if we are given a hyperplane distribution H and $J \in \operatorname{End}(H)$ satisfying $J^2 = -\mathrm{id}_H$, then we obtain an almost CR structure $T^{1,0}M$ by defining it as the subbundle of $\mathbb{C}H$ that consists of i-eigenvectors of J. In this sense, we may consider such a pair (H, J) as the real presentation of an almost CR structure. The partial integrability condition is equivalent to

$$[X, Y] - [JX, JY] \in C^\infty(M, H), \qquad X,\, Y \in C^\infty(M, H), \tag{3.1}$$

and the integrability holds if and only if the following is also satisfied:

$$[X, Y] - [JX, JY] + J([JX, Y] + [X, JY]) = 0, \qquad X,\, Y \in C^\infty(M, H). \tag{3.2}$$

The left-hand side of (3.2) is $C^\infty(M)$-linear, and hence it defines a $(1,2)$-tensor over the subbundle H. We call $1/4$ times this tensor the *Nijenhuis tensor* of the partially integrable CR structure and denote it by N. In complex terms,

$$N(X,Y) = [X_{1,0}, Y_{1,0}]_{0,1} + [X_{0,1}, Y_{0,1}]_{1,0}, \qquad X,\, Y \in C^\infty(M, \mathbb{C}H),$$

where the subscripts "$1,0$" and "$0,1$" denote the projections from $\mathbb{C}H = T^{1,0}M \oplus \overline{T^{1,0}M}$ onto the each summand.

3.2. The Levi form. Nondegeneracy and strict pseudoconvexity. Let $T^{1,0}M$ be partially integrable. If θ is any (possibly locally-defined) 1-form on M whose kernel is H, then the partial integrability condition (1.1) is equivalent to that its exterior derivative $d\theta$ vanishes on $T^{1,0}M \otimes T^{1,0}M$ (and hence also on $\overline{T^{1,0}M} \otimes \overline{T^{1,0}M}$). The *Levi form* h is defined as the remaining component:

$$h(Z, \overline{W}) := -i\, d\theta(Z, \overline{W}) = i\, \theta([Z, \overline{W}]), \qquad Z,\, W \in C^\infty(M, T^{1,0}M). \tag{3.3}$$

This is a hermitian form on $T^{1,0}M$ in the sense that

$$h(Z, \overline{W}) = \overline{h(W, \overline{Z})}.$$

The real presentation is as follows: (3.3) is the restriction to $T^{1,0}M \otimes \overline{T^{1,0}M}$ of the $\mathbb{C}$-bilinear extension of

$$h(X,Y) := d\theta(X, JY) = -\theta([X, JY]), \qquad X,\, Y \in C^\infty(M, H). \tag{3.4}$$

This real 2-form is actually symmetric by 3.1. When it is complexified, this has a chance to have nonzero values only on $T^{1,0}M \otimes \overline{T^{1,0}M}$ and $\overline{T^{1,0}M} \otimes T^{1,0}M$. So one can regard (3.4) as the "real extension" of (3.3). This real symmetric 2-form is also identified with the Levi form.

The Levi form depends on θ, but it is immediate from the last expression in (3.3) or (3.4) that it just scales when we take another θ. Invariantly, we define the $\mathbb{C}$-linear map

$$T^{1,0}M \otimes \overline{T^{1,0}M} \longrightarrow \mathbb{C}(TM/H), \qquad Z \otimes \overline{W} \longmapsto (i[Z, \overline{W}] \bmod H).$$

It is natural to call this $\mathbb{C}(TM/H)$-valued hermitian form the *weighted Levi form,* because the complex line bundle $\mathbb{C}(TM/H)$ is the density bundle of biweight $(1,1)$—see Subsection 3.6.

We say that a partially integrable CR structure $T^{1,0}M$ is *nondegenerate* if the Levi form is a nondegenerate hermitian form at each point on M. This is equivalent to that H is a contact distribution. So, when $T^{1,0}M$ is nondegenerate, any choice of θ exactly annihilating H is called a *contact form* for $T^{1,0}M$. Since θ defines a hermitian form h on $T^{1,0}M$, it is also called a *pseudohermitian structure.* From the viewpoint of comparing CR structures with conformal structures, choosing a contact form corresponds to choosing a representative Riemannian metric from the given conformal class.

If the Levi form has definite signature, then $T^{1,0}M$ is *strictly* (or *strongly*) *pseudoconvex*. In this case, we can always take a globally-defined pseudohermitian structure θ for which h is positive definite.

3.3. Index notation of tensors. A symmetry of the Nijenhuis tensor. Let $T^{1,0}M$ be a partially integrable CR structure. If $\{ Z_\alpha \}$ is a local frame of $T^{1,0}M$, we put $Z_{\overline{\alpha}} = \overline{Z_\alpha}$ and express various tensors over H by components with respect to $\{ Z_\alpha, Z_{\overline{\alpha}} \}$ and its dual frame. For example, the Nijenhuis tensor N is a $(1,2)$-tensor over H, so N is represented by the collection of $(2n)^3$ functions, each of which is denoted by N associated with two lower indices and a single upper index. In this case, because of an obvious property of the Nijenhuis tensor, the only components that can be nonzero are $N_{\alpha\beta}{}^{\overline{\gamma}}$ and $N_{\overline{\alpha}\overline{\beta}}{}^{\gamma}$, which are defined by

$$N(Z_\alpha, Z_\beta) = N_{\alpha\beta}{}^{\overline{\gamma}} Z_{\overline{\gamma}} \qquad \text{and} \qquad N(Z_{\overline{\alpha}}, Z_{\overline{\beta}}) = N_{\overline{\alpha}\overline{\beta}}{}^{\gamma} Z_\gamma.$$

Since N is a real tensor, these components satisfy

$$\overline{N_{\alpha\beta}{}^{\overline{\gamma}}} = N_{\overline{\alpha}\overline{\beta}}{}^{\gamma}.$$

Moreover, since $N(Z,W) = -N(W,Z)$, the components has the following symmetry:

$$N_{\alpha\beta}{}^{\overline{\gamma}} = -N_{\beta\alpha}{}^{\overline{\gamma}}, \qquad \text{or equivalently,} \qquad N_{\alpha\beta}{}^{\overline{\gamma}} = N_{[\alpha\beta]}{}^{\overline{\gamma}}. \tag{3.5}$$

As we saw above, the tensor N has components only in $(T^{1,0}M)^* \otimes (T^{1,0}M)^* \otimes \overline{T^{1,0}M}$ and $(\overline{T^{1,0}M})^* \otimes (\overline{T^{1,0}M})^* \otimes T^{1,0}M$. The component with values in the first is sometimes expressed as "the tensor $N_{\alpha\beta}{}^{\overline{\gamma}}$." This is an example of the *abstract index notation* [**PR**], and the idea is as follows. When this expression is used, the symbol $N_{\alpha\beta}{}^{\overline{\gamma}}$ does not stand for a function anymore. Instead, it should be considered as representing the rule that determines the coefficients of $Z_\alpha \otimes Z_\beta \otimes \theta^{\overline{\gamma}}$ in the expansion of N when a local frame $\{ Z_\alpha \}$ is given. The characters α, β, γ have no special meaning—even if we write $N_{\alpha'\beta'}{}^{\overline{\gamma'}}$, the tensor it refers to remains the same, although $N_{\alpha\beta}{}^{\overline{\gamma}}$ and $N_{\alpha'\beta'}{}^{\overline{\gamma'}}$ should not be directly compared. The vector bundle $(T^{1,0}M)^* \otimes (T^{1,0}M)^* \otimes \overline{T^{1,0}M}$ in which $N_{\alpha\beta}{}^{\overline{\gamma}}$ takes values is denoted by $E_{\alpha\beta}{}^{\overline{\gamma}}$. The space of sections is denoted by $\mathcal{E}_{\alpha\beta}{}^{\overline{\gamma}}$, so it makes sense to write

$$N_{\alpha\beta}{}^{\overline{\gamma}} \in \mathcal{E}_{\alpha\beta}{}^{\overline{\gamma}}.$$

Equation (3.5) is regarded as an abstract expression of the skew-symmetry of N rather than a mere relation of specific components. The bundle of tensors with this type of symmetry, namely $\bigwedge^2 (T^{1,0}M)^* \otimes \overline{T^{1,0}M}$, is denoted by $E_{[\alpha\beta]}{}^{\overline{\gamma}}$, so (3.5) can also be described as

$$N_{\alpha\beta}{}^{\overline{\gamma}} \in \mathcal{E}_{[\alpha\beta]}{}^{\overline{\gamma}}.$$

The abstract index notation is freely used in the sequel.

If $T^{1,0}M$ is nondegenerate and a contact form θ is given, then we have the Levi form h. Because it is a hermitian form, we write it $h_{\alpha\overline{\beta}}$ using indices. Since

h is a nondegenerate form, the *dual* of h is defined. This is a hermitian form on $(T^{1,0}M)^*$, denoted by $h^{\alpha\overline{\beta}}$, satisfying

$$h_{\alpha\overline{\gamma}}h^{\beta\overline{\gamma}} = \delta_\alpha{}^\beta,$$

where δ is the Kronecker delta and the Einstein summation convention is observed. The hermitian forms $h_{\alpha\overline{\beta}}$ and $h^{\alpha\overline{\beta}}$ are used to lower and raise indices of other tensors. For example, we define $N_{\alpha\beta\gamma} := h_{\gamma\overline{\sigma}}N_{\alpha\beta}{}^{\overline{\sigma}}$.

PROPOSITION 3.1. *For any contact form θ, it holds that*

$$N_{\alpha\beta\gamma} + N_{\beta\gamma\alpha} + N_{\gamma\alpha\beta} = 0. \tag{3.6}$$

PROOF. Let $\{\theta^\alpha\}$ be a set of 1-forms on M annihilating $\overline{T^{1,0}M}$ such that its restriction to $T^{1,0}M$ gives the dual frame for $\{Z_\alpha\}$. We set $\theta^{\overline{\alpha}} := \overline{\theta^\alpha}$. Then, by the definition of the Nijenhuis tensor,

$$d\theta^{\overline{\gamma}}(Z_\alpha, Z_\beta) = -N_{\alpha\beta}{}^{\overline{\gamma}}.$$

Therefore, by differentiating

$$d\theta = ih_{\alpha\overline{\beta}}\,\theta^\alpha \wedge \theta^{\overline{\beta}} \mod \theta$$

and considering types we obtain $N_{[\alpha\beta\gamma]} = 0$. Then (3.6) follows by (3.5). $\square$

In the case of tensors over TM, we specify one vector field T that is transverse to H by using a choice of a contact form θ. This special vector field, called the *Reeb vector field*, is characterized by

$$\theta(T) = 1, \qquad T \lrcorner\, d\theta = 0.$$

By Cartan's formula we obtain $\mathcal{L}_T\theta = 0$. In particular, T is a contact vector field. The complexified tangent bundle is decomposed into the sum

$$\mathbb{C}TM = \mathbb{C}T \oplus T^{1,0}M \oplus \overline{T^{1,0}M}. \tag{3.7}$$

If $\{Z_\alpha\}$ is a local frame of $T^{1,0}M$, then the *admissible coframe* $\{\theta^\alpha\}$ is the collection of 1-forms vanishing on $\mathbb{C}T \oplus \overline{T^{1,0}M}$ such that $\{\theta^\alpha|_{T^{1,0}M}\}$ is the dual coframe for $\{Z_\alpha\}$. This makes $\{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\}$ into the dual coframe for $\{T, Z_\alpha, Z_{\overline{\alpha}}\}$. The index 0 is used for components corresponding with T or θ.

3.4. Tanaka–Webster connection. Now we prove the existence theorem of the *Tanaka–Webster connection,* which is a canonical connection in pseudohermitian geometry. For partially integrable CR structures, this connection seems to be first considered by Mizner [**M**]. A similar, but different, generalization of the Tanaka–Webster connection is given by Tanno [**Tno**], and this is used in [**BaD**], [**BlD**], and [**S**]. Compared to it, our connection has an advantage that it preserves J. Instead, there is an additional component of the torsion tensor that can be nonzero: $\mathrm{Tor}_{\alpha\beta}{}^{\overline{\gamma}}$. But this is just the Nijenhuis tensor as we shall see after the proof.

PROPOSITION 3.2. *Let $T^{1,0}M$ be a nondegenerate partially integrable CR structure and θ a contact form. Then there is a unique connection ∇ on TM satisfying the following conditions:*

(i) *H, T, J, h are all parallel with respect to ∇;*
(ii) *The torsion tensor $\mathrm{Tor}(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$ satisfies*

$$\mathrm{Tor}(X,JY) - \mathrm{Tor}(JX,Y) = 2\,h(X,Y)T, \qquad X,Y \in \Gamma(H), \tag{3.8a}$$

$$\mathrm{Tor}(T,JX) = -J\,\mathrm{Tor}(T,X), \qquad X \in \Gamma(H). \tag{3.8b}$$

The conditions $\nabla J = 0$ and $\nabla h = 0$ make sense because H is parallel.

We discuss in the same way as in the integrable case, e.g., [**Tnk**]. Note first that, since ∇ preserves H, J, and T, it is necessary that the complex-linear extension of ∇ respects the decomposition (3.7). The restriction of ∇ to the first summand is determined by $\nabla T = 0$. Moreover, since ∇ is originally a connection on TM, its restriction to $T^{1,0}M$ automatically determines that to $\overline{T^{1,0}M}$. Therefore it suffices to prove that we can uniquely determine a complex-linear connection ∇ on $T^{1,0}M$ so that $\nabla h = 0$ and (3.8) are satisfied. In the following, Z, W, and V denote arbitrary smooth sections of $T^{1,0}M$. In terms of complex vector fields, (3.8) is rewritten as follows:

$$\mathrm{Tor}(Z,\overline{W}) = ih(Z,\overline{W})T, \tag{3.9a}$$

$$\mathrm{Tor}(T,Z) \in C^\infty(M,\overline{T^{1,0}M}). \tag{3.9b}$$

PROOF OF PROPOSITION 3.2. We first discuss the uniqueness. We prove that, if ∇ satisfies $\nabla h = 0$ and (3.9), then it should satisfy

$$\nabla_{\overline{Z}} W = [\overline{Z},W]_{1,0}, \tag{3.10a}$$

$$h(\nabla_Z W,\overline{V}) = Z(h(W,\overline{V})) - h(W,[Z,\overline{V}]_{0,1}), \tag{3.10b}$$

$$\nabla_T W = [T,W]_{1,0}, \tag{3.10c}$$

where the subscript "$1,0$" (resp. "$0,1$") denotes the projection onto $T^{1,0}M$ (resp. $\overline{T^{1,0}M}$) with respect to (3.7). Since $\mathrm{Tor}(\overline{Z},W) = -ih(\overline{Z},W)T = -\theta([\overline{Z},W])T$,

$$\nabla_{\overline{Z}} W - \nabla_W \overline{Z} = [\overline{Z},W] + \mathrm{Tor}(\overline{Z},W) = [\overline{Z},W]_{1,0} + [\overline{Z},W]_{0,1}.$$

Therefore (3.10a) follows, and (3.10b) is an immediate consequence from (3.10a) and $\nabla h = 0$. Because $\nabla_T W = [T,W] + \mathrm{Tor}(T,W)$ by the definition of the torsion tensor, (3.9b) implies (3.10c).

Now we prove that, if we define ∇ on $T^{1,0}M$ by (3.10), then $\nabla h = 0$ and (3.9) follow. From (3.10a) and (3.10b), it is immediate that $\nabla_Z h = \nabla_{\overline{Z}} h = 0$. We compute $\nabla_T h$ as follows using $T \lrcorner d\theta = 0$:

$$\begin{aligned}
0 &= -id^2\theta(T, Z, \overline{W}) \\
&= -i(T(d\theta(Z, \overline{W})) - d\theta([T, Z], \overline{W}) - d\theta([\overline{W}, T], Z)) \\
&= T(h(Z, \overline{W})) - h([T, Z]_{1,0}, \overline{W}) - h(Z, [T, \overline{W}]_{0,1}) \\
&= T(h(Z, \overline{W})) - h(\nabla_T Z, \overline{W}) - h(Z, \nabla_T \overline{W}) \\
&= (\nabla_T h)(Z, \overline{W}).
\end{aligned}$$

The torsion condition (3.9) is obvious from (3.10a) and (3.10c). □

Let us further examine the torsion of the Tanaka–Webster connection. Equation (3.9a) shows that $\mathrm{Tor}(Z, \overline{W})$ is completely determined by the Levi form, and (3.9b) gives a local invariant $A = \mathrm{Tor}(T, \cdot)$ of pseudohermitian structure, which is the *pseudohermitian torsion tensor:*

$$\mathrm{Tor}_{\alpha\overline{\beta}}{}^{0} = ih_{\alpha\overline{\beta}}, \qquad \mathrm{Tor}_{0\alpha}{}^{\overline{\beta}} = -\mathrm{Tor}_{\alpha 0}{}^{\overline{\beta}} =: A_{\alpha}{}^{\overline{\beta}}.$$

The remaining component $\mathrm{Tor}(Z, W)$ is actually what we already know. This can be seen from $d^2\theta = 0$ as follows. First, by the partial integrability, it is immediate that $\theta(\mathrm{Tor}(Z, W)) = 0$. Moreover,

$$\begin{aligned}
0 &= -i\, d^2\theta(Z, W, \overline{V}) \\
&= Z(h(W, \overline{V})) - W(h(Z, \overline{V})) - h([Z, W], \overline{V}) + h(Z, [W, \overline{V}]_{0,1}) - h(W, [Z, \overline{V}]_{0,1}) \\
&= h(\nabla_Z W, \overline{V}) - h(\nabla_W Z, \overline{V}) - h([Z, W], \overline{V}) \\
&= h(\mathrm{Tor}(Z, W), \overline{V}),
\end{aligned}$$

from which we conclude that $\mathrm{Tor}(Z, W) \in C^\infty(M, \overline{T^{1,0}M})$. This is actually the Nijenhuis tensor because $\mathrm{Tor}(Z, W) = \mathrm{Tor}(Z, W)_{0,1} = -[Z, W]_{0,1} = -N(Z, W)$. In index notation,

$$\mathrm{Tor}_{\alpha\beta}{}^{\overline{\gamma}} = -N_{\alpha\beta}{}^{\overline{\gamma}}. \tag{3.11}$$

The consideration above leads to the following first structure equation, where $\{\, \theta^\alpha \,\}$ is the admissible coframe to $\{\, Z_\alpha \,\}$ and $\{\, \omega_\alpha{}^\beta \,\}$ is the connection forms:

$$d\theta = ih_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}, \tag{3.12}$$

$$d\theta^\gamma = \theta^\alpha \wedge \omega_\alpha{}^\gamma - A_{\overline{\alpha}}{}^\gamma \theta^{\overline{\alpha}} \wedge \theta - \tfrac{1}{2} N_{\overline{\alpha}\overline{\beta}}{}^\gamma \theta^{\overline{\alpha}} \wedge \theta^{\overline{\beta}}. \tag{3.13}$$

Proposition 3.3. *Let Z be a section of $T^{1,0}M$. Then, the Tanaka–Webster torsion tensor A is given by*

$$A(Z) = -[T, Z]_{0,1}. \tag{3.14}$$

In particular, A vanishes if and only if T defines a transverse symmetry, i.e., $T^{1,0}M$ is invariant under the flow generated by T. Furthermore, A has the following *symmetry:*

$$A_{\alpha\beta} = A_{\beta\alpha}. \tag{3.15}$$

Proof. Equation (3.14) is immediate from $A(Z) = \mathrm{Tor}(T, Z) = \nabla_T Z - [T, Z]$, because we already know that it is a section of $\overline{T^{1,0}M}$. Therefore, A vanishes if and only if $[T, Z] = \mathcal{L}_T Z$ is a section of $T^{1,0}M$ for any $Z \in C^\infty(M, T^{1,0}M)$, i.e., $T^{1,0}M$ is invariant by the Reeb flow. Moreover,

$$A_{\alpha\beta} = h(A(Z_\alpha), Z_\beta) = -id\theta([T, Z_\alpha], Z_\beta),$$

and combining $d^2\theta = 0$ and $T \lrcorner d\theta = 0$ we obtain $d\theta([T, Z_\alpha], Z_\beta) - d\theta([T, Z_\beta], Z_\alpha)$. Hence we conclude that (3.15) holds. □

The following lemma is useful when we prove identities involving covariant derivatives. This is a generalization of [**L2**, Lemma 2.1] to the partially-integrable case, and the proof needs no modification.

Lemma 3.4. *Let $T^{1,0}M$ be a nondegenerate partially integrable CR structure and θ a contact form. In a neighborhood of any point $p \in M$, there exists a local frame $\{ Z_\alpha \}$ of $T^{1,0}M$ for which the Tanaka–Webster connection forms $\{ \omega_\alpha{}^\beta \}$ vanish at p.*

For example, the exterior derivative of a 1-form $\sigma = \sigma_\alpha \theta^\alpha$ is given by

$$d\sigma = \sigma_{\alpha,\beta}\theta^\beta \wedge \theta^\alpha + \sigma_{\alpha,\overline{\beta}}\theta^{\overline{\beta}} \wedge \theta^\alpha + \sigma_{\alpha,0}\theta \wedge \theta^\alpha - A_{\overline{\beta}}{}^\alpha \sigma_\alpha \theta^{\overline{\beta}} \wedge \theta - \tfrac{1}{2} N_{\overline{\beta}\overline{\gamma}}{}^\alpha \sigma_\alpha \theta^{\overline{\beta}} \wedge \theta^{\overline{\gamma}}.$$

Here the covariant derivatives of tensors are denoted by indices after commas. In the case of covariant derivatives of a scalar-valued function we omit the comma; e.g., $\nabla_\alpha u = u_\alpha$ and $\nabla_{\overline{\beta}} \nabla_\alpha u = u_{\alpha\overline{\beta}}$. By Lee's argument [**L2**], one can show the following lemma.

Lemma 3.5. *The second covariant derivatives of a scalar-valued function u satisfy the following:*

$$u_{\alpha\overline{\beta}} - u_{\overline{\beta}\alpha} = ih_{\alpha\overline{\beta}} u_0, \qquad u_{\alpha\beta} - u_{\beta\alpha} = -N_{\alpha\beta}{}^{\overline{\gamma}} u_{\overline{\gamma}}, \qquad u_{0\alpha} - u_{\alpha 0} = A_\alpha{}^{\overline{\beta}} u_{\overline{\beta}}. \tag{3.16}$$

Next we study the curvature:

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

If we set $\Pi_\alpha{}^\beta = d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta$, we have $R(X, Y)Z_\alpha = \Pi_\alpha{}^\beta(X, Y)Z_\beta$. Let

$$\begin{aligned} \Pi_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}\sigma\overline{\tau}}\theta^\sigma \wedge \theta^{\overline{\tau}} + W_{\alpha\overline{\beta}\gamma}\,\theta^\gamma \wedge \theta + W_{\alpha\overline{\beta}\overline{\gamma}}\,\theta^{\overline{\gamma}} \wedge \theta \\ + V_{\alpha\overline{\beta}\sigma\tau}\,\theta^\sigma \wedge \theta^\tau + V_{\alpha\overline{\beta}\overline{\sigma}\overline{\tau}}\,\theta^{\overline{\sigma}} \wedge \theta^{\overline{\tau}}, \end{aligned} \tag{3.17}$$

where $V_\alpha{}^\beta{}_{(\sigma\tau)} = V_\alpha{}^\beta{}_{(\overline{\sigma}\overline{\tau})} = 0$. Since $\nabla h = 0$ we have $\Pi_{\alpha\overline{\beta}} + \Pi_{\overline{\beta}\alpha} = 0$, and hence

$$R_{\alpha\overline{\beta}\sigma\overline{\tau}} = R_{\overline{\beta}\alpha\overline{\tau}\sigma}, \qquad W_{\alpha\overline{\beta}\overline{\gamma}} = -W_{\overline{\beta}\alpha\overline{\gamma}}, \qquad V_{\alpha\overline{\beta}\sigma\tau} = -V_{\overline{\beta}\alpha\sigma\tau}. \tag{3.18}$$

We substitute (3.17) into the exterior derivative of (3.13) and compare the coefficients to obtain

$$R_{\alpha\overline{\beta}\sigma\overline{\tau}} - R_{\sigma\overline{\beta}\alpha\overline{\tau}} = -N_{\alpha\sigma}{}^{\overline{\gamma}} N_{\overline{\tau}\overline{\gamma}\overline{\beta}}, \tag{3.19a}$$

$$W_{\alpha\overline{\beta}\gamma} = A_{\alpha\gamma,\overline{\beta}} - N_{\gamma\sigma\alpha} A_{\overline{\beta}}{}^{\sigma}, \tag{3.19b}$$

$$V_{\alpha\overline{\beta}\sigma\tau} = \tfrac{i}{2}(h_{\sigma\overline{\beta}} A_{\alpha\tau} - h_{\tau\overline{\beta}} A_{\alpha\sigma}) + \tfrac{1}{2} N_{\sigma\tau\alpha,\overline{\beta}}. \tag{3.19c}$$

The component $R_{\alpha\overline{\beta}\rho\overline{\sigma}}$ is called the *pseudohermitian curvature tensor*. The *pseudohermitian Ricci* and *scalar* curvatures are defined by

$$R_{\alpha\overline{\beta}} := R_{\alpha}{}^{\gamma}{}_{\gamma\overline{\beta}} \qquad \text{and} \qquad R := R_{\alpha}{}^{\alpha}.$$

It is seen from the first identity of (3.18) that $R_{\alpha\overline{\beta}} = R_{\overline{\beta}\alpha}$. We should be very careful to the indices that are contracted, because (3.19a) implies

$$R_{\gamma}{}^{\gamma}{}_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}} + N_{\alpha\sigma\tau} N_{\overline{\beta}}{}^{\tau\sigma}. \tag{3.20}$$

3.5. Change of contact forms. A choice of a contact form θ defines the connection and supplies various pseudohermitian invariants. If a certain pseudohermitian invariant is also preserved by any change of contact form, then it is called a CR invariant. To investigate such invariants, we need the transformation law of the connection and relevant quantities. As we do not need the full curvature tensor, we make the following summary.

PROPOSITION 3.6. *Let θ and $\hat{\theta} = e^{2\Upsilon}\theta$, $\Upsilon \in C^\infty(M)$, be two contact forms for a nondegenerate partially integrable CR structure $T^{1,0}M$. Then, the Tanaka–Webster connection forms, the torsions and the Ricci tensors are related as follows:*

$$\begin{aligned} \hat{\omega}_{\alpha}{}^{\beta} &= \omega_{\alpha}{}^{\beta} + 2(\Upsilon_{\alpha}\theta^{\beta} - \Upsilon^{\beta}\theta_{\alpha}) + 2\delta_{\alpha}{}^{\beta}\Upsilon_{\gamma}\theta^{\gamma} \\ &\quad + 2i(\Upsilon^{\beta}{}_{\alpha} + 2\Upsilon_{\alpha}\Upsilon^{\beta} + 2\delta_{\alpha}{}^{\beta}\Upsilon_{\gamma}\Upsilon^{\gamma})\theta, \end{aligned} \tag{3.21a}$$

$$\hat{A}_{\alpha\beta} = A_{\alpha\beta} + i(\Upsilon_{\alpha\beta} + \Upsilon_{\beta\alpha}) - 4i\Upsilon_{\alpha}\Upsilon_{\beta} + i(N_{\gamma\alpha\beta} + N_{\gamma\beta\alpha})\Upsilon^{\gamma}, \tag{3.21b}$$

$$\hat{R}_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}} - (n+2)(\Upsilon_{\alpha\overline{\beta}} + \Upsilon_{\overline{\beta}\alpha}) - (\Upsilon_{\gamma}{}^{\gamma} + \Upsilon_{\overline{\gamma}}{}^{\overline{\gamma}} + 4(n+1)\Upsilon_{\gamma}\Upsilon^{\gamma})h_{\alpha\overline{\beta}}. \tag{3.21c}$$

PROOF. The new Reeb vector field is $\hat{T} = e^{-2\Upsilon}(T - 2i\Upsilon^{\alpha}Z_{\alpha} + 2i\Upsilon^{\overline{\alpha}}Z_{\overline{\alpha}})$ and the admissible coframe dual to $\{ Z_\alpha \}$ is $\{ \hat{\theta}^\alpha = \theta^\alpha + 2i\Upsilon^\alpha\theta \}$. To establish (3.21a) and (3.21b), it suffices to check that

$$d\hat{h}_{\alpha\overline{\beta}} = \hat{h}_{\gamma\overline{\beta}}\hat{\omega}_{\alpha}{}^{\gamma} + \hat{h}_{\alpha\overline{\gamma}}\hat{\omega}_{\overline{\beta}}{}^{\overline{\gamma}}$$

and

$$d\hat{\theta}^{\gamma} = \hat{\theta}^{\alpha} \wedge \hat{\omega}_{\alpha}{}^{\gamma} - \hat{h}^{\gamma\overline{\beta}}\hat{A}_{\overline{\alpha}\overline{\beta}}\hat{\theta}^{\overline{\alpha}} \wedge \hat{\theta} - \tfrac{1}{2}N_{\overline{\alpha}\overline{\beta}}{}^{\gamma}\hat{\theta}^{\overline{\alpha}} \wedge \hat{\theta}^{\overline{\beta}}.$$

They are shown straightforward using (3.16). We compute $\hat{\Pi}_{\gamma}{}^{\gamma} = d\hat{\omega}_{\gamma}{}^{\gamma}$ modulo $\hat{\theta}^\alpha \wedge \hat{\theta}^\beta$, $\hat{\theta}^{\overline{\alpha}} \wedge \hat{\theta}^{\overline{\beta}}$, $\hat{\theta}$, or equivalently, modulo $\theta^\alpha \wedge \theta^\beta$, $\theta^{\overline{\alpha}} \wedge \theta^{\overline{\beta}}$, θ. By the first identity of (3.16) we obtain, modulo $\hat\theta^\alpha\wedge\hat\theta^\beta$, $\hat\theta^{\overline{\alpha}}\wedge\hat\theta^{\overline{\beta}}$, $\hat\theta$,

$$\begin{aligned}\hat\Pi_\gamma{}^\gamma &\equiv \Pi_\gamma{}^\gamma - ((n+2)(\Upsilon_{\alpha\overline{\beta}}+\Upsilon_{\overline{\beta}\alpha}) + (\Upsilon_\gamma{}^\gamma + \Upsilon_{\overline{\gamma}}{}^{\overline{\gamma}} + 4(n+1)\Upsilon_\gamma\Upsilon^\gamma)h_{\alpha\overline{\beta}})\theta^\alpha\wedge\theta^{\overline{\beta}}\\ &\equiv (R_{\alpha\overline{\beta}} - (n+2)(\Upsilon_{\alpha\overline{\beta}}+\Upsilon_{\overline{\beta}\alpha}) - (\Upsilon_\gamma{}^\gamma + \Upsilon_{\overline{\gamma}}{}^{\overline{\gamma}} + 4(n+1)\Upsilon_\gamma\Upsilon^\gamma)h_{\alpha\overline{\beta}})\hat\theta^\alpha\wedge\hat\theta^{\overline{\beta}}.\end{aligned}$$

This proves (3.21c). □

3.6. Density bundles over CR manifolds. Here we sketch the concept of density bundles following [**GG**] only in the case of integral biweights. Suppose that we can take an $(n+2)^{\mathrm{nd}}$ root of the CR canonical bundle $K=\bigwedge^{n+1}(\overline{T^{1,0}M})^\perp$ (which is always possible locally). We fix such a line bundle, and write its dual $E(1,0)$. We set

$$E(w,w') := E(1,0)^{\otimes w}\otimes\overline{E(1,0)}^{\otimes w'},\qquad w,\,w'\in\mathbb{Z}.$$

We call $E(w,w')$ the *density bundle of biweight* (w,w'). The space of $E(w,w')$ is denoted by $\mathcal{E}(w,w')$, and its elements are called *densities*. In particular, $E(0,0)=\underline{\mathbb{C}}:=\mathbb{C}\times M$ and $\mathcal{E}(0,0)=C^\infty(M)$, which are also denoted by E and $\mathcal{E}$. Since there is a specified isomorphism $E(-n-2,0)\cong K$, we can uniquely define a connection ∇ on $E(1,0)$ so that the induced connection on $E(w,w')$ agrees with the Tanaka–Webster connection on K when $w=-n-2$. The bundles and the spaces of density-weighted tensors is indicated by adding the weight after the usual symbols. For example,

$$E_{\alpha\overline{\beta}}(w,w') := E_{\alpha\overline{\beta}}\otimes E(w,w'),\qquad \mathcal{E}_{\alpha\overline{\beta}}(w,w') := \mathcal{E}_{\alpha\overline{\beta}}\otimes\mathcal{E}(w,w').$$

If there is no fear of confusion, density-weighted tensors are just called tensors.

Suppose that the line bundle $H^\perp\subset T^*M$ of contact forms is oriented; in the rest of this subsection, contact forms are positive with respect to this orientation. Farris [**Fa**] observed that, if ζ is a locally-defined nonvanishing section of K, there is a unique contact form θ satisfying

$$\theta\wedge(d\theta)^n = i^{n^2} n!(-1)^q\theta\wedge(T\lrcorner\zeta)\wedge(T\lrcorner\overline{\zeta}),\tag{3.22}$$

where q is the number of the negative eigenvalues of the Levi form. We say that this θ is *volume-normalized* by ζ. If we replace ζ with $\lambda\zeta$, $\lambda\in C^\infty(M,\mathbb{C}^\times)$, then θ changes to $|\lambda|^{2/(n+2)}\theta$. We set

$$|\zeta|^{2/(n+2)} = \zeta^{1/(n+2)}\otimes\overline{\zeta}^{1/(n+2)}\in\mathcal{E}(-1,-1),$$

which is independent of the choice of the $(n+2)^{\mathrm{nd}}$ root of ζ and linearly corresponds to θ. Let $|\zeta|^{-2/(n+2)}\in\mathcal{E}(1,1)$ be its inverse. Then we obtain a CR-invariant section $\boldsymbol{\theta}$ of $T^*M\otimes E(1,1)$:

$$\boldsymbol{\theta} := \theta\otimes|\zeta|^{-2/(n+2)}.$$

LEMMA 3.7. $\nabla\boldsymbol{\theta}=0$*, where* ∇ *is the Tanaka–Webster connection for any* θ*.*

PROOF. The volume normalization condition implies $\nabla\zeta = 0$. Hence $\nabla|\zeta|^2 = 0$, and therefore $\nabla|\zeta|^{-2/(n+2)}$ should be zero. $\square$

Since $\boldsymbol{\theta}$ determines a trivialization $\mathbb{C}H^\perp \otimes E(1,1) \longrightarrow \underline{\mathbb{C}}$, there is a canonical identification

$$\mathbb{C}H^\perp \cong E(-1,-1). \tag{3.23}$$

This is compatible with the Tanaka–Webster connection because $\nabla\boldsymbol{\theta} = 0$. Dually, there is an identification

$$\mathbb{C}(TM/H) \cong E(1,1), \qquad (v \bmod \mathbb{C}H) \longmapsto \boldsymbol{\theta}(v). \tag{3.24}$$

We may as well be taking these isomorphisms as the definition of $E(-1,-1)$ or $E(1,1)$ when $H^\perp$ is oriented. Then $E(w,w)$ for $w \in \mathbb{Z}$ is defined only by the contact distribution, and is globally defined if the orientation is given globally.

Since the Levi form $h_{\alpha\overline{\beta}}$ and θ have the same scaling factor,

$$\boldsymbol{h}_{\alpha\overline{\beta}} := h_{\alpha\overline{\beta}} \otimes \theta^{-1} \in \mathcal{E}_{\alpha\overline{\beta}}(1,1)$$

is a parallel CR-invariant tensor, where θ is considered as a density in $\mathcal{E}(-1,-1)$ via (3.23). Its dual is $\boldsymbol{h}^{\alpha\overline{\beta}} \in \mathcal{E}^{\alpha\overline{\beta}}(-1,-1)$. Indices of density-weighted tensors are lowered and raised by $\boldsymbol{h}_{\alpha\overline{\beta}}$ and $\boldsymbol{h}^{\alpha\overline{\beta}}$ unless otherwise stated.

The weighted versions of the Nijenhuis tensor, the pseudohermitian torsion and curvature tensors are defined by

$$\boldsymbol{N}_{\alpha\beta}{}^{\overline{\gamma}} := N_{\alpha\beta}{}^{\overline{\gamma}}, \qquad \boldsymbol{A}_{\alpha\beta} := A_{\alpha\beta}, \qquad \boldsymbol{R}_{\alpha}{}^{\beta}{}_{\sigma\overline{\tau}} := R_{\alpha}{}^{\beta}{}_{\sigma\overline{\tau}}.$$

When we deal with weighted tensors, ∇_α, $\nabla_{\overline{\alpha}}$ and $\nabla_{\mathbf{0}}$ denote the components of ∇ relative to θ^α, $\theta^{\overline{\alpha}}$ and $\boldsymbol{\theta}$. Since the transformation law (3.21a) of the Tanaka–Webster connection forms does not contain the Nijenhuis tensor, that of covariant derivatives of weighted tensors are just the same as in the integrable case. For completeness, we include here the formulae from [**GG**].

PROPOSITION 3.8. *Let $f \in \mathcal{E}(w,w')$ and $\sigma^\alpha \in \mathcal{E}^\alpha$. If θ and $\hat{\theta} = e^{2\Upsilon}\theta$ are two contact forms, then the associated Tanaka–Webster connections ∇, $\hat{\nabla}$ transform as follows:*

$$\begin{aligned}
\hat{\nabla}_\alpha f &= \nabla_\alpha f + w\Upsilon_\alpha f,\\
\hat{\nabla}_{\overline{\alpha}} f &= \nabla_{\overline{\alpha}} f + w'\Upsilon_{\overline{\alpha}} f,\\
\hat{\nabla}_{\mathbf{0}} f &= \nabla_{\mathbf{0}} f + i\Upsilon^{\overline{\gamma}}\nabla_{\overline{\gamma}} f - i\Upsilon^\gamma\nabla_\gamma f\\
&\qquad + \frac{1}{n+2}((w+w')\Upsilon_{\mathbf{0}} + iw\Upsilon_{\overline{\gamma}}{}^{\overline{\gamma}} - iw'\Upsilon_\gamma{}^\gamma + i(w'-w)\Upsilon_\gamma\Upsilon^\gamma)f,\\
\hat{\nabla}_\alpha\tau_\beta &= \nabla_\alpha\tau_\beta - \Upsilon_\alpha\tau_\beta - \Upsilon_\beta\tau_\alpha,\\
\hat{\nabla}_{\overline{\alpha}}\tau_\beta &= \nabla_{\overline{\alpha}}\tau_\beta + \boldsymbol{h}_{\beta\overline{\alpha}}\Upsilon^\gamma\tau_\gamma,\\
\hat{\nabla}_{\mathbf{0}}\tau_\beta &= \nabla_{\mathbf{0}}\tau_\beta + i\Upsilon^{\overline{\gamma}}\nabla_{\overline{\gamma}}\tau_\beta - i\Upsilon^\gamma\nabla_\gamma\tau_\beta - i(\Upsilon^\gamma{}_\beta - \Upsilon_\beta\Upsilon^\gamma)\tau_\gamma.
\end{aligned}$$

3.7. Deformations of partially integrable CR structures. A partially integrable CR structure $T^{1,0}M$ with underlying hyperplane distribution H is naturally regarded as a section of the Grassmannian bundle of n-dimensional subspaces associated to $\mathbb{C}H$. Suppose $\hat{T}^{1,0}$ is another almost CR structure on the same distribution H. If $\hat{T}^{1,0}$ projects onto $T^{1,0}M$ by the projection map $\mathbb{C}H = T^{1,0}M \oplus \overline{T^{1,0}M} \longrightarrow T^{1,0}M$, then it is described by a $\mathbb{C}$-homomorphism $\varphi\colon T^{1,0}M \longrightarrow \overline{T^{1,0}M}$ as follows:

$$\hat{T}^{1,0} = \coprod_{p\in M} \hat{T}^{1,0}_p, \qquad \hat{T}^{1,0}_p = \{\, Z + \varphi_p(Z) \mid Z \in T^{1,0}_p M \,\}. \tag{3.25}$$

In this sense, a sufficiently small almost CR deformation of $T^{1,0}M$ is described by $\varphi \in \operatorname{Hom}(T^{1,0}M, \overline{T^{1,0}M})$. For the converse, let $\iota\colon \overline{T^{1,0}M} \longrightarrow T^{1,0}M$ be the complex conjugation. Then, if the pointwise eigenvalues of $\iota\circ\varphi$ are all less than 1, then the subbundle $\hat{T}^{1,0}$ defined by (3.25) is an almost CR structure. Note that $\mathrm{id} + \varphi\colon T^{1,0}M \longrightarrow \hat{T}^{1,0}$ is an isomorphism under this condition on the eigenvalues.

PROPOSITION 3.9. *Let $T^{1,0}M$ be a nondegenerate partially integrable CR structure and $\varphi \in \operatorname{Hom}(T^{1,0}M, \overline{T^{1,0}M})$ a $\mathbb{C}$-homomorphism such that the pointwise eigenvalues of $\iota\circ\varphi$ are all less than 1. Then, $\hat{T}^{1,0}$ is partially integrable if and only if*

$$\varphi_{\alpha\beta} = \varphi_{\beta\alpha},$$

where

$$\varphi_{\alpha\beta} := \boldsymbol{h}_{\beta\overline{\gamma}}\varphi_{\alpha}{}^{\overline{\gamma}} \in \mathcal{E}_{\alpha\beta}(1,1)$$

and $\boldsymbol{h}_{\alpha\overline{\beta}}$ is the weighted Levi form of $T^{1,0}M$.

PROOF. Let $\{\, Z_\alpha \,\}$ be a local frame of the original partially integrable CR structure $T^{1,0}M$. Then $\{\, \hat{Z}_\alpha = Z_\alpha + \varphi_\alpha{}^{\overline{\beta}} Z_{\overline{\beta}} \,\}$ is a local frame of $\hat{T}^{1,0}$. The latter almost CR structure is partially integrable if and only if

$$\theta([\hat{Z}_\alpha, \hat{Z}_\beta]) = \theta([Z_\alpha + \varphi_\alpha{}^{\overline{\sigma}} Z_{\overline{\sigma}}, Z_\beta + \varphi_\beta{}^{\overline{\tau}} Z_{\overline{\tau}}]) = 0,$$

where θ is any contact form. Since θ annihilates $[Z_\alpha, Z_\beta] = [Z_{\overline{\sigma}}, Z_{\overline{\tau}}]$ by the partial integrability, this is equivalent to

$$\theta([Z_{\overline{\sigma}}, Z_\beta])\varphi_\alpha{}^{\overline{\sigma}} + \theta([Z_\alpha, Z_{\overline{\tau}}])\varphi_\beta{}^{\overline{\tau}} = 0,$$

or $\varphi_{\alpha\beta} - \varphi_{\beta\alpha} = 0$. □

Suppose $\hat{T}^{1,0}_t$ is a smooth 1-parameter family of nondegenerate partially integrable CR structures defined for small $|t|$ sharing the same underlying contact distribution (this condition for the underlying contact distribution is always implicitly assumed in the sequel). Suppose $\hat{T}^{1,0}_t$ is *based at* $T^{1,0}M$, i.e., $\hat{T}^{1,0}_0 = T^{1,0}M$. Then, after restricting for smaller $|t|$ if necessary, $\hat{T}^{1,0}_t$ is described by a family

$\varphi_t \in \mathrm{Hom}(T^{1,0}M, \overline{T^{1,0}M})$ such that $(\varphi_t)_{\alpha\beta}$ is symmetric. So we make the following definition. The CR invariance, and the meaning, of the condition (3.26) will become clear in Proposition 3.11.

DEFINITION 3.10. Let $T^{1,0}M$ be a nondegenerate partially integrable CR structure. Then any tensor $\psi_{\alpha\beta} \in \mathcal{E}_{(\alpha\beta)}(1,1)$ is said to determine an *infinitesimal deformation of partially integrable CR structure.* If $T^{1,0}M$ is integrable, then $\psi_{\alpha\beta}$ is called *integrable* if the following holds:

$$\nabla_{[\alpha}\psi_{\beta]\gamma} = 0. \tag{3.26}$$

When $\hat{T}^{1,0}_t$ is a smooth 1-parameter family of partially integrable CR structures based at $T^{1,0}M$, then it is said to be *tangent* to $\psi_{\alpha\beta}$ if $\varphi^\bullet_{\alpha\beta} = \psi_{\alpha\beta}$, where $\varphi^\bullet_{\alpha\beta}$ is the derivative at $t = 0$ of $(\varphi_t)_{\alpha\beta}$ described in the last paragraph.

We want to compute the variations of various tensors associated to partially integrable CR structure. In order to do this, we use the isomorphism $\mathrm{id}+\varphi\colon T^{1,0}M \to \hat{T}^{1,0}$. For example, the Levi form h_t (with respect to a fixed θ) is a hermitian form on $\hat{T}^{1,0}_t$, so we consider the pullback of this form by $\mathrm{id} + \varphi_t$. Then h_t becomes a hermitian form $T^{1,0}M$ for all t (we still use the symbol h_t for the pullback). If $\{ Z_\alpha \}$ is a local frame of $T^{1,0}M$, then

$$\begin{aligned}(h_t)_{\alpha\overline{\beta}} &= i\, d\theta(\hat{Z}_\alpha, \hat{Z}_{\overline{\beta}}) = i\, d\theta(Z_\alpha + t\psi_\alpha{}^{\overline{\sigma}} Z_{\overline{\sigma}}, Z_{\overline{\beta}} + t\psi_{\overline{\beta}}{}^{\tau} Z_\tau) + O(t^2)\\ &= i\, d\theta(Z_\alpha, Z_{\overline{\beta}}) + O(t^2) = h_{\alpha\overline{\beta}} + O(t^2).\end{aligned}$$

Hence we obtain

$$h^\bullet_{\alpha\overline{\beta}} = 0. \tag{3.27}$$

Other tensors will be treated similarly in the sequel. If there is contravariant factors, then we push them forward by $(\mathrm{id} + \varphi)^{-1}$.

PROPOSITION 3.11. *Let $T^{1,0}M$ be a nondegenerate integrable CR structure and $\hat{T}^{1,0}_t$ is a 1-parameter family of partially integrable CR structures based at $T^{1,0}M$ that is tangent to $\psi_{\alpha\beta}$. Then the variation of the Nijenhuis tensor is given by*

$$N^\bullet_{\alpha\beta}{}^{\overline{\gamma}} = 2\nabla_{[\alpha}\psi_{\beta]}{}^{\overline{\gamma}}. \tag{3.28}$$

PROOF. Let $\{ Z_\alpha \}$ be a special local frame stated in Lemma 3.4, for which the Tanaka–Webster connection forms $\{ \omega_\alpha{}^\beta \}$ vanish at $p \in M$. Recall that $(N_t)_{\alpha\beta}{}^{\overline{\gamma}} = \hat{\theta}^{\overline{\gamma}}([\hat{Z}_\alpha, \hat{Z}_\beta]) = -d\hat{\theta}^{\overline{\gamma}}(\hat{Z}_\alpha, \hat{Z}_\beta)$. Because $\hat{\theta}^{\overline{\gamma}} = \theta^{\overline{\gamma}} - t\psi_\sigma{}^{\overline{\gamma}}\theta^\sigma + O(t^2)$,

$$(N_t)_{\alpha\beta}{}^{\overline{\gamma}} = \theta^{\overline{\gamma}}([\hat{Z}_\alpha, \hat{Z}_\beta]) - t\psi_\sigma{}^{\overline{\gamma}}\theta^\sigma([Z_\alpha, Z_\beta]) + O(t^2).$$

The second term vanishes at p because of (3.13) and that $\theta^\sigma([Z_\alpha, Z_\beta]) = -d\theta^\sigma(Z_\alpha, Z_\beta)$. The first term is computed at p as follows, again using (3.13) and the fact that

$T^{1,0}M$ is integrable:

$$\begin{aligned}\theta^{\overline{\gamma}}([\hat{Z}_\alpha, \hat{Z}_\beta]) &= \theta^{\overline{\gamma}}([Z_\alpha, Z_\beta]) + t\theta^{\overline{\gamma}}([\psi_\alpha{}^{\overline{\sigma}} Z_{\overline{\sigma}}, Z_\beta]) + t\theta^{\overline{\gamma}}([Z_\alpha, \psi_\beta{}^{\overline{\tau}} Z_{\overline{\tau}}]) + O(t^2)\\ &= t\theta^{\overline{\gamma}}(\psi_\alpha{}^{\overline{\sigma}}[Z_{\overline{\sigma}}, Z_\beta] - (Z_\beta \psi_\alpha{}^{\overline{\sigma}}) Z_{\overline{\sigma}})\\ &\quad + t\theta^{\overline{\gamma}}(\psi_\beta{}^{\overline{\tau}}[Z_\alpha, Z_{\overline{\tau}}] + (Z_\alpha \psi_\beta{}^{\overline{\tau}}) Z_{\overline{\tau}}) + O(t^2)\\ &= t(Z_\alpha \psi_\beta{}^{\overline{\gamma}} - Z_\beta \psi_\alpha{}^{\overline{\gamma}}) + O(t^2) = t(\nabla_\alpha \psi_\beta{}^{\overline{\gamma}} - \nabla_\beta \psi_\alpha{}^{\overline{\gamma}}) + O(t^2).\end{aligned}$$

Therefore (3.28) follows. □

3.8. Deformations on the Heisenberg group. Let $M = \mathcal{H}$ be the Heisenberg group of dimension $2n+1$:

$$\mathcal{H} := \{\, z = (z', w) \in \mathbb{C}^n \times \mathbb{C} \mid \operatorname{Im} w = |z'|^2 \,\}.$$

By setting $t := \operatorname{Re} w$, we can identify $\mathcal{H}$ with $\mathbb{C}^n \times \mathbb{R} = \{\, (z', t) \,\}$. If we write $z' = (z^\alpha) = (z^1, \dots, z^n)$, the standard CR structure $T^{1,0}M$ is spanned by

$$Z_\alpha := \frac{\partial}{\partial z^\alpha} + i\overline{z}^\alpha \frac{\partial}{\partial t}, \qquad \alpha = 1, \dots, n.$$

The frame $\{\, Z_\alpha \,\}$ is called the *standard frame*. The following is the *standard contact form*:

$$\theta := \frac{1}{2}\left(dt - i \sum_{\alpha=1}^{n} (\overline{z}^\alpha \, dz^\alpha - z^\alpha \, d\overline{z}^\alpha) \right).$$

The associated Reeb vector field $T = 2\,\partial/\partial t$ and $\{\, \theta^\alpha = dz^\alpha \,\}$ is the admissible coframe for the standard frame $\{\, Z_\alpha \,\}$. The Levi form is given by

$$h_{\alpha\overline{\beta}} = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Since $d\theta^\alpha = 0$, the Tanaka–Webster connection forms $\omega_\alpha{}^\beta$ and the pseudohermitian torsion tensor $A_\alpha{}^{\overline{\beta}}$ all vanish identically.

Proposition 3.12. *Let $M = \mathcal{H}$ be the $(2n+1)$-dimensional Heisenberg group. Suppose $\hat{T}^{1,0}_t$ is a family of partially integrable CR structure based at the standard CR structure that is tangent to $\psi_{\alpha\beta}$. Then, the variations of the pseudohermitian torsion tensor $A_{\alpha\beta}$ and the pseudohermitian Ricci tensor $R_{\alpha\overline{\beta}}$ associated to the standard contact form θ are given by*

$$A^\bullet_{\alpha\beta} = -\nabla_0 \psi_{\alpha\beta}, \qquad R^\bullet_{\alpha\overline{\beta}} = -\nabla_\alpha \nabla^{\overline{\sigma}} \psi_{\overline{\beta}\overline{\sigma}} - \nabla_{\overline{\beta}} \nabla^\tau \psi_{\alpha\tau}. \tag{3.29}$$

Proof. We use the standard frame $\{\, Z_\alpha \,\}$ for computation. Then,

$$[T, \hat{Z}_\alpha] = t(\nabla_0 \psi_\alpha{}^{\overline{\beta}}) Z_{\overline{\beta}} + O(t^2),$$

and hence, by (3.14),

$$(A_t)_\alpha{}^{\overline{\beta}} = -\hat{\theta}^{\overline{\beta}}([T, \hat{Z}_\alpha]) = -\theta^{\overline{\beta}}([T, \hat{Z}_\alpha]) + O(t^2) = -\nabla_0 \psi_\alpha{}^{\overline{\beta}} + O(t^2).$$

Moreover, because

$$[\hat{Z}_\alpha, \hat{Z}_{\overline{\beta}}] = t((\nabla_\alpha \psi_{\overline{\beta}}{}^{\sigma}) Z_\sigma - (\nabla_{\overline{\beta}} \psi_\alpha{}^{\overline{\tau}}) Z_{\overline{\tau}}) + O(t^2),$$

by (3.10a)–(3.10c) we have

$$\begin{aligned}
(\omega_t)_\alpha{}^\beta(\hat{Z}_{\overline{\gamma}}) &= -t\nabla_\alpha \psi_{\overline{\gamma}}{}^\beta + O(t^2), \\
(\omega_t)_\alpha{}^\beta(Z_\gamma) &= t\nabla^\beta \psi_{\alpha\gamma} + O(t^2), \\
(\omega_t)_\alpha{}^\beta(T) &= O(t^2).
\end{aligned}$$

Because the Nijenhuis tensor is $O(t)$, from (3.20) we obtain

$$\begin{aligned}
(R_t)_{\alpha\overline{\beta}} &= (R_t)_\gamma{}^\gamma{}_{\alpha\overline{\beta}} + O(t^2) = (d(\omega_t)_\gamma{}^\gamma)(\hat{Z}_\alpha, \hat{Z}_{\overline{\beta}}) + O(t^2) \\
&= \hat{Z}_\alpha (\omega_t)_\gamma{}^\gamma(\hat{Z}_{\overline{\beta}}) - \hat{Z}_{\overline{\beta}} (\omega_t)_\gamma{}^\gamma(\hat{Z}_\alpha) - (\omega_t)_\gamma{}^\gamma([\hat{Z}_\alpha, \hat{Z}_{\overline{\beta}}]) + O(t^2) \\
&= -t(\nabla_\alpha \nabla^{\overline{\gamma}} \psi_{\overline{\beta}\overline{\gamma}} + \nabla_{\overline{\beta}} \nabla^\gamma \psi_{\alpha\gamma}) + O(t^2).
\end{aligned}$$

This completes the proof. □

4. Θ-structures and asymptotically complex hyperbolic metrics

4.1. Θ-structures. Here we introduce the notion of Θ-structure due to Epstein–Melrose–Mendoza [**EMM**]. The description in [**GS**] is also helpful.

Let $\overline{X}$ be a smooth manifold-with-boundary with dimension $2n+2$, $n \geq 1$. Suppose we are given a section of $T^*\overline{X}$ defined only along the boundary:

$$\Theta \in C^\infty(\partial X, T^*\overline{X}|_{\partial X}).$$

We assume the following conditions on Θ, where $\iota \colon \partial X \hookrightarrow \overline{X}$ is the inclusion map:

(i) $\iota^*\Theta$ is a nowhere vanishing 1-form on ∂X;

(ii) Moreover, the kernel of $\theta = \iota^*\Theta$ is a contact distribution on ∂X, i.e., $\theta \wedge (d\theta)^n$ is nowhere vanishing.

Some of the following discussion still works without (ii), but for simplicity we take it for granted from the beginning.

A subset $\mathcal{V}_\Theta$ of $C^\infty(\overline{X}, T\overline{X})$ is defined as follows. Let $\tilde{\Theta} \in C^\infty(\overline{X}, T^*\overline{X})$ be any smooth extension of Θ. Then, for an arbitrary boundary defining function $\rho \in C^\infty(\overline{X})$, a vector field $V \in C^\infty(\overline{X}, T\overline{X})$ belongs to $\mathcal{V}_\Theta$ if and only if

$$V \in \rho C^\infty(\overline{X}, T\overline{X}), \qquad \tilde{\Theta}(V) \in \rho^2 C^\infty(\overline{X}).$$

By the first condition, whether the second is satisfied or not is determined only by Θ. Moreover, it is clear that $\mathcal{V}_\Theta$ depends only on the conformal class of Θ. So, we define the notion of Θ-structure as follows; if a Θ-structure is given, then $\mathcal{V}_\Theta$ is defined without ambiguity.

Definition 4.1. A Θ-*structure* on a smooth manifold-with-boundary $\overline{X}$ is a conformal class $[\Theta]$ of elements of $C^\infty(\partial X, T^*\overline{X}|_{\partial X})$ satisfying conditions (i) and

(ii) above. A pair $(\overline{X}, [\Theta])$ is called a Θ-*manifold.* Any contact form that belongs to the class $\iota^*[\Theta]$ is called a *compatible contact form* on ∂X.

There is a canonical smooth vector bundle ${}^\Theta T\overline{X}$ of rank $2n+2$ over $\overline{X}$, whose global sections are identified with the elements of $\mathcal{V}_\Theta$. Over the interior X, $({}^\Theta T\overline{X})|_X$ is isomorphic to the usual tangent bundle TX. To illustrate the structure near $p \in \partial X$, let $\{ N, T, Y_i \} = \{ N, T, Y_1, \dots, Y_{2n} \}$ be a local frame of $T\overline{X}$ in a neighborhood of p dual to a certain coframe of the form $\{ d\rho, \tilde{\Theta}, \alpha^i \}$, where $\tilde{\Theta}$ is an extension of some representative $\Theta \in [\Theta]$. Then any $V \in \mathcal{V}_\Theta$ is, near p, expressed as

$$V = a\rho N + b\rho^2 T + c^i \rho Y_i, \qquad a,\ b,\ c^i \in C^\infty(X). \tag{4.1}$$

Hence $\{ \rho N, \rho^2 T, \rho Y_i \}$ extends to a frame of ${}^\Theta T\overline{X}$ near $p \in \partial X$. One can also see from this that $\mathcal{V}_\Theta$ is a Lie subalgebra of $C^\infty(\overline{X}, T\overline{X})$.

Definition 4.2. The vector bundle ${}^\Theta T\overline{X}$ is called the Θ-*tangent bundle*, and its dual vector bundle is the Θ-*cotangent bundle* ${}^\Theta T^*\overline{X}$. Their sections are called Θ-*vector fields* and *1-*Θ-*forms*, respectively. Sections of tensor products of ${}^\Theta T\overline{X}$s and ${}^\Theta T^*\overline{X}$s are called Θ-*tensors* in general.

For $p \in \partial X$, let F_p be the set of Θ-vector fields that vanish at p. Then the fiber ${}^\Theta T_p\overline{X}$ is naturally identified with the quotient vector space $\mathcal{V}_\Theta / F_p$. The point is that F_p is an ideal of $\mathcal{V}_\Theta$, as it is verified using (4.1), and therefore ${}^\Theta T_p\overline{X}$ is a Lie algebra. Thanks to the contact condition (ii), its derived series is

$$K_{2,p} \subset K_{1,p} \subset {}^\Theta T_p\overline{X},$$

where

$$K_{1,p} := \langle \rho^2 T, \rho Y_1, \dots, \rho Y_{2n} \rangle / F_p, \qquad K_{2,p} := \langle \rho^2 T \rangle / F_p.$$

Collecting these subspaces, we obtain two subbundles of ${}^\Theta T\overline{X}|_{\partial X}$, which we write K_1 and K_2. The line bundle K_2 has a canonical orientation, with respect to which $t_p \in K_{2,p} \setminus \{ 0 \}$ is positive if and only if $\tilde{\Theta}(\tilde{t}) \geq 0$ near p, where $\tilde{\Theta}$ and $\tilde{t}$ are arbitrary extensions of Θ and t_p, respectively.

Any fiber metric g of ${}^\Theta T\overline{X}$ is called a Θ-*metric.* Since ${}^\Theta T\overline{X}|_X$ is canonically identified with TX, g can be regarded as a Riemannian metric defined on X, and hence it determines the Levi-Civita connection ∇. Actually, ∇ has a property that is suitable for manipulations involving Θ-tensor bundles as follows.

Definition 4.3. A Θ-*connection* on a Θ-manifold $(\overline{X}, [\Theta])$ is an $\mathbb{R}$-linear mapping

$$\nabla \colon C^\infty(\overline{X}, {}^\Theta T\overline{X}) \longrightarrow C^\infty(\overline{X}, {}^\Theta T^*\overline{X} \otimes {}^\Theta T\overline{X}) = C^\infty(\overline{X}, \operatorname{End}({}^\Theta T\overline{X})),$$

such that $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$ for $f \in C^\infty(X)$, where we write $(\nabla Y)(X) = \nabla_X Y$.

PROPOSITION 4.4. *Let g be a Θ-metric on a Θ-manifold. Then its Levi-Civita connection ∇ satisfies*

$$X,\, Y \in \mathcal{V}_\Theta \Longrightarrow \nabla_X Y \in \mathcal{V}_\Theta.$$

Therefore, ∇ is naturally regarded as a Θ-connection.

PROOF. If $\Gamma^K{}_{IJ}$ is the Christoffel symbol for the Levi-Civita connection of g with respect to a local frame $\{\, \boldsymbol{Y}_I \,\}$ of ${}^\Theta T\overline{X}$, then $\Gamma_{KIJ} := g_{KL}\Gamma^L{}_{IJ}$ is a linear combination of the derivatives $\boldsymbol{Y}_I g_{KL}$ and C_{KIJ}, where $[\boldsymbol{Y}_I, \boldsymbol{Y}_J] = C^K{}_{IJ} Y_K$ and $C_{KIJ} := g_{KL} C^L{}_{IJ}$. Since $\mathcal{V}_\Theta$ is closed under the Lie bracket, the coefficients $C^K{}_{IJ}$ are smooth up to the boundary of $\overline{X}$. □

4.2. ACH metrics. Let g be a Θ-metric with possibly indefinite signature. Recall the canonical filtration $K_2 \subset K_1 \subset {}^\Theta T\overline{X}|_{\partial X}$. We would like to consider the orthogonal decomposition

$$ {}^\Theta T\overline{X} = R \oplus K_1, \qquad K_1 = K_2 \oplus L; \tag{4.2}$$

this is always possible if g is positive definite. In the case of indefinite metrics, we assume that this is possible. The 1-Θ-form $d\rho/\rho$ is, if restricted to ∂X, independent of the choice of a boundary defining function ρ. This $(d\rho/\rho)|_{\partial X}$ is a canonical section of the line bundle $K_1^\perp \subset {}^\Theta T^*\overline{X}$. We assume that the following two conditions are satisfied:

$$\left|\frac{d\rho}{\rho}\right|_g^2 = \frac{1}{4} \qquad \text{at } \partial X, \tag{4.3}$$

$$g \text{ is positive-definite on } K_2. \tag{4.4}$$

If we define $r \in C^\infty(\partial X, R)$ by $(d\rho/\rho)(r) = 1$, then (4.3) is equivalent to $|r|_g^2 = 4$. Note that if these conditions are satisfied then the orthogonal decomposition (4.2) is automatically possible.

When we fix $\Theta \in [\Theta]$ and ρ, then $(\tilde{\Theta}/\rho^2)|_{K_2}$ is a well-defined section of K_2^*. It is also true that this is determined by $\theta = \iota^*\Theta$ and ρ. Let t be the section of K_2 given by the condition $(\tilde{\Theta}/\rho^2)(t) = 1$. We say that ρ is an *admissible boundary defining function* for θ if $|t|_g^2 = 1$. Let $\mathcal{F}_\theta$ be the set of preferred boundary defining functions for θ. Condition (4.4) guarantees that $\mathcal{F}_\theta$ is not empty for any θ.

LEMMA 4.5. *Let g be a fixed Θ-metric satisfying* (4.4). *Then, the first jet of $\rho \in \mathcal{F}_\theta$ is uniquely determined by θ. This gives a one-to-one correspondence between compatible contact forms and the first jets of boundary defining functions. Furthermore, for given θ, $\mathcal{F}_\theta$ is determined only by $g|_{\partial X}$.*

PROOF. Two preferred boundary defining functions ρ_1 and ρ_2 for the same θ should satisfy $\tilde{\Theta}/\rho_1^2 = \tilde{\Theta}/\rho_2^2$ along ∂X, and this is equivalent to that $\rho_1 = \rho_2 + O(\rho^2)$. The second statement is because, for any boundary defining function ρ, there is a compatible contact form θ for which ρ is preferred. The last assertion is obvious by the definition. $\square$

Let $H \subset T(\partial X)$ be the contact distribution associated to $[\Theta]$. Given a boundary defining function ρ, there is a vector-bundle isomorphism

$$\lambda_\rho \colon H \to L, \qquad Y_p \mapsto \pi_p(\rho\tilde{Y} \bmod F_p), \tag{4.5}$$

where $\tilde{Y} \in C^\infty(\overline{X}, T\overline{X})$ is any extension of $Y_p \in H_p$ and $\pi_p \colon K_{1,p} \to L_p$ is the projection with respect to the decomposition (4.2). Again, λ_ρ depends only on the first jet of ρ.

Definition 4.6. Let $(\overline{X}, [\Theta])$ a Θ-manifold and $M = \partial X$. An *ACH metric* is a Θ-metric g satisfying (4.3), (4.4) and the following additional conditions:

(i) For any $p \in \partial X$, the map

$$L_p \longrightarrow {}^\Theta T_p\overline{X}, \qquad Z_p \longmapsto [r_p, Z_p], \tag{4.6}$$

is the identity map onto L_p;

(ii) There is a nondegenerate partially integrable CR structure $T^{1,0}M$ with underlying contact structure H such that, for some (hence for any) compatible contact form θ on M, $\lambda_\rho^*(g|_L)$ agrees with the Levi form, where ρ is a preferred boundary defining function for θ.

If g is an ACH metric, then the triple $(\overline{X}, [\Theta], g)$ is called an *ACH manifold.*

On condition (ii), the assumption of partial integrability is actually not restrictive here, since if $\lambda_\rho^*(g|_L) = (d\theta)|_H(\cdot, J\cdot)$ holds for the endomorphism J giving an almost CR structure $T^{1,0}M$, then $(d\theta)|_H(\cdot, J\cdot)$ is symmetric, which implies that $T^{1,0}M$ is partially integrable. Furthermore, because of the contact condition, J is uniquely determined. So we can define as follows.

Definition 4.7. We say that $(M, T^{1,0}M)$ is the *infinity* of the ACH manifold, or that $T^{1,0}M$ is *induced* by the ACH metric g.

Here is a technical lemma used in the next subsection.

Lemma 4.8. *Let g be a Θ-metric satisfying* (4.3) *and* (4.4). *Suppose that there is a local frame $\{ N, T, Y_j \}$ of $T\overline{X}$ around $p \in \partial X$, which is dual to $\{ d\rho, \tilde{\Theta}, \alpha^j \}$, such that*

$$d\tilde{\Theta}(N, Y_j) = -\tilde{\Theta}([N, Y_j]) = O(\rho) \qquad \textit{and} \qquad R_p = \langle \rho N \rangle / F_p.$$

Then, the map (4.6) *is the identity onto L_p if and only if $L_p = \langle \rho Y_1, \dots, \rho Y_{2n} \rangle / F_p$.*

Proof. Since $r_p = (\rho N)_p$, $[\rho N, \rho^2 T] \equiv 2\rho^2 T$ and $[\rho N, \rho Y_j] \equiv \rho Y_j$ modulo F_p. $\square$

4.3. Normalization of ACH metrics. Let $(\overline{X}, [\Theta], g)$ be an ACH manifold and $M = \partial X$. We have seen in Lemma 4.5 that a first jet of boundary defining functions is uniquely determined for each θ. There is a canonical way to extend it to a germ along ∂X as shown below. This (germ of) boundary defining function(s) is called the *model boundary defining function* following the terminology of [**GS**].

LEMMA 4.9. *Let g be a Θ-metric satisfying* (4.3), (4.4). *Then, for any compatible contact form θ, there exists a boundary defining function ρ whose first jet corresponds to θ via Lemma 4.5 that satisfies*

$$\left|\frac{d\rho}{\rho}\right|_g^2 = \frac{1}{4} \qquad \textit{near } \partial X. \tag{4.7}$$

The germ of ρ along ∂X is unique.

PROOF. Take any admissible boundary defining function ρ' as a reference, and set $\rho = e^{\psi}\rho'$. Then $|d\rho/\rho|_g^2 = 1/4$ is equivalent to

$$\frac{2X_{\rho'}}{\rho'}\psi + \rho\left|\frac{d\psi}{\rho'}\right|_g^2 = \frac{1}{\rho'}\left(\frac{1}{4} - \left|\frac{d\rho'}{\rho'}\right|_g^2\right), \tag{4.8}$$

where $X_{\rho'} := \sharp_g(d\rho'/\rho')$ is the dual Θ-vector field of $d\rho'/\rho'$ with respect to g. If we write

$$X_{\rho'} = a\rho N + b\rho^2 T + c^i \rho Y_i, \qquad a,\, b,\, c^i \in C^\infty(\overline{X}),$$

then the assumption (4.3) implies that $(d\rho'/\rho')(X_{\rho'}) = 1/4$ on ∂X, and hence $a = 1/4$ on ∂X. Hence (4.8) is a noncharacteristic first-order PDE. The first-jet condition implies that ψ should be zero along ∂X, and thus we obtain a unique solution of (4.8) near ∂X. □

Fix any compatible contact form θ. Let ρ be a model boundary defining function associated to θ and $X_\rho := \sharp_g(d\rho/\rho)$. We consider the smooth map induced by the flow Fl_t of the vector field $4X_\rho/\rho$, which is transverse to ∂X:

$$\Phi\colon (\text{an open neighborhood of } M \text{ in } M \times [0,\infty)) \longrightarrow \overline{X}, \qquad (p,t) \longmapsto \mathrm{Fl}_t(p).$$

Here M is identified with $M \times \{0\}$. The manifold-with-boundary $M \times [0,\infty)$ carries a standard Θ-structure $[\Theta]_{\mathrm{std}}$, which is given by extending the class of contact forms in such a way that it annihilates ∂_t. On the other hand, $[\Theta]$ annihilates $4X_\rho/\rho$ along ∂X because

$$\tilde{\Theta}(4X_\rho/\rho) = 4\rho g(d\rho/\rho, \tilde{\Theta}/\rho^2) = O(\rho).$$

Therefore, we conclude that Φ is a Θ-*diffeomorphism,* i.e., a diffeomorphism preserving Θ-structures, onto its image. Since $d\rho(4X_\rho/\rho) = 4g(d\rho, d\rho/\rho^2) = 1$, it holds that $t = \Phi^*\rho$ and that $t\partial_t$ is orthogonal to $\ker(dt/t)$ with respect to the induced Θ-metric Φ^*g. Moreover, $|t\partial_t|_{\Phi^*g}^2 = |4X_\rho|_g^2 = 4$, and hence t is the model boundary defining function on $(M \times [0,\infty)_t, [\Theta]_{\mathrm{std}}, \Phi^*g)$ for θ.

DEFINITION 4.10. Let $(M, T^{1,0}M)$ be a nondegenerate partially integrable CR manifold and $\overline{X} := M \times [0,\infty)_\rho$ equipped with the standard Θ-structure. Let θ a contact form on $(M, T^{1,0}M)$. Then a *normal-form ACH metric* g for $(M, T^{1,0}M, \theta)$ is an ACH metric defined near the boundary of $\overline{X}$ satisfying the following conditions:

(i) $\rho\partial_\rho$ is orthogonal to $\ker(d\rho/\rho)$ with respect to g;
(ii) ρ is a model boundary defining function on $(\overline{X}, [\Theta]_{\mathrm{std}}, g)$ for θ;
(iii) the induced partially integrable CR structure at infinity is the original one.

The triple $(\overline{X}, [\Theta]_{\mathrm{std}}, g)$ is called a *normal-form ACH manifold* for $(M, T^{1,0}M, \theta)$.

The discussion so far in this subsection is summed up as follows.

PROPOSITION 4.11. *Let $(\overline{X}, [\Theta], g)$ be an ACH manifold with C^∞-smooth ACH metric g, and $(M, T^{1,0}M)$ its infinity. Then, for any choice of a compatible contact form θ, a sufficiently small neighborhood of the boundary of $(\overline{X}, [\Theta], g)$ is identified with a normal-form ACH manifold for $(M, T^{1,0}M, \theta)$ via a boundary-fixing Θ-diffeomorphism.*

PROPOSITION 4.12. *Let $(M, T^{1,0}M)$ be a nondegenerate partially integrable CR manifold and $\overline{X} := M \times [0,\infty)_\rho$ carrying the standard Θ-structure. Let $\{ Z_\alpha \}$ be a local frame of $T^{1,0}M$, $\{ \theta^\alpha \}$ a family of 1-forms on M such that $\{ \theta^\alpha \mid_{T^{1,0}M} \}$ is the dual coframe for $\{ Z_\alpha \}$. Let θ be a contact form on $(M, T^{1,0}M)$. The 1-forms θ, θ^α and $\theta^{\overline{\alpha}}$ are extended in such a way that they annihilate ∂_ρ and are killed by $\mathcal{L}_{\partial_\rho}$. Then, a Θ-metric g on $\overline{X}$ is a normal-form ACH metric for $(M, T^{1,0}M, \theta)$ if and only if it is of the form*

$$g = 4\left(\frac{d\rho}{\rho}\right)^2 + g_{00}\left(\frac{\theta}{\rho^2}\right)^2 + 2g_{0A}\frac{\theta}{\rho^2}\frac{\theta^A}{\rho} + g_{AB}\frac{\theta^A}{\rho}\frac{\theta^B}{\rho}, \tag{4.9}$$

where the indices A, B run $\{ 1, \dots, n, \overline{1}, \dots, \overline{n} \}$, and satisfies

$$g_{00}|_M = 1, \qquad g_{0\alpha}|_M = 0, \qquad g_{\alpha\overline{\beta}}|_M = h_{\alpha\overline{\beta}}, \qquad \text{and} \qquad g_{\alpha\beta}|_M = 0, \tag{4.10}$$

where $h_{\alpha\overline{\beta}}$ is the Levi form associated with θ.

PROOF. The condition $\rho\partial_\rho \perp_g \ker(d\rho/\rho)$, together with that ρ is a model boundary defining function, implies that g is of the form (4.9). Because ρ is preferred for θ, g_{00} must be 1 at M. By Lemma 4.8, the condition (i) in Definition 4.6 is equivalent to $g_{0\alpha}|_M = 0$ in this situation. The given partially integrable CR structure $T^{1,0}M$ is induced by g if and only if $g_{\alpha\overline{\beta}}|_M = h_{\alpha\overline{\beta}}$ and $g_{\alpha\beta}|_M = 0$. □

COROLLARY 4.13. *Let $(\overline{X}, [\Theta])$ be a compact Θ-manifold and $T^{1,0}M$ is a compatible strictly pseudoconvex partially integrable CR structure on the boundary. Take* *a C^∞-smooth ACH metric g_0 that is, after normalized by some contact form θ, written as follows, where θ and $\{\,\theta^\alpha\,\}$ on $M\times[0,\infty)$ are taken as in Proposition 4.12:*

$$g_0 = 4\left(\frac{d\rho}{\rho}\right)^2 + \left(\frac{\theta}{\rho^2}\right)^2 + 2h_{\alpha\overline{\beta}}\frac{\theta^\alpha}{\rho}\frac{\theta^{\overline{\beta}}}{\rho}. \tag{4.11}$$

Here $h_{\alpha\overline{\beta}}$ is the Levi form of $(M, T^{1,0}M, \theta)$. Then, a C^∞-smooth Θ-metric g on $(\overline{X}, [\Theta])$ is an ACH metric that induces $T^{1,0}M$ if and only if

$$|g-g_0|^2_{g_0} = O(\rho). \tag{4.12}$$

Proof. Suppose g is a C^∞-smooth ACH metric that induces $T^{1,0}M$. Then, since the normalization of g with respect to θ is the form (4.9) with (4.10), $g-g_0$ vanishes at ∂X as a Θ-tensor. Hence (4.12) holds. Conversely, if (4.12) is satisfied, then $g-g_0$ vanishes at ∂X as a Θ-tensor (since g is positive definite in this case). Therefore, one can check that g is an ACH metric directly by the definition. □

4.4. Complex hyperbolic metric. We close this section with a brief observation on the complex hyperbolic space $\mathbb{C}H^n$. There are two popular models for this space, namely, the ball model and the Siegel domain model. As a metric on the unit ball $B\subset\mathbb{C}^{n+1}$, the complex hyperbolic metric g is the Kähler metric given by

$$g = 4\frac{\partial^2}{\partial z^i\partial\overline{z}^j}\left(\log\frac{1}{1-|z|^2}\right)dz^i d\overline{z}^j,$$

where the constant factor is chosen so that g solves $\mathrm{Ric}_{i\overline{j}} = -\frac{1}{4}(n+2)g_{i\overline{j}}$. If we consider this as a metric on the Siegel domain

$$D := \{\, z=(z',w)\in\mathbb{C}^n\times\mathbb{C} \mid \operatorname{Im} w > |z'|^2\,\},$$

then

$$g = 4\frac{\partial^2}{\partial z^i\partial\overline{z}^j}\left(\log\frac{1}{\operatorname{Im} w - |z'|^2}\right)dz^i d\overline{z}^j.$$

If we write $r = \operatorname{Im} w - |z'|^2$, then

$$g_{i\overline{j}} = 4\left(\frac{r_i r_{\overline{j}}}{r^2} - \frac{r_{i\overline{j}}}{r}\right).$$

Let us identify g with a Riemannian metric in the standard manner. We adopt the following convention: if $V = V^i\partial_i + V^{\overline{i}}\partial_{\overline{i}}$ and $W = W^i\partial_i + W^{\overline{i}}\partial_{\overline{i}}$ are real tangent vectors, we set

$$g(V,W) := \frac{1}{2}(g_{i\overline{j}}V^iW^{\overline{j}} + g_{i\overline{j}}W^iV^{\overline{j}}).$$

The symmetric 2-tensor that represents this Riemannian metric is $g_{i\overline{j}}dz^id\overline{z}^j$, where $dz^id\overline{z}^j$ is now regarded as the symmetric product of dz^i and $d\overline{z}^j$. Note that the Ricci tensor is $\mathrm{Ric}_{i\overline{j}}\,dz^id\overline{z}^j + \mathrm{Ric}_{i\overline{j}}\,d\overline{z}^jdz^i = 2\,\mathrm{Ric}_{i\overline{j}}\,dz^id\overline{z}^j$, so this Riemannian metric solves $\mathrm{Ric} = -\frac{1}{2}(n+2)g$.

The boundary of the Siegel domain D is the Heisenberg group $\mathcal{H}$. Recall that we can identify $\mathcal{H}$ with $\mathbb{C}^n \times \mathbb{R}$ as we did in Subsection 3.8. We consider the following identification of D and $\mathcal{H} \times (0,\infty) = \mathbb{C}^n \times \mathbb{R} \times (0,\infty)$:

$$D \longrightarrow \mathbb{C}^n \times \mathbb{R} \times (0,\infty), \qquad (z', w) \longmapsto (z', t, r) = (z', \operatorname{Re} w, \operatorname{Im} w - |z'|^2).$$

Let Φ be the inverse of this mapping: $\Phi(z', t, r) = (z', t + i(r + |z'|^2))$. Then,

$$\begin{aligned}
\Phi^*(r_i dz^i) &= \Phi^*\left(-\sum_{\alpha=1}^n \overline{z}^\alpha dz^\alpha + \frac{1}{2i} dw\right) \\
&= -\sum_{\alpha=1}^n \overline{z}^\alpha dz^\alpha - \frac{i}{2} dt + \frac{1}{2} dr + \frac{1}{2} \sum_{\alpha=1}^n (\overline{z}^\alpha dz^\alpha + z^\alpha d\overline{z}^\alpha) \\
&= \frac{1}{2} dr - \frac{i}{2} dt - \frac{1}{2} \sum_{\alpha=1}^n (\overline{z}^\alpha dz^\alpha - z^\alpha d\overline{z}^\alpha) \\
&= \frac{1}{2} dr - i\theta,
\end{aligned}$$

where θ is the standard contact form. Therefore we obtain

$$\Phi^* g = \frac{dr^2}{r^2} + 4\frac{\theta^2}{r^2} + \frac{4}{r} \sum_{\alpha=1}^n dz^\alpha d\overline{z}^\alpha.$$

So, if we set $\rho = (r/2)^{1/2}$, this is

$$\Phi^* g = 4\frac{d\rho^2}{\rho^2} + \frac{\theta^2}{\rho^4} + \frac{2}{\rho^2} \sum_{\alpha=1}^n dz^\alpha d\overline{z}^\alpha. \tag{4.13}$$

The final result (4.13) shows, by Proposition 4.12, that $\Phi^* g$ is a normal-form ACH metric on $\mathcal{H} \times [0,\infty) = \{ (z', t, \rho) \}$ for the Heisenberg group associated with the standard contact form. Since g exactly solves $\operatorname{Ric} = -\frac{1}{2}(n+2)g$, we can conclude from this that the CR obstruction tensor $\mathcal{O}_{\alpha\beta}$ vanishes for the Heisenberg group.

We remark that we had to take the square root of a boundary defining function. This means that the C^∞-structure of $\overline{D} = D \cap \mathcal{H} \cong \mathcal{H} \times [0,\infty) = \{ (z', w, r) \}$ and that of $\mathcal{H} \times [0,\infty) = \{ (z', w, \rho) \}$ are different. This generalizes to the square root construction, which will be described in Subsection 5.1, for an arbitrary complex manifold-with-boundary.

5. Bergman-type metrics

5.1. Bergman-type metrics and square root construction. Let Ω be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^{n+1}$. Fefferman proved in [**F1**] that the Bergman kernel function $B(z, \overline{z})$ restricted on the diagonal admits the following asymptotic expansion at $\partial\Omega$, where $r \in C^\infty(\overline{\Omega})$ is an arbitrary boundary defining function:

$$B(z, \overline{z}) = r(z)^{-n-2} F(z) + \log r(z) \cdot G(z), \qquad F, G \in C^\infty(\overline{\Omega}).$$

The function $\log B(z,\overline{z})$ is strictly plurisubharmonic, and the Kähler metric associated to it is called the Bergman metric on Ω. Since $F(z) > 0$ for $z \in \partial\Omega$ by Hörmander [**Hö**], if we set $\varphi(z) = B(z,\overline{z})^{-1/(n+2)}$, this function is a boundary defining function, which is $C^{n+2,\varepsilon}$-smooth for any $\varepsilon \in (0,1)$. The Bergman metric has $(n+2)\log(1/\varphi)$ as a Kähler potential function. We also remark that one can write

$$\varphi \sim r\left(\Phi + \sum_{k=1}^{\infty}(r^{n+2}\log r)^k\Phi_k\right), \qquad \Phi,\ \Phi_k \in C^\infty(\overline{\Omega}). \tag{5.1}$$

We call (5.1) a *conormal expansion* of the function φ.

Similarly, there is a function φ_{MA} called the solution to the complex Monge–Ampère equation, which is discussed in the next subsection. Lee–Melrose [**LM**] proved that it admits a conormal expansion (5.1). The metric g given by the potential $\log(1/\varphi_{\mathrm{MA}})$ is a negatively curved Kähler-Einstein metric.

Now let $\overline{\Omega}$ be an arbitrary complex manifold-with-boundary such that $\partial\Omega$ carries nondegenerate CR structure. Following Epstein–Melrose–Mendoza [**EMM**], we call a Kähler metric g defined in the interior Ω a *Bergman-type metric* if g has, up to a positive constant factor, a potential function $\log(1/\varphi)$ if we choose some boundary defining function φ. If $\partial\Omega$ is not strictly pseudoconvex, g should have indefinite signature—such a situation is also taken into consideration. In the sequel we assume that φ is at least C^2-smooth up to the boundary, and for simplicity, that the second derivatives of φ have C^∞-smooth boundary values (as is the case when φ has conormal expansion).

We normalize g so that $4\log(1/\varphi)$ is a potential for g. In a local chart

$$g = g_{i\overline{j}}dz^i d\overline{z}^j, \qquad g_{i\overline{j}} = 4\frac{\partial^2}{\partial z^i \partial\overline{z}^j}\left(\log\frac{1}{\varphi}\right) = 4\left(\frac{\varphi_i\varphi_{\overline{j}}}{\varphi^2} - \frac{\varphi_{i\overline{j}}}{\varphi}\right), \tag{5.2}$$

and this is continuous. This metric is identified with a Riemannian metric on Ω in the manner described in Subsection 4.4.

We would like to see that g can be regarded as an ACH metric if properly recognized. In order for that, we need to replace the C^∞-structure of $\overline{\Omega}$ and take a Θ-structure appropriately.

Let $\mathcal{C}^\infty$ be the sheaf of germs of smooth functions on $\overline{\Omega}$. We define a presheaf $\mathcal{C}^\infty_{1/2,\mathrm{pre}}$ as follows. Take a set of boundary coordinate charts $\{\,(U_\lambda;(x_\lambda,y^i_\lambda))\,\}$ that covers $\partial\Omega$, where $(x_\lambda,y^i_\lambda)\colon U_\lambda \longrightarrow \mathbb{R}_{\geq 0}\times\mathbb{R}^{2n+1}$ is the coordinate map. If $V \subset \overline{\Omega}$ is an open subset of some U_λ, then the ring $\mathcal{C}^\infty_{1/2,\mathrm{pre}}(V)$ is obtained by adjoining $x_\lambda^{1/2}$ to $\mathcal{C}^\infty(V)$. Since $(x_\lambda/x_\mu)^{1/2}$ is smooth on $U_\lambda \cap U_\mu$ in the original C^∞-structure, $\mathcal{C}^\infty_{1/2,\mathrm{pre}}(V)$ is well-defined. For any other open set $W \subset \overline{\Omega}$, $\mathcal{C}^\infty_{1/2,\mathrm{pre}}(W)$ is equal to $\mathcal{C}^\infty(W)$. Let $\mathcal{C}^\infty_{1/2}$ be the sheafification of $\mathcal{C}^\infty_{1/2,\mathrm{pre}}$. We define a new C^∞-structure on $\overline{\Omega}$ by $\mathcal{C}^\infty_{1/2}$. In other words, a function f in $\overline{\Omega}$ is smooth with respect to the new C^∞-structure if and only if f is C^∞-smooth in Ω with respect to the original structure and, for each boundary coordinate chart $(U_\lambda;(x_\lambda,y_\lambda^i))$, f is C^∞-smooth as the function of $(x_\lambda^{1/2},y_\lambda^i)$.

For a C^∞-smooth boundary defining function r of the original $\overline{\Omega}$, we set $\tilde{\theta}=\frac{i}{2}(\partial r-\overline{\partial}r)$, and define $\Theta_{\mathrm{pre}}=\tilde{\theta}|_{\partial\Omega}$. If $\hat{r}=e^{2\tilde{\Upsilon}}r$ is another C^∞-smooth boundary defining function, we get $\hat{\Theta}_{\mathrm{pre}}=e^{2\Upsilon}\Theta_{\mathrm{pre}}$, where $\Upsilon=\tilde{\Upsilon}|_{\partial\Omega}$.

Definition 5.1. Let $\overline{\Omega}$ be a complex manifold-with-boundary such that the induced CR structure on $\partial\Omega$ is nondegenerate. The *square root* of $\overline{\Omega}$, denoted by $\overline{\Omega}_{1/2}$, is the manifold-with-boundary that is identical to $\overline{\Omega}$ as a topological manifold and equipped with the C^∞-structure given by $\mathcal{C}^\infty_{1/2}$. The *canonical* Θ-*structure* on $\overline{\Omega}_{1/2}$ is defined to be the conformal class of $i_{1/2}^*\Theta_{\mathrm{pre}}$, where $i_{1/2}\colon\overline{\Omega}_{1/2}\longrightarrow\overline{\Omega}$ is the identity map.

Let ξ the $(1,0)$-vector field near $\partial\Omega$ characterized by

$$\xi\lrcorner\,\partial\overline{\partial}r=0\mod\overline{\partial}r,\qquad\partial r(\xi)=1.$$

If we set $\nu:=\operatorname{Re}\xi$ and $\tilde{T}:=2\operatorname{Im}\xi$, then

$$dr(\nu)=1,\qquad dr(\tilde{T})=0,\qquad\tilde{\theta}(\nu)=0,\qquad\text{and}\qquad\tilde{\theta}(\tilde{T})=1.$$

Let $\tilde{Z}_1,\dots,\tilde{Z}_n$ be $(1,0)$-vector fields defined near a point of $\partial\Omega$ spanning $\ker\partial r\subset T^{1,0}\overline{\Omega}$, and $\tilde{Z}_{\overline{\alpha}}:=\overline{\tilde{Z}_\alpha}$. We take $\tilde{\theta}^\alpha$s so that $\{\,dr,\tilde{\theta},\tilde{\theta}^\alpha,\tilde{\theta}^{\overline{\alpha}}\,\}$ becomes the dual coframe for $\{\,\nu,\tilde{T},\tilde{Z}_\alpha,\tilde{Z}_{\overline{\alpha}}\,\}$, where $\tilde{\theta}^{\overline{\alpha}}:=\overline{\tilde{\theta}^\alpha}$.

Define $\tilde{\Theta}$ (resp. $\tilde{\Theta}^\alpha$, $\tilde{\Theta}^{\overline{\alpha}}$) as the pullback of $\tilde{\theta}$ (resp. $\tilde{\theta}^\alpha$, $\tilde{\theta}^{\overline{\alpha}}$) by $i_{1/2}$. If we set $\rho=r^{1/2}$, then ρ is a smooth boundary defining function of $\overline{\Omega}_{1/2}$, and $\{\,d\rho/\rho,\tilde{\Theta}/\rho^2,\tilde{\Theta}^\alpha/\rho,\tilde{\Theta}^{\overline{\alpha}}/\rho\,\}$ is a local frame of the (complexified) Θ-cotangent bundle $\mathbb{C}^\Theta T^*\overline{\Omega}_{1/2}$. Since this is the pullback of $\{\,\frac{1}{2}r^{-1}dr,r^{-1}\tilde{\theta},r^{-1/2}\tilde{\theta}^\alpha,r^{-1/2}\tilde{\theta}^{\overline{\alpha}}\,\}$, $\mathbb{C}^\Theta T^*\overline{\Omega}_{1/2}$ can be thought as the vector bundle spanned by this set of 1-forms. One can also take $\{\,r^{-1}\partial r,r^{-1}\overline{\partial}r,r^{-1/2}\tilde{\theta}^\alpha,r^{-1/2}\tilde{\theta}^{\overline{\alpha}}\,\}$ as a local frame of $\mathbb{C}^\Theta T^*\overline{\Omega}_{1/2}$. Systematic use of this special coframe was the approach that Roth [**R**] took—in the current language, he computed on $\mathbb{C}^\Theta T^*\overline{\Omega}_{1/2}$ rather than on $\mathbb{C}T^*\overline{\Omega}$.

Proposition 5.2. *If g is a Bergman-type metric on Ω, then up to a constant factor, $i_{1/2}^*g$ is an ACH metric on $\overline{\Omega}_{1/2}$ equipped with the canonical Θ-structure. If g is defined by a boundary defining function φ and $\theta=\frac{i}{2}(\partial\varphi-\overline{\partial}\varphi)|_{T\partial\Omega}$, then $\rho=(r/2)^{1/2}\in\mathcal{F}_\theta$ for any C^∞-smooth boundary defining function $r\in C^\infty(\overline{\Omega})$ such that $\varphi=r+o(r^2)$.*

Proof. By the smoothness assumption for φ, one can take a C^∞-smooth boundary defining function $r\in C^\infty(\overline{\Omega})$ such that $\varphi=r+o(r^2)$. Using r, we take $\{\,\nu,\tilde{T},\tilde{Z}_\alpha,\tilde{Z}_{\overline{\alpha}}\,\}$ and $\{\,dr,\tilde{\theta},\tilde{\theta}^\alpha,\tilde{\theta}^{\overline{\alpha}}\,\}$ as above. We further define the real-valued function κ as follows:

$$\xi\lrcorner\,\partial\overline{\partial}r=\kappa\,\overline{\partial}r,\qquad\text{or}\qquad\kappa=\partial\overline{\partial}r(\xi,\overline{\xi}).$$

Then by [**G1**, Equation (1.4)], for some set of functions $\{\tilde{h}_{\alpha\overline{\beta}}\}$,

$$\partial\overline{\partial}r = \kappa\,\partial r \wedge \overline{\partial}r - \tilde{h}_{\alpha\overline{\beta}}\tilde{\theta}^{\alpha} \wedge \tilde{\theta}^{\overline{\beta}} = i\kappa\,dr \wedge \tilde{\theta} - \tilde{h}_{\alpha\overline{\beta}}\tilde{\theta}^{\alpha} \wedge \tilde{\theta}^{\overline{\beta}}. \tag{5.3}$$

Since $\partial\overline{\partial}\varphi - \partial\overline{\partial}r$ is $o(1)$ in local coordinates, we conclude that

$$g = g_0 + o(1), \qquad g_0 = 4\left(\frac{\partial r\overline{\partial}r}{r^2} + \tilde{h}_{\alpha\overline{\beta}}\frac{\tilde{\theta}^{\alpha}\tilde{\theta}^{\overline{\beta}}}{r}\right) = \frac{dr^2 + 4\tilde{\theta}^2}{r^2} + 4\tilde{h}_{\alpha\overline{\beta}}\frac{\tilde{\theta}^{\alpha}\tilde{\theta}^{\overline{\beta}}}{r}. \tag{5.4}$$

Here $o(1)$ denotes a certain 2-tensor whose components with respect to the coframe $\{r^{-1}\partial r, r^{-1}\overline{\partial}r, r^{-1/2}\tilde{\theta}^{\alpha}, r^{-1/2}\tilde{\theta}^{\overline{\alpha}}\}$ are $o(1)$. Let $\tilde{\Theta}$, $\tilde{\Theta}^{\alpha}$, and $\tilde{\Theta}^{\overline{\alpha}}$ as above, and here we define ρ by $\rho = (r/2)^{1/2}$. Then it is immediate from (5.4) that

$$i_{1/2}^{*}g = G_0 + o(1), \qquad G_0 = 4\frac{d\rho^2}{\rho^2} + \frac{\tilde{\Theta}^2}{\rho^4} + 2\tilde{h}_{\alpha\overline{\beta}}\frac{\tilde{\Theta}^{\alpha}\tilde{\Theta}^{\overline{\beta}}}{\rho^2}. \tag{5.5}$$

This is a Θ-metric on $\overline{\Omega}_{1/2}$. Condition (4.4) is trivially true, and it follows from (5.5) that (4.3) and the second condition of Definition 4.6 are satisfied. Moreover, if $\{N, \tilde{T}, \tilde{Z}_{\alpha}, \tilde{Z}_{\overline{\alpha}}\}$ is the dual frame for $\{d\rho, \tilde{\Theta}, \tilde{\Theta}^{\alpha}, \tilde{\Theta}^{\overline{\alpha}}\}$, then (5.5) shows that

$$R_p = \langle \rho N \rangle / F_p \qquad \text{and} \qquad (L_p)_{\mathbb{C}} = \langle \rho\tilde{Z}_1, \dots, \rho\tilde{Z}_n, \rho\tilde{Z}_{\overline{1}}, \dots, \rho\tilde{Z}_{\overline{n}} \rangle / F_p.$$

Therefore, by Lemma 4.8 we only have to check $d\tilde{\Theta}(N, \tilde{Z}_{\alpha}) = O(\rho)$ to prove that the first condition in Definition 4.6 holds. Since it follows from (5.3) that $d\tilde{\theta}$ does not contain $dr \wedge \tilde{\theta}^{\alpha}$ term, $d\tilde{\Theta}(N, \tilde{Z}_{\alpha})$ is actually zero. $\square$

5.2. Complex Monge–Ampère equation. Asymptotic solutions. Consider the special case where $\Omega \subset \mathbb{C}^{n+1}$ is a domain with smooth boundary carrying nondegenerate CR structure. Let g be a Bergman-type metric given by a boundary defining function φ that is at least C^4-smooth so that the curvature tensor is defined. If we set $G = 4\log(1/\varphi)$, $g_{i\overline{j}} = \partial_i\partial_{\overline{j}}G$. Let $\omega = (i/2)g_{i\overline{j}}dz^i \wedge d\overline{z}^j$ be the associated Kähler form and $dV_g = \omega^{n+1}/(n+1)!$ the volume form. We consider the following equation, where dV_{Euc} is the volume form of the Euclidean metric on $\mathbb{C}^{n+1}$:

$$e^{-(n+2)G}|dV_g| = dV_{\mathrm{Euc}}. \tag{5.6}$$

If this is satisfied, then

$$\mathrm{Ric}_{i\overline{j}} = -\partial_i\partial_{\overline{j}}\log|\det g| = -\frac{n+2}{4}g_{i\overline{j}}.$$

Equation (5.6) is the *complex Monge–Ampère equation* that we consider here. If q is the number of negative eigenvalues of the Levi form, we can compute $|dV_g|/dV_{\mathrm{Euc}}$

as follows:

$$
\begin{aligned}
\frac{|dV_g|}{dV_{\mathrm{Euc}}} &= (-1)^q \det(g_{i\bar{j}}) = (-1)^q \det\left(\frac{\varphi_i\varphi_{\bar{j}}}{\varphi^2} - \frac{\varphi_{i\bar{j}}}{\varphi}\right) \\
&= (-1)^{n+1+q} \det\begin{pmatrix} 1 & \varphi^{-1}\varphi_{\bar{j}} \\ \varphi^{-1}\varphi_i & \varphi^{-1}\varphi_{i\bar{j}} \end{pmatrix} \\
&= (-1)^{n+1+q}\varphi^{-n-2} \det\begin{pmatrix} \varphi & \varphi_{\bar{j}} \\ \varphi_i & \varphi_{i\bar{j}} \end{pmatrix}.
\end{aligned} \tag{5.7}
$$

Since $e^{-(n+2)G} = \varphi^{n+2}$, equation (5.6) is equivalent to

$$
J[\varphi] = 1, \qquad \text{where} \quad J[\varphi] := (-1)^{n+1+q} \det\begin{pmatrix} \varphi & \varphi_{\bar{j}} \\ \varphi_i & \varphi_{i\bar{j}} \end{pmatrix}. \tag{5.8}
$$

There is a simple method given by Fefferman [**F2**] to obtain an approximate solution of this equation.

LEMMA 5.3. *Let $\varphi \in C^\infty(\overline{\Omega})$ be a boundary defining function and $\eta \in C^\infty(\overline{\Omega})$ an arbitrary function. Take another boundary defining function $r \in C^\infty(\overline{\Omega})$.*

(1) $J[\varphi] > 0$ on $\partial\Omega$, and $J[\eta\varphi] = \eta^{n+2}J[\varphi] + O(r)$.

(2) If $J[\varphi] = 1 + O(r^{s-1})$ for some $s \geq 2$, then

$$
J[\varphi + \eta\varphi^s] = J[\varphi] + s(n+3-s)\eta\varphi^{s-1} + O(r^s).
$$

PROOF. Let $a \in \mathbb{C}$, $(b^j) \in \mathbb{C}^{n+1}$, and suppose that

$$
\begin{pmatrix} 0 & \varphi_{\bar{j}} \\ \varphi_i & \varphi_{i\bar{j}} \end{pmatrix} \begin{pmatrix} a \\ b^{\bar{j}} \end{pmatrix} = 0 \quad \text{on } \partial\Omega.
$$

The fact that the first component is zero implies that $\varphi_{\bar{j}} b^{\bar{j}} = 0$, so the nondegeneracy implies that $\varphi_{i\bar{j}} b^{\bar{j}}$ is nonzero unless $(b^j) = 0$. Therefore the matrix above is nonsingular and thus $J[\varphi] \neq 0$ on $\partial\Omega$. As one can see from (5.7), on $\Omega = \{\, \varphi > 0 \,\}$, the sign of $J[\varphi]$ is equal to that of

$$
(-1)^q \det\left(\frac{\varphi_i\varphi_{\bar{j}}}{\varphi^2} - \frac{\varphi_{i\bar{j}}}{\varphi}\right).
$$

Since the hermitian matrix $(\varphi^{-2}\varphi_i\varphi_{\bar{j}} - \varphi^{-1}\varphi_{i\bar{j}})$ has signature $(n+1-q, q)$ near $\partial\Omega$, $J[\varphi] > 0$ near $\partial\Omega$. Therefore $J[\varphi]$ should be positive on $\partial\Omega$, too. The two equalities in the statement are proved by elementary matrix operations. See [**F2**] for details. □

PROPOSITION 5.4. *There exists a smooth boundary defining function $\varphi \in C^\infty(\overline{\Omega})$ such that*

$$
J[\varphi] = 1 + O(r^{n+2}). \tag{5.9}
$$

Such a φ is unique modulo $O(r^{n+2})$.

Proof. By Lemma 5.3 (1), one can choose φ_1 so that $J[\varphi_1] = 1 + O(r)$. This condition determines φ_1 up to $O(r^2)$. Then we use (2) of the same lemma inductively as follows. Suppose φ_{s-1} satisfies $J[\varphi_{s-1}] = 1 + O(r^{s-1})$, and φ_{s-1} is unique modulo $O(r^{s-1})$. Next we set $\varphi_s = \varphi_{s-1} + \eta\varphi_{s-1}^s$. Then we obtain $J[\varphi_s] = J[\varphi_{s-1}] + s(n+3-s)\eta\varphi_{s-1}^{s-1} + O(r^s)$. So, if $s \neq n+3$, one can uniquely determine the boundary value of η so that $J[\varphi_s] = O(r^s)$. Thus φ_s is determined modulo $O(r^s)$. $\square$

A boundary defining function $\varphi \in C^\infty(\overline{\Omega})$ satisfying (5.9) is called a *Fefferman approximate solution* to (5.8). This is the best possible C^∞-smooth approximation. A detailed analysis of the obstruction, which is the boundary value of $\Phi_1(z)$ in the expansion (5.1) when Ω is a bounded strictly pseudoconvex domain, is given by Graham [**G1, G2**].

Proposition 5.5. *Let $\Omega \subset \mathbb{C}^{n+1}$ be a domain with smooth boundary whose induced CR structure is nondegenerate. Take a Fefferman approximate solution φ to the complex Monge–Ampère equation* (5.8)*, and let g be the Bergman-type metric with Kähler potential $4\log(1/\varphi)$. Then, if we consider the induced ACH metric on the square root $\overline{\Omega}_{1/2}$ equipped with the canonical Θ-structure, which we also write g by abusing notation, then the tensor $E = \mathrm{Ric} + \frac{1}{2}(n+2)g$ satisfies*

$$E = O(\rho^{2n+4}) \tag{5.10}$$

as a Θ-tensor. Here ρ is an arbitrary boundary defining function of $\overline{\Omega}_{1/2}$.

Proof. By the definition of $J[\varphi]$, it follows that

$$dV_g = \varphi^{-n-2}J[\varphi]dV_{\mathrm{Euc}}.$$

Therefore, the usual formula of the Ricci tensor implies

$$\mathrm{Ric}_{j\overline{k}} = \frac{\partial^2}{\partial z^j\partial\overline{z}^k}((n+2)\varphi - \log J(\varphi)) = -\frac{1}{4}(n+2)g_{j\overline{k}} - \frac{\partial^2}{\partial z^j\partial\overline{z}^k}\log J(r).$$

Since (5.9) is satisfied, we can write $\log J(r) = r^{n+2}f$ with $f \in C^\infty(\overline{\Omega})$. Then,

$$\frac{\partial^2}{\partial z^j\partial\overline{z}^k}\log J(r) = (n+2)(n+1)r^n f r_j r_{\overline{k}} + O(r^{n+1}),$$

and hence, as a Θ-tensor, E is given by

$$E = -2(n+2)(n+1)r^n f\,\partial r\,\overline{\partial} r + O(\rho^{2n+4}).$$

Since $\{\, r^{-1}\partial r, r^{-1}\overline{\partial} r, r^{-1/2}\tilde{\theta}^\alpha, r^{-1/2}\tilde{\theta}^{\overline{\alpha}} \,\}$ is a local frame of $\mathbb{C}^\Theta T^*\overline{\Omega}_{1/2}$, the first term on the right-hand side is also $O(\rho^{2n+4})$ as a Θ-tensor. $\square$

Since the construction of a Fefferman approximate solution is local, the following proposition is also true.

PROPOSITION 5.6. *Let $\Omega \subset \mathbb{C}^{n+1}$ be a domain, and M a nondegenerate smooth real hypersurface of Ω with induced CR structure $T^{1,0}M$ such that $\Omega \setminus M$ is the union of two connected open sets Ω_+ and Ω_-, $\Omega_+ \cap \Omega_- = \emptyset$. Let $\overline{X}$ be the square root of $\Omega_+ \cup M$. Then there exists a C^∞-smooth ACH metric g on $\overline{X}$ that induces $T^{1,0}M$ for which $E = \mathrm{Ric} + \frac{1}{2}(n+2)g$ satisfies $E = O(\rho^{2n+4})$, where ρ is a boundary defining function of $\overline{X}$.*

5.3. CR Q-curvature. In the case of integrable CR structures, the original definition of the CR Q-curvature used the so-called Fefferman construction. If M is a $(2n+1)$-dimensional nondegenerate integrable CR manifold with trivial CR canonical bundle K_M, then this construction gives a canonical conformal class on the S^1-bundle $(K_M^*)^{1/(n+2)}/\mathbb{R}_+$, where $K_M^* := K_M \setminus$ (the zero section). Moreover, there is a canonical way to get an S^1-invariant representative metric for any choice of a contact form θ. Therefore, the associated conformal Q-curvature on $(K_M^*)^{1/(n+2)}/\mathbb{R}_+$ descends to a function on M. This is the CR Q-curvature of Fefferman–Hirachi [**FH**], which we write Q^{FH}.

Compared to the Q-curvature in conformal geometry, there is a special phenomenon about the CR Q-curvature: Q^{FH} always vanishes for a special class of contact forms called invariant contact forms. Suppose we are given a nondegenerate integrable CR manifold $(M, T^{1,0}M)$ together with an embedding $M \hookrightarrow \mathbb{C}^{n+1}$. Then, if we take a Fefferman approximate solution φ to (5.6), which is highly dependent to the embedding, $\theta = \frac{i}{2}(\partial\varphi - \overline{\partial}\varphi)|_{TM}$ is called the associated *invariant contact form.*

Fefferman–Hirachi's CR Q-curvature Q^{FH} is characterized by the following properties:

(Q1) Q^{FH} is determined by a finite jet of $T^{1,0}M$ and θ;
(Q2) Q^{FH} vanishes for any invariant contact form θ;
(Q3) If θ and $\hat{\theta} = e^{2\Upsilon}\theta$ are two contact forms, then

$$e^{2(n+1)\Upsilon}\hat{Q}^{\mathrm{FH}} = Q^{\mathrm{FH}} + c_n P_{n+1}\Upsilon, \tag{5.11}$$

where c_n is a certain constant depending on n and P_{n+1} is the $(n+1)^{\mathrm{st}}$ CR-invariant power of the sublaplacian.

In [**FH**], the definition of P_{n+1} is given by ambient-metric construction. By [**GG**, Proposition 5.4], this operator is equal to what we defined in Definition 2.17. Hence (5.11) is the same as (2.20) up to the constant c_n. The constant c_n depends on the normalization of P_{n+1} and Q^{FH}, so we do not discuss this point in detail.

Since any nondegenerate integrable CR structure can be formally embedded at a given point $p \in M$, (Q1) reduces the computation of CR Q-curvatures to the case of embedded CR structures. Since an embedded CR structure admits an invariant contact form, by using (5.11), we can compute Q^{FH} for any given θ. Therefore Q^{FH} is uniquely determined by these properties.

If one wants to prove that Q in the sense of Definition 2.17 generalizes Q^{FH}, it suffices to check that Q satisfies (Q1)–(Q3) in the integrable case. Property (Q1) follows by the fact that Q admits a universal expression as a local pseudohermitian invariant, while (Q3) is true up to a constant as already mentioned in Theorem 2.15. Therefore, it remains to prove that Q vanishes for invariant contact forms when $(M, T^{1,0}M)$ is embedded. This will be done in Proposition 10.2.

CHAPTER 3

Asymptotic solutions of the Einstein equation

6. Construction of asymptotic solutions

6.1. Choice of frame. Rule for the index notation. Let $(M, T^{1,0}M)$ be a nondegenerate partially integrable CR manifold. We have seen in Subsection 4.3 that any Θ-manifold with C^∞-smooth ACH metric can be, by fixing a contact form θ, identified with a normal-form ACH manifold near the boundary. Therefore, to prove Theorems 2.1 and 2.2, it suffices to work on normal-form ACH metrics. Namely, we first prove the following theorem in this section.

THEOREM 6.1. *Let $(M, T^{1,0}M)$ be a nondegenerate partially integrable CR manifold. Then, for any compatible contact form θ, there exists a C^∞-smooth normal-form ACH metric g for $(M, T^{1,0}M, \theta)$ on $\overline{X} = M \times [0,\infty)$ that satisfies* (2.1)*. Such a metric g is unique modulo $O(\rho^{2n+2})$. Moreover, g can be taken so that* (2.2) *is also satisfied. Then g is unique modulo $O(\rho^{2n+2})$ tensor with $O(\rho^{2n+3})$ trace.*

In this section (and also when we deal with normal-form ACH metrics in the later sections), $\overline{X} = M \times [0,\infty)$ and ρ is always the coordinate for the second factor. Let $\{ \boldsymbol{Z}_I \}$ be a local frame of the Θ-tangent bundle ${}^\Theta T\overline{X}$ given by

$$\{ \boldsymbol{Z}_I \} := \{ \rho\partial_\rho, \rho^2 T, \rho Z_\alpha, \rho Z_{\overline{\alpha}} \}, \tag{6.1}$$

where T is the Reeb vector field and $\{ Z_\alpha \}$ is a local frame of $T^{1,0}M$, both extended constantly in the ρ-direction. In other words, we define θ and θ^α to be the pullbacks of corresponding 1-forms on M by the projection to the first factor $\pi\colon \overline{X} \longrightarrow M$ and T, Z_α are defined so that $\{ \partial_\rho, T, Z_\alpha, Z_{\overline{\alpha}} \}$ is the dual frame to $\{ d\rho, \theta, \theta^\alpha, \theta^{\overline{\alpha}} \}$. The indices corresponding to (6.1) are ∞, 0, 1, $\dots$, n, $\overline{1}$, $\dots$, $\overline{n}$. The following indexing rule is used in the rest of this chapter:

- α, β, γ, σ, τ run $\{ 1, \dots, n \}$ and $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$, $\overline{\sigma}$, $\overline{\tau}$ run $\{ \overline{1}, \dots, \overline{n} \}$;
- i, j, k run $\{ 0, 1, \dots, n, \overline{1}, \dots, \overline{n} \}$;
- I, J, K, L run $\{ \infty, 0, 1, \dots, n, \overline{1}, \dots, \overline{n} \}$.

The index ∞ is said to be *normal*, while the other indices are *tangential*.

The components of any Θ-tensor on $\overline{X}$ are classified by the number of tangential indices, and those in each class can be considered as representing a tensor on M, not on $\overline{X}$, *with coefficients in* $C^\infty(\overline{X})$. For example, if $\varphi = \varphi_{IJ}$ is a symmetric 2-Θ-tensor on $\overline{X}$, then there are three "normal-tangential" types for its components. We introduce the symbol θ^i by setting $\theta^0 = \theta$, and set

$$\varphi^{(0)} := \varphi_{\infty\infty}, \qquad \varphi^{(1)} := \varphi_{\infty i}\theta^i, \qquad \varphi^{(2)} := \varphi_{ij}\theta^i\theta^j.$$

Because of the tensorial transformation law of the coefficients, these are well-defined regardless of a particular choice of $\{ Z_\alpha \}$. Therefore, $\varphi^{(0)}$, $\varphi^{(1)}$, and $\varphi^{(2)}$ are a 0-tensor, a 1-tensor, and a symmetric 2-tensor globally defined on M with coefficients in $C^\infty(\overline{X})$, respectively. By abusing notation, in the sequel we just write $\varphi_{\infty\infty}$, $\varphi_{\infty i}$, and φ_{ij} to represent these tensors on M. The Tanaka–Webster connection ∇ can be applied to them in the obvious way. Lowercase Greek indices and their complex conjugates in these tensors can be raised and lowered by the Levi form.

Let g be a C^∞-smooth normal-form ACH metric on $\overline{X}$. By Proposition 4.12, the ACH condition is equivalent to the following boundary value condition of the components of g with respect to (6.1):

$$g_{\infty\infty} = 4, \qquad g_{\infty 0} = 0, \qquad g_{\infty\alpha} = 0,$$
$$g_{00} = 1 + O(\rho), \qquad g_{0\alpha} = O(\rho), \qquad g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + O(\rho), \qquad g_{\alpha\beta} = O(\rho).$$

Let φ be g minus the boundary values. That is, the symmetric 2-tensor φ_{ij} on M with coefficients in $C^\infty(\overline{X})$ are defined by

$$g_{00} = 1 + \varphi_{00}, \qquad g_{0\alpha} = \varphi_{0\alpha}, \qquad g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + \varphi_{\alpha\overline{\beta}}, \qquad g_{\alpha\beta} = \varphi_{\alpha\beta}. \tag{6.2}$$

This tensor is exactly what we can use to control the Ricci tensor of g.

6.2. Extension of the Tanaka–Webster connection. In order to relate the Levi-Civita connection of g and the Tanaka–Webster connection on the boundary, we need some extension of the latter to $\overline{X}$. In the case of bounded domains in $\mathbb{C}^{n+1}$, Graham–Lee [**GL1**] introduced such an extension, called the ambient connection, using the CR structures on the level sets of any given boundary defining function. However, on general Θ-manifolds this idea does not work because there is no complex structure, or not even almost complex structure, inside. So our approach here is more primitive. We define the connection $\overline{\nabla}$ of $T\overline{X}$ by setting

$$\overline{\nabla}\partial_\rho = 0, \qquad \overline{\nabla}_{\partial_\rho} Z_j = 0, \qquad \overline{\nabla}_{Z_i} Z_j = \Gamma^k{}_{ij} Z_k,$$

where $\Gamma^k{}_{ij}$ is the Christoffel symbol of the Tanaka–Webster connection associated to θ with respect to $\{ Z_i \}$. Then we get

$$\begin{aligned}
&\overline{\nabla}_{\rho\partial_\rho}(\rho\partial_\rho) = \rho\partial_\rho, &&\overline{\nabla}_{\rho\partial_\rho}(\rho^2 T) = 2\rho^2 T, &&\overline{\nabla}_{\rho\partial_\rho}(\rho Z_\beta) = \rho Z_\beta,\\
&\overline{\nabla}_{\rho^2 T}(\rho\partial_\rho) = 0, &&\overline{\nabla}_{\rho^2 T}(\rho^2 T) = 0, &&\overline{\nabla}_{\rho^2 T}(\rho Z_\beta) = \rho^3\Gamma^\gamma{}_{0\beta} Z_\gamma,\\
&\overline{\nabla}_{\rho Z_\alpha}(\rho\partial_\rho) = 0, &&\overline{\nabla}_{\rho Z_\alpha}(\rho^2 T) = 0, &&\overline{\nabla}_{\rho Z_\alpha}(\rho Z_\beta) = \rho^2\Gamma^\gamma{}_{\alpha\beta} Z_\gamma,\\
&\overline{\nabla}_{\rho Z_{\overline{\alpha}}}(\rho\partial_\rho) = 0, &&\overline{\nabla}_{\rho Z_{\overline{\alpha}}}(\rho^2 T) = 0, &&\overline{\nabla}_{\rho Z_{\overline{\alpha}}}(\rho Z_\beta) = \rho^2\Gamma^\gamma{}_{\overline{\alpha}\beta} Z_\gamma;
\end{aligned}$$

we have omitted $\overline{\nabla}(\rho Z_{\overline{\beta}})$ because this is just the complex conjugate of $\overline{\nabla}(\rho Z_\beta)$. Therefore $\overline{\nabla}_{\boldsymbol{Z}_K}\boldsymbol{Z}_I$ is a linear combination of $\boldsymbol{Z}_J$s with coefficients in $C^\infty(\overline{X})$, and hence we are allowed to regard $\overline{\nabla}$ as a Θ-connection. The nonzero Christoffel symbols are, with respect to the frame $\{\, \boldsymbol{Z}_I \,\}$,

$$\begin{aligned} &\overline{\Gamma}^\infty{}_{\infty\infty} = 1, &&\overline{\Gamma}^0{}_{\infty 0} = 2, &&\overline{\Gamma}^\gamma{}_{\infty\beta} = \delta_\beta{}^\gamma, \\ &\overline{\Gamma}^\gamma{}_{0\beta} = \rho^2\Gamma^\gamma{}_{0\beta}, &&\overline{\Gamma}^\gamma{}_{\alpha\beta} = \rho\Gamma^\gamma{}_{0\beta}, &&\overline{\Gamma}^\gamma{}_{\overline{\alpha}\beta} = \rho\Gamma^\gamma{}_{0\beta}. \end{aligned}$$

Note that we again omitted the complex conjugates, and that $\Gamma^k{}_{ij}$ denotes the Christoffel symbol of ∇ *with respect to* $\{\, Z_i \,\}$.

The nontrivial components of the torsion of the connection $\overline{\nabla}$ are

$$\overline{T}^0{}_{\alpha\overline{\beta}} = ih_{\alpha\overline{\beta}}, \qquad \overline{T}^{\overline{\gamma}}{}_{0\beta} = \rho^2 A_\beta{}^{\overline{\gamma}}, \qquad \overline{T}^{\overline{\gamma}}{}_{\alpha\beta} = -\rho N_{\alpha\beta}{}^{\overline{\gamma}}. \tag{6.3}$$

Of course, the torsion $\overline{T}^K{}_{IJ}$ is skew-symmetric with respect to I and J, so there are more nonzero components than displayed above. Taking complex conjugates provides further nonzero components. The components which are not obtained in this way are all zero.

Let us describe the action of $\overline{\nabla}$ on Θ-tensors more closely. Suppose $S = S_{IJ}$ is a 2-Θ-tensor for example. As we did in the previous subsection, we classify the components by the normal-tangential type. Then S_{IJ} is considered as a quadruple $(S_{\infty\infty}, S_{\infty j}, S_{i\infty}, S_{ij})$ which consists of a 0-tensor, two 1-tensors, and a 2-tensor on M, all with coefficients in $C^\infty(\overline{X})$. We can consider their covariant derivatives with respect to the Tanaka–Webster connection: $\nabla_k S_{\infty\infty}$, $\nabla_k S_{\infty j}$, $\nabla_k S_{i\infty}$, $\nabla_k S_{ij}$. On the other hand, $\overline{\nabla}_K S_{IJ}$ is a 3-Θ-tensor, and this can be considered as an 8-tuple of tensors on M. If

$$\#(I_1, \dots, I_p) := p + (\text{the number of 0s in the index list } I_1,\, \dots,\, I_p),$$

then we have

$$\overline{\nabla}_\infty S_{IJ} = (\rho\partial_\rho - \#(I,J))S_{IJ}, \tag{6.4a}$$
$$\overline{\nabla}_0 S_{IJ} = \rho^2\nabla_0 S_{IJ}, \tag{6.4b}$$
$$\overline{\nabla}_\alpha S_{IJ} = \rho\nabla_\alpha S_{IJ}. \tag{6.4c}$$

This generalizes to an arbitrary Θ-tensor as follows.

LEMMA 6.2. *Let $S = S_{I_1\cdots I_p}{}^{J_1\cdots J_q}$ be an arbitrary Θ-tensor. Then, its covariant derivative $\overline{\nabla}S$ with respect to the extended Tanaka–Webster connection is given by*

$$\overline{\nabla}_\infty S_{I_1\cdots I_p}{}^{J_1\cdots J_q} = (\rho\partial_\rho - \#(I_1, \dots, I_p) + \#(J_1, \dots, J_p))S_{I_1\cdots I_p}{}^{J_1\cdots J_q}, \tag{6.5a}$$
$$\overline{\nabla}_0 S_{I_1\cdots I_p}{}^{J_1\cdots J_q} = \rho^2\nabla_0 S_{I_1\cdots I_p}{}^{J_1\cdots J_q}, \tag{6.5b}$$
$$\overline{\nabla}_\alpha S_{I_1\cdots I_p}{}^{J_1\cdots J_q} = \rho\nabla_\alpha S_{I_1\cdots I_p}{}^{J_1\cdots J_q}. \tag{6.5c}$$

If we decompose ${}^{\Theta}T\overline{X}$ into the direct sum of the two line bundles spanned by $\boldsymbol{Z}_\infty$, $\boldsymbol{Z}_0$ and the two vector bundles spanned by $\{\,\boldsymbol{Z}_\alpha\,\}$, $\{\,\boldsymbol{Z}_{\overline{\alpha}}\,\}$, then the connection $\overline{\nabla}$ respects this decomposition. Clearly $\overline{\nabla}$ is flat on the two line bundles, and hence the components of the curvature $\overline{R}_I{}^J{}_{KL}$ is zero unless I and J are simultaneously lowercase Greek or conjugate lowercase Greek. Here we compute the lowercase Greek case only because the other is just its complex conjugate. Since $\overline{\nabla}_{\partial_\rho} Z_j = 0$ and $[\partial_\rho, Z_j] = 0$, $\overline{R}_\alpha{}^\beta{}_{KL}$ is zero if at least one of K and L is ∞. So we only have to compute $\overline{R}_\alpha{}^\beta{}_{kl}$. Because the covariant differentiation in the direction of Z_i with respect to $\overline{\nabla}$ is just the trivial extension of that with respect to the Tanaka–Webster connection ∇, the expression of the curvature tensor is the same. Paying attention to that we are using the frame $\{\,\boldsymbol{Z}_I\,\}$ to define the components of $\overline{R}$, we obtain

$$\overline{R}_\alpha{}^\beta{}_{\sigma\overline{\tau}} = \rho^2 R_\alpha{}^\beta{}_{\sigma\overline{\tau}}, \qquad \overline{R}_\alpha{}^\beta{}_{0\gamma} = -\rho^3 W_\alpha{}^\beta{}_\gamma\,, \qquad \overline{R}_\alpha{}^\beta{}_{0\overline{\gamma}} = \rho^3 W^\beta{}_{\alpha\overline{\gamma}}\,,$$
$$\overline{R}_\alpha{}^\beta{}_{\sigma\tau} = \rho^2 V_\alpha{}^\beta{}_{\sigma\tau}\,, \qquad \overline{R}_\alpha{}^\beta{}_{\overline{\sigma}\overline{\tau}} = -\rho^2 V^\beta{}_{\alpha\overline{\sigma}\overline{\tau}}\,,$$

where $R_{\alpha\overline{\beta}\sigma\overline{\tau}}$, $W_{\alpha\overline{\beta}\gamma}$ and $V_{\alpha\overline{\beta}\sigma\tau}$ are the curvature components of ∇ defined in (3.17). In (3.19b), W and V are given explicitly in terms of the Nijenhuis tensor and the Tanaka–Webster torsion tensor. The nontrivial components of the Ricci tensor $\overline{R}_{IJ}$, defined by $\overline{R}_{IJ} := \overline{R}_I{}^K{}_{KJ}$, is given by

$$\overline{R}_{\alpha 0} = \rho^3 A_{\alpha\beta,}{}^\beta, \qquad \overline{R}_{\alpha\overline{\beta}} = \rho^2 R_{\alpha\overline{\beta}}, \qquad \overline{R}_{\alpha\beta} = \rho^2 (i(n-1)A_{\alpha\beta} + N_{\gamma\beta\alpha,}{}^\gamma). \tag{6.6}$$

The tensor $\overline{R}_{IJ}$ is not necessarily symmetric. In fact, $\overline{R}_{0\alpha}$ is zero despite of the first equation of (6.6). The other components, namely $\overline{R}_{\infty I}$, $\overline{R}_{I\infty}$, and $\overline{R}_{00}$, are also zero.

6.3. Levi-Civita connection. Recall from Proposition 4.4 that the Levi-Civita connection induced by a Θ-metric is a Θ-connection. Let $D = D^K{}_{IJ}$ be the 3-Θ-tensor representing the difference between $\overline{\nabla}$ and the Levi-Civita connection ∇^g determined by g; that is, if we write $D(\boldsymbol{Z}_I, \boldsymbol{Z}_J) = D^K{}_{IJ}\boldsymbol{Z}_K$,

$$\nabla^g_{\boldsymbol{Z}_I}\boldsymbol{Z}_J = \overline{\nabla}_{\boldsymbol{Z}_I}\boldsymbol{Z}_J + D(\boldsymbol{Z}_I, \boldsymbol{Z}_J).$$

Then we obtain

$$\begin{aligned}\nabla^g_{\boldsymbol{Z}_K}\nabla^g_{\boldsymbol{Z}_L}\boldsymbol{Z}_J &= \overline{\nabla}_{\boldsymbol{Z}_K}(\overline{\nabla}_{\boldsymbol{Z}_L}\boldsymbol{Z}_J + D(\boldsymbol{Z}_L, \boldsymbol{Z}_J)) + D(\boldsymbol{Z}_K, \overline{\nabla}_{\boldsymbol{Z}_L}\boldsymbol{Z}_J + D(\boldsymbol{Z}_L, \boldsymbol{Z}_J))\\ &= \overline{\nabla}_{\boldsymbol{Z}_K}\overline{\nabla}_{\boldsymbol{Z}_L}\boldsymbol{Z}_J + (\overline{\nabla}_{\boldsymbol{Z}_K}D)(\boldsymbol{Z}_L, \boldsymbol{Z}_J) + D(\overline{\nabla}_{\boldsymbol{Z}_K}\boldsymbol{Z}_L, \boldsymbol{Z}_J) + D(\boldsymbol{Z}_L, \overline{\nabla}_{\boldsymbol{Z}_K}\boldsymbol{Z}_J)\\ &\quad + D(\boldsymbol{Z}_K, \overline{\nabla}_{\boldsymbol{Z}_L}\boldsymbol{Z}_J) + D(\boldsymbol{Z}_K, D(\boldsymbol{Z}_L, \boldsymbol{Z}_J))\end{aligned}$$

and therefore the Riemann curvature tensor is

$$\begin{aligned}
R(\boldsymbol{Z}_K,\boldsymbol{Z}_L)(\boldsymbol{Z}_I) &= (\nabla^g_{\boldsymbol{Z}_K}\nabla^g_{\boldsymbol{Z}_L} - \nabla^g_{\boldsymbol{Z}_L}\nabla^g_{\boldsymbol{Z}_K} - \nabla^g_{[\boldsymbol{Z}_K,\boldsymbol{Z}_L]})\boldsymbol{Z}_I \\
&= (\overline{\nabla}_{\boldsymbol{Z}_K}\overline{\nabla}_{\boldsymbol{Z}_L} - \overline{\nabla}_{\boldsymbol{Z}_L}\overline{\nabla}_{\boldsymbol{Z}_K} - \overline{\nabla}_{[\boldsymbol{Z}_K,\boldsymbol{Z}_L]})\boldsymbol{Z}_I \\
&\quad + D(\overline{\nabla}_{\boldsymbol{Z}_K}\boldsymbol{Z}_L - \overline{\nabla}_{\boldsymbol{Z}_L}\boldsymbol{Z}_K - [\boldsymbol{Z}_K,\boldsymbol{Z}_L],\boldsymbol{Z}_I) \\
&\quad + (\overline{\nabla}_{\boldsymbol{Z}_K}D)(\boldsymbol{Z}_L,\boldsymbol{Z}_I) - (\overline{\nabla}_{\boldsymbol{Z}_L}D)(\boldsymbol{Z}_K,\boldsymbol{Z}_I) \\
&\quad + D(\boldsymbol{Z}_K, D(\boldsymbol{Z}_L,\boldsymbol{Z}_I)) - D(\boldsymbol{Z}_L, D(\boldsymbol{Z}_K,\boldsymbol{Z}_I)) \\
&= \overline{R}(\boldsymbol{Z}_K,\boldsymbol{Z}_L)(\boldsymbol{Z}_I) + D(\overline{T}(\boldsymbol{Z}_K,\boldsymbol{Z}_L),\boldsymbol{Z}_I) \\
&\quad + (\overline{\nabla}_{\boldsymbol{Z}_K}D)(\boldsymbol{Z}_L,\boldsymbol{Z}_I) - (\overline{\nabla}_{\boldsymbol{Z}_L}D)(\boldsymbol{Z}_K,\boldsymbol{Z}_I) \\
&\quad + D(\boldsymbol{Z}_K, D(\boldsymbol{Z}_L,\boldsymbol{Z}_I)) - D(\boldsymbol{Z}_L, D(\boldsymbol{Z}_K,\boldsymbol{Z}_I)).
\end{aligned}$$

In the index notation,

$$R_I{}^J{}_{KL} = \overline{R}_I{}^J{}_{KL} + \overline{\nabla}_K D^J{}_{LI} - \overline{\nabla}_L D^J{}_{KI} + D^J{}_{KM}D^M{}_{LI} - D^J{}_{LM}D^M{}_{KI} + \overline{T}^M{}_{KL}D^J{}_{MI}.$$

Since ∇^g is torsion-free,

$$\overline{T}^M{}_{KL} + D^M{}_{KL} - D^M{}_{LK} = 0. \tag{6.7}$$

Hence the Ricci tensor of g is (using its symmetry) given by

$$\mathrm{Ric}_{IJ} = \overline{R}_{JI} + \overline{\nabla}_K D^K{}_{IJ} - \overline{\nabla}_I D^K{}_{KJ} - D^L{}_{KI}D^K{}_{LJ} + D^L{}_{IJ}D^K{}_{KL}. \tag{6.8}$$

Thus the computation of the Ricci tensor reduces to that of $D^K{}_{IJ}$. We set $D_{KIJ} := g_{KL}D^L{}_{IJ}$. Then, as in the case of the usual formula, one can derive

$$D_{KIJ} = \frac{1}{2}(\overline{\nabla}_I g_{JK} + \overline{\nabla}_J g_{IK} - \overline{\nabla}_K g_{IJ} + \overline{T}_{IJK} + \overline{T}_{JIK} - \overline{T}_{KIJ}), \tag{6.9}$$

where $\overline{T}_{KIJ} := g_{KL}\overline{T}^L{}_{IJ}$. One can write D_{KIJ} down explicitly in terms of φ_{ij} using (6.2), (6.3), and (6.5).

The computation of $D^K{}_{IJ}$, the tensor D with indices in the original positions, involves the inverse g^{IJ} of the metric. But in order to express g^{IJ} exactly, we have to use infinite series of φ, where φ is defined by (6.2), which is too complicated to handle. So we give up the exact computation and are satisfied with an approximation. *We omit any term which contains φ_{ij} as a factor, and other than that, an $O(\rho)$ factor.* Since φ_{ij} itself is an $O(\rho)$ tensor, nonlinear terms in φ_{ij} can be all omitted, and moreover something like $\rho\varphi_{ij}$ is also negligible. Then g^{IJ} is given by

$$\begin{aligned}
&g^{\infty\infty} \equiv \tfrac{1}{4}, \qquad g^{\infty 0} \equiv g^{\infty\alpha} \equiv 0, \\
&\quad g^{00} \equiv 1 - \varphi_{00}, \qquad g^{0\alpha} \equiv -\varphi_0{}^{\alpha}, \qquad g^{\alpha\overline{\beta}} \equiv h^{\alpha\overline{\beta}} - \varphi^{\alpha\overline{\beta}}, \qquad g^{\alpha\beta} \equiv -\varphi^{\alpha\beta}.
\end{aligned} \tag{6.10}$$

This somewhat crude computation is enough for our purpose, namely, determining the expansion of φ_{ij} inductively so that g satisfies the Einstein equation (to high order). This is because, if φ_{ij} is perturbed by adding ψ_{ij} which is $O(\rho^m)$, then the change of the omitted terms is $O(\rho^{m+1})$.

Lemma 6.3. *The tensor D_{KIJ} is, modulo φ_{ij} times $O(\rho)$, as in Table 6.1. (Since we know (6.7), we did not list $D_{\infty JI}$ in the table if $D_{\infty IJ}$ is present, and we have also omitted complex conjugates.)*

Type	*Value*
$D_{\infty\infty\infty}$	-4
$D_{\infty\infty 0}$	0
$D_{\infty\infty\alpha}$	0
$D_{\infty 00}$	$2-\frac{1}{2}(\rho\partial_\rho-4)\varphi_{00}$
$D_{\infty 0\alpha}$	$-\frac{1}{2}(\rho\partial_\rho-3)\varphi_{0\alpha}$
$D_{\infty\alpha\overline{\beta}}$	$h_{\alpha\overline{\beta}}-\frac{1}{2}(\rho\partial_\rho-2)\varphi_{\alpha\overline{\beta}}$
$D_{\infty\alpha\beta}$	$-\frac{1}{2}(\rho\partial_\rho-2)\varphi_{\alpha\beta}$

Type	*Value*
$D_{0\infty\infty}$	0
$D_{0\infty 0}$	$-2+\frac{1}{2}(\rho\partial_\rho-4)\varphi_{00}$
$D_{0\infty\alpha}$	$\frac{1}{2}(\rho\partial_\rho-3)\varphi_{0\alpha}$
D_{000}	0
$D_{00\alpha}$	0
$D_{0\alpha\overline{\beta}}$	$-\frac{i}{2}h_{\alpha\overline{\beta}}-\frac{i}{2}h_{\alpha\overline{\beta}}\varphi_{00}$
$D_{0\alpha\beta}$	$-\rho^2 A_{\alpha\beta}$

Type	*Value*
$D_{\overline{\gamma}\infty\infty}$	0
$D_{\overline{\gamma}\infty 0}$	$\frac{1}{2}(\rho\partial_\rho-3)\varphi_{0\overline{\gamma}}$
$D_{\overline{\gamma}\infty\alpha}$	$-h_{\alpha\overline{\gamma}}+\frac{1}{2}(\rho\partial_\rho-2)\varphi_{\alpha\overline{\gamma}}$
$D_{\overline{\gamma}\infty\overline{\alpha}}$	$\frac{1}{2}(\rho\partial_\rho-2)\varphi_{\overline{\alpha}\overline{\gamma}}$
$D_{\overline{\gamma}00}$	0
$D_{\overline{\gamma}0\alpha}$	$\frac{i}{2}h_{\alpha\overline{\gamma}}+\frac{i}{2}h_{\alpha\overline{\gamma}}\varphi_{00}$
$D_{\overline{\gamma}0\overline{\alpha}}$	0
$D_{\overline{\gamma}\alpha\beta}$	$\frac{i}{2}(h_{\alpha\overline{\gamma}}\varphi_{0\beta}+h_{\beta\overline{\gamma}}\varphi_{0\alpha})$
$D_{\overline{\gamma}\alpha\overline{\beta}}$	$-\frac{i}{2}(h_{\alpha\overline{\beta}}\varphi_{0\overline{\gamma}}-h_{\alpha\overline{\gamma}}\varphi_{0\overline{\beta}})$
$D_{\overline{\gamma}\overline{\alpha}\overline{\beta}}$	$\rho N_{\overline{\gamma}\overline{\beta}\overline{\alpha}}$

Table 6.1. D_{KIJ}, with φ_{ij} times $O(\rho)$ omitted (Lemma 6.3)

Proof. First we consider the case where ∞ appears as an index. A direct computation shows $D_{\infty\infty\infty}=-4$. For the other 13 components (see the table), (6.5b) and (6.5c) implies that $\overline{\nabla}_I g_{JK}$ is negligible unless $I=\infty$, and is zero if only one of J and K is ∞. The torsion component $\overline{T}_{KIJ}$ is zero if ∞ appears in the indices. Therefore, $D_{K\infty\infty}\equiv D_{\infty\infty J}\equiv D_{\infty I\infty}\equiv 0$, and

$$D_{\infty IJ}\equiv -D_{I\infty J}\equiv -D_{JI\infty}\equiv -\tfrac{1}{2}(\rho\partial_\rho-\#(I,J))g_{IJ}.$$

On the other hand, if ∞ does not appear in the indices, D_{KIJ} is

$$D_{KIJ}\equiv \tfrac{1}{2}(\overline{T}_{IJK}+\overline{T}_{JIK}-\overline{T}_{KIJ}),$$

and this is computed on a type by type basis. The details are omitted. □

Lemma 6.4. *The tensor $D^K{}_{IJ}$ is, modulo φ_{ij} times $O(\rho)$, as in Table 6.2 (again some types are omitted because of symmetry and complex conjugacy).*

Proof. One just computes by the formula $D^K{}_{IJ}=g^{KL}D_{LIJ}$ using Table 6.3 and (6.10). The details are omitted. □

Type	*Value*
${D^\infty}_{\infty\infty}$	-1
${D^\infty}_{\infty 0}$	0
${D^\infty}_{\infty\alpha}$	0
${D^\infty}_{00}$	$\frac{1}{2}-\frac{1}{8}(\rho\partial_\rho-4)\varphi_{00}$
${D^\infty}_{0\alpha}$	$-\frac{1}{8}(\rho\partial_\rho-3)\varphi_{0\alpha}$
${D^\infty}_{\alpha\overline{\beta}}$	$\frac{1}{4}h_{\alpha\overline{\beta}}-\frac{1}{8}(\rho\partial_\rho-2)\varphi_{\alpha\overline{\beta}}$
${D^\infty}_{\alpha\beta}$	$-\frac{1}{8}(\rho\partial_\rho-2)\varphi_{\alpha\beta}$

Type	*Value*
${D^0}_{\infty\infty}$	0
${D^0}_{\infty 0}$	$-2+\frac{1}{2}\rho\partial_\rho\varphi_{00}$
${D^0}_{\infty\alpha}$	$\frac{1}{2}(\rho\partial_\rho-1)\varphi_{0\alpha}$
${D^0}_{00}$	0
${D^0}_{0\alpha}$	$-\frac{i}{2}\varphi_{0\alpha}$
${D^0}_{\alpha\overline{\beta}}$	$-\frac{i}{2}h_{\alpha\overline{\beta}}$
${D^0}_{\alpha\beta}$	$-\rho^2 A_{\alpha\beta}$

Type	*Value*
${D^\gamma}_{\infty\infty}$	0
${D^\gamma}_{\infty 0}$	$\frac{1}{2}(\rho\partial_\rho+1){\varphi_0}^\gamma$
${D^\gamma}_{\infty\alpha}$	$-{\delta_\alpha}^\gamma+\frac{1}{2}\rho\partial_\rho{\varphi_\alpha}^\gamma$
${D^\gamma}_{\infty\overline{\alpha}}$	$\frac{1}{2}\rho\partial_\rho{\varphi_{\overline{\alpha}}}^\gamma$
${D^\gamma}_{00}$	0
${D^\gamma}_{0\alpha}$	$\frac{i}{2}{\delta_\alpha}^\gamma+\frac{i}{2}{\delta_\alpha}^\gamma\varphi_{00}$
${D^\gamma}_{0\overline{\alpha}}$	$\frac{i}{2}{\varphi_{\overline{\alpha}}}^\gamma$
${D^\gamma}_{\alpha\beta}$	$\frac{i}{2}({\delta_\alpha}^\gamma\varphi_{0\beta}+{\delta_\beta}^\gamma\varphi_{0\alpha})$
${D^\gamma}_{\alpha\overline{\beta}}$	$\frac{i}{2}{\delta_\alpha}^\gamma\varphi_{0\overline{\beta}}$
${D^\gamma}_{\overline{\alpha}\overline{\beta}}$	$\rho N^\gamma{}_{\overline{\beta}\overline{\alpha}}$

TABLE 6.2. ${D^K}_{IJ}$, with φ_{ij} times $O(\rho)$ omitted (Lemma 6.4)

6.4. The Ricci tensor. According to equation (6.8), we need to compute the following to obtain a formula of the Ricci tensor:

$$\overline{\nabla}_K {D^K}_{IJ}, \qquad \overline{\nabla}_I {D^K}_{KJ}, \qquad {D^L}_{KI}{D^K}_{LJ}, \qquad \text{and} \qquad {D^L}_{IJ}{D^K}_{KL}.$$

Because we know in advance that Ric_{IJ} is symmetric in I and J, we only have to consider the cases where (I,J) is either (∞,∞), $(\infty,0)$, (∞,α), $(0,0)$, $(0,\alpha)$, $(\alpha,\overline{\beta})$, or (α,β). A direct computation using Lemma 6.4 leads to

(6.11a) $$\overline{\nabla}_K {D^K}_{\infty\infty} \equiv 1,$$

(6.11b) $$\overline{\nabla}_K {D^K}_{\infty 0} \equiv 0,$$

(6.11c) $$\overline{\nabla}_K {D^K}_{\infty\alpha} \equiv 0,$$

(6.11d) $$\overline{\nabla}_K {D^K}_{00} \equiv -\tfrac{3}{2}-\tfrac{1}{8}(\rho\partial_\rho-3)(\rho\partial_\rho-4)\varphi_{00},$$

(6.11e) $$\overline{\nabla}_K {D^K}_{0\alpha} \equiv -\tfrac{1}{8}(\rho\partial_\rho-2)(\rho\partial_\rho-3)\varphi_{0\alpha},$$

(6.11f) $$\overline{\nabla}_K {D^K}_{\alpha\overline{\beta}} \equiv -\tfrac{1}{4}h_{\alpha\overline{\beta}}-\tfrac{1}{8}(\rho\partial_\rho-1)(\rho\partial_\rho-2)\varphi_{\alpha\overline{\beta}},$$

(6.11g) $$\overline{\nabla}_K {D^K}_{\alpha\beta} \equiv \rho^2 N_{\gamma\beta\alpha,}{}^{\gamma}-\rho^4 A_{\alpha\beta,0}-\tfrac{1}{8}(\rho\partial_\rho-1)(\rho\partial_\rho-2)\varphi_{\alpha\beta}.$$

One can see from the same lemma that ${D^K}_{KI}$ is

(6.12a) $${D^K}_{K\infty} \equiv -(2n+3)+\frac{1}{2}\rho\partial_\rho\varphi_{00}+\rho\partial_\rho{\varphi_\alpha}^\alpha,$$

(6.12b) $${D^K}_{K0} \equiv 0,$$

(6.12c) $${D^K}_{K\alpha} \equiv 0,$$

and therefore

$$\overline{\nabla}_\infty D^K{}_{K\infty} \equiv 2n+3+\tfrac{1}{2}\rho\partial_\rho(\rho\partial_\rho-1)\varphi_{00}+\rho\partial_\rho(\rho\partial_\rho-1)\varphi_\alpha{}^\alpha; \tag{6.13}$$

all the other types of $\overline{\nabla}_I D^K{}_{KJ}$ are negligible. The third term $D^L{}_{KI}D^K{}_{LJ}$ is computed directly using Lemma 6.4, and is as follows:

$$D^L{}_{K\infty}D^K{}_{L\infty} \equiv 2n+5-2\rho\partial_\rho\varphi_{00}-2\rho\partial_\rho\varphi_\alpha{}^\alpha, \tag{6.14a}$$

$$D^L{}_{K\infty}D^K{}_{L0} \equiv 0, \tag{6.14b}$$

$$D^L{}_{K\infty}D^K{}_{L\beta} \equiv \tfrac{i}{2}(\rho\partial_\rho+1)\varphi_{0\beta}, \tag{6.14c}$$

$$D^L{}_{K0}D^K{}_{L0} \equiv -\tfrac{1}{2}(n+4)+(\rho\partial_\rho-n-2)\varphi_{00}+\varphi_\alpha{}^\alpha, \tag{6.14d}$$

$$D^L{}_{K0}D^K{}_{L\beta} \equiv -\rho^3 N_\beta{}^{\overline{\sigma\tau}}A_{\overline{\sigma\tau}}+\tfrac{1}{4}(3\rho\partial_\rho-2n-5)\varphi_{0\beta}, \tag{6.14e}$$

$$D^L{}_{K\alpha}D^K{}_{L\overline{\beta}} \equiv \rho^2 N^{\overline{\tau}}{}_{\alpha\sigma}N^\sigma{}_{\overline{\beta}\overline{\tau}}+\tfrac{1}{2}(\rho\partial_\rho-2)\varphi_{\alpha\overline{\beta}}+\tfrac{1}{2}h_{\alpha\overline{\beta}}\varphi_{00}, \tag{6.14f}$$

$$D^L{}_{K\alpha}D^K{}_{L\beta} \equiv i\rho^2 A_{\alpha\beta}+\tfrac{1}{2}\rho\partial_\rho\varphi_{\alpha\beta}. \tag{6.14g}$$

The fourth term $D^L{}_{IJ}D^K{}_{KL}$ can be computed by Lemma 6.4 and equation (6.12). Since $D^K{}_{KL}$ is nonzero only for $L=\infty$, we obtain $D^L{}_{IJ}D^K{}_{KL}=D^\infty{}_{IJ}D^K{}_{K\infty}$, and therefore

$$D^L{}_{\infty\infty}D^K{}_{KL} \equiv 2n+3-\tfrac{1}{2}\rho\partial_\rho\varphi_{00}-\rho\partial_\rho\varphi_\alpha{}^\alpha, \tag{6.15a}$$

$$D^L{}_{\infty 0}D^K{}_{KL} \equiv 0, \tag{6.15b}$$

$$D^L{}_{\infty\alpha}D^K{}_{KL} \equiv 0, \tag{6.15c}$$

$$\begin{aligned} D^L{}_{00}D^K{}_{KL} \equiv &-\tfrac{1}{2}(2n+3)+\tfrac{1}{4}\rho\partial_\rho\varphi_{00}+\tfrac{1}{2}\rho\partial_\rho\varphi_\alpha{}^\alpha \\ &+\tfrac{1}{8}(2n+3)(\rho\partial_\rho-4)\varphi_{00}, \end{aligned} \tag{6.15d}$$

$$D^L{}_{0\alpha}D^K{}_{KL} \equiv \tfrac{1}{8}(2n+3)(\rho\partial_\rho-3)\varphi_{0\alpha}, \tag{6.15e}$$

$$\begin{aligned} D^L{}_{\alpha\overline{\beta}}D^K{}_{KL} \equiv &-\tfrac{1}{4}(2n+3)h_{\alpha\overline{\beta}}+\tfrac{1}{8}h_{\alpha\overline{\beta}}\rho\partial_\rho\varphi_{00}+\tfrac{1}{4}h_{\alpha\overline{\beta}}\rho\partial_\rho\varphi_\gamma{}^\gamma \\ &+\tfrac{1}{8}(2n+3)(\rho\partial_\rho-2)\varphi_{\alpha\overline{\beta}}, \end{aligned} \tag{6.15f}$$

$$D^L{}_{\alpha\beta}D^K{}_{KL} \equiv \tfrac{1}{8}(2n+3)(\rho\partial_\rho-2)\varphi_{\alpha\beta}. \tag{6.15g}$$

By combining all these results, we can compute the Ricci tensor. Since we are actually interested in the difference of the Ricci tensor and $-\frac{1}{2}(n+2)g$, the result in the following lemma is stated with respect to their difference.

LEMMA 6.5. *If g is a C^∞-smooth normal-form ACH metric for $(M,T^{1,0}M,\theta)$ whose components are expressed as* (6.2)*, then the components of $E:=\mathrm{Ric}+\frac{1}{2}(n+2)g$ is given as follows, where we omit φ_{ij} times $O(\rho)$:*

$$\begin{aligned} E_{\infty\infty} &\equiv -\tfrac{1}{2}\rho\partial_\rho(\rho\partial_\rho-4)\varphi_{00}-\rho\partial_\rho(\rho\partial_\rho-2)\varphi_\alpha{}^\alpha, \\ E_{\infty 0} &\equiv 0, \\ E_{\infty\alpha} &\equiv -\tfrac{i}{2}(\rho\partial_\rho+1)\varphi_{0\alpha}, \\ E_{00} &\equiv -\tfrac{1}{8}((\rho\partial_\rho)^2-(2n+4)\rho\partial_\rho-4n)\varphi_{00}+\tfrac{1}{2}(\rho\partial_\rho-2)\varphi_\alpha{}^\alpha, \end{aligned}$$

$$
\begin{aligned}
E_{0\alpha} &\equiv \rho^3 A_{\alpha\beta,}{}^{\beta} + \rho^3 N_{\alpha}{}^{\overline{\beta}\overline{\gamma}} A_{\overline{\beta}\overline{\gamma}} - \tfrac{1}{8}(\rho\partial_\rho + 1)(\rho\partial_\rho - 2n - 3)\varphi_{0\alpha},\\
E_{\alpha\overline{\beta}} &\equiv \rho^2 R_{\alpha\overline{\beta}} - 2\rho^2 N_{\alpha}{}^{\overline{\gamma}}{}_{\rho} N_{\overline{\beta}}{}^{\rho}{}_{\overline{\gamma}} - \tfrac{1}{8}((\rho\partial_\rho)^2 - (2n+2)\rho\partial_\rho - 8)\varphi_{\alpha\overline{\beta}}\\
&\qquad + \tfrac{1}{8} h_{\alpha\overline{\beta}}(\rho\partial_\rho - 4)\varphi_{00} + \tfrac{1}{4} h_{\alpha\overline{\beta}}\rho\partial_\rho\varphi_{\gamma}{}^{\gamma},\\
E_{\alpha\beta} &\equiv in\rho^2 A_{\alpha\beta} + \rho^2(N_{\gamma\alpha\beta,}{}^{\gamma} + N_{\gamma\beta\alpha,}{}^{\gamma}) - \rho^4 A_{\alpha\beta,0} - \tfrac{1}{8}\rho\partial_\rho(\rho\partial_\rho - 2n - 2)\varphi_{\alpha\beta}.
\end{aligned}
$$

In particular, the tensor E is automatically $O(\rho)$.

6.5. The Bianchi identity. While φ has $(2n+1)^2$ components, the Einstein equation is a system of $(2n+2)^2$ differential equations between the components. However, these equations are not all independent because of the contracted Bianchi identity:

$$d\,\mathrm{Scal} = -2\delta\,\mathrm{Ric}\,. \tag{6.16}$$

Here δ is the divergence operator. In the case of ACH metrics, we can derive the following useful lemma from (6.16).

LEMMA 6.6. *Let $m \geq 1$ be a positive integer. Suppose that g is a C^∞-smooth normal-form ACH metric for which $E := \mathrm{Ric} + \frac{1}{2}(n+2)g$ satisfies $E = O(\rho^m)$. Then we have*

$$O(\rho^{m+1}) = (m - 4n - 4)E_{\infty\infty} - 4(m-4)E_{00} - 8(m-2)E_{\alpha}{}^{\alpha}, \tag{6.17a}$$

$$O(\rho^{m+1}) = (m - 2n - 4)E_{\infty 0}, \tag{6.17b}$$

$$O(\rho^{m+1}) = (m - 2n - 3)E_{\infty\alpha} - 4iE_{0\alpha}. \tag{6.17c}$$

Here, $E_{\alpha}{}^{\alpha} = h^{\alpha\overline{\beta}}E_{\alpha\overline{\beta}}$.

PROOF. Since the Levi-Civita connection ∇^g kills the metric, from (6.16) we also obtain $d(\mathrm{tr}_g\, E) = -2\delta E$, or

$$g^{IJ}\nabla^g_K E_{IJ} = 2g^{IJ}\nabla^g_I E_{JK}.$$

In terms of the extended Tanaka–Webster connection $\overline{\nabla}$ and the tensor D, we can rewrite this identity as

$$g^{IJ}(\overline{\nabla}_K E_{IJ} - 2D^L{}_{KI}E_{JL}) = 2g^{IJ}(\overline{\nabla}_I E_{JK} - D^L{}_{IJ}E_{LK} - D^L{}_{IK}E_{JL}),$$

or equivalently,

$$0 = g^{IJ}(\overline{\nabla}_K E_{IJ} - 2\overline{\nabla}_I E_{JK} + 2D^L{}_{IJ}E_{KL} - 2\overline{T}^L{}_{IK}E_{JL}). \tag{6.18}$$

Since $\overline{\nabla}$ is a Θ-connection, $E = O(\rho^m)$ implies $\overline{\nabla}E = O(\rho^m)$ and so we obtain

$$
\begin{aligned}
O(\rho^{m+1}) &= \tfrac{1}{4}(\overline{\nabla}_K E_{\infty\infty} - 2\overline{\nabla}_\infty E_{\infty K} + 2D^L{}_{\infty\infty}E_{KL} - 2\overline{T}^L{}_{\infty K}E_{\infty L})\\
&\quad + 1\cdot(\overline{\nabla}_K E_{00} - 2\overline{\nabla}_0 E_{0K} + 2D^L{}_{00}E_{KL} - 2\overline{T}^L{}_{0K}E_{0L})\\
&\quad + 2h^{\beta\overline{\gamma}}(\overline{\nabla}_K E_{\beta\overline{\gamma}} - \overline{\nabla}_\beta E_{\overline{\gamma}K} - \overline{\nabla}_{\overline{\gamma}} E_{\beta K} + (D^L{}_{\beta\overline{\gamma}} + D^L{}_{\overline{\gamma}\beta})E_{KL}\\
&\qquad\qquad - \overline{T}^L{}_{\beta K}E_{\overline{\gamma}L} - \overline{T}^L{}_{\overline{\gamma}K}E_{\beta L}).
\end{aligned}
\tag{6.19}
$$

Since $\overline{\nabla}_0 E_{IJ}$ and $\overline{\nabla}_\alpha E_{IJ}$ are $O(\rho^{m+1})$, this further simplifies to

$$\begin{aligned}O(\rho^{m+1}) = {} & \tfrac{1}{4}(\overline{\nabla}_K E_{\infty\infty} - 2\overline{\nabla}_\infty E_{\infty K} + 2D^L{}_{\infty\infty} E_{KL} - 2\overline{T}^L{}_{\infty K} E_{\infty L}) \\ & + (\overline{\nabla}_K E_{00} + 2D^L{}_{00} E_{KL} - 2\overline{T}^L{}_{0K} E_{0L}) \\ & + 2h^{\beta\overline{\gamma}}(\overline{\nabla}_K E_{\beta\overline{\gamma}} + (D^L{}_{\beta\overline{\gamma}} + D^L{}_{\overline{\gamma}\beta}) E_{KL} - \overline{T}^L{}_{\beta K} E_{\overline{\gamma}L} - \overline{T}^L{}_{\overline{\gamma}K} E_{\beta L}).\end{aligned}$$

Recall the boundary values of $D^K{}_{IJ}$ from Table 6.4 and those of $\overline{T}^K{}_{IJ}$ from (6.3): on $M = M \times \{0\}$, they are

$$D^\infty{}_{\infty\infty} = -1, \qquad D^\infty{}_{00} = \tfrac{1}{2}, \qquad D^\infty{}_{\alpha\overline{\beta}} = \tfrac{1}{4} h_{\alpha\overline{\beta}}, \tag{6.20a}$$

$$D^0{}_{\infty 0} = -2, \qquad D^0{}_{\alpha\overline{\beta}} = \tfrac{i}{2} h_{\alpha\overline{\beta}}, \tag{6.20b}$$

$$D^\gamma{}_{\infty\alpha} = -\delta_\alpha{}^\gamma, \qquad D^\gamma{}_{0\alpha} = \tfrac{i}{2}\delta_\alpha{}^\gamma \tag{6.20c}$$

and

$$\overline{T}^0{}_{\alpha\overline{\beta}} = i h_{\alpha\overline{\beta}};$$

all the nontrivial components that are not shown are zero on M. Therefore, our equality again reduces to

$$\begin{aligned}O(\rho^{m+1}) = {} & \tfrac{1}{4}(\overline{\nabla}_K E_{\infty\infty} - 2\overline{\nabla}_\infty E_{\infty K} - 2E_{\infty K}) + (\overline{\nabla}_K E_{00} + E_{\infty K}) \\ & + 2h^{\beta\overline{\gamma}}(\overline{\nabla}_K E_{\beta\overline{\gamma}} + \tfrac{1}{2} h_{\beta\overline{\gamma}} E_{\infty K} - \overline{T}^0{}_{\beta K} E_{0\overline{\gamma}} - \overline{T}^0{}_{\overline{\gamma}K} E_{0\beta}).\end{aligned}$$

Substituting $K = \infty$, $K = 0$ and $K = \alpha$ into this formula, by (6.4a) we find that

$$O(\rho^{m+1}) = -\tfrac{1}{4}(\rho\partial_\rho - 4n - 4)E_{\infty\infty} + (\rho\partial_\rho - 4)E_{00} + 2(\rho\partial_\rho - 2)E_\alpha{}^\alpha, \tag{6.21a}$$

$$O(\rho^{m+1}) = -\tfrac{1}{2}(\rho\partial_\rho - 2n - 4)E_{\infty 0}, \tag{6.21b}$$

$$O(\rho^{m+1}) = -\tfrac{1}{2}(\rho\partial_\rho - 2n - 3)E_{\infty\alpha} + 2iE_{0\alpha}, \tag{6.21c}$$

which imply (6.17). □

6.6. Construction of asymptotic solution. Now we prove Theorem 6.1. Let $E := \mathrm{Ric} + \frac{1}{2}(n+2)g$. Recall from Lemma 6.5 that $E = O(\rho)$ for any normal-form ACH metric g. We shall inductively show that there exists a normal-form ACH metric $g^{(m)}$ satisfying $E = O(\rho^m)$ for each $m \le 2n+2$,, and for such $g^{(m)}$ its components $g^{(m)}_{ij}$ are unique modulo $O(\rho^m)$.

Suppose we have a normal-form ACH metric $g^{(m)}$ such that $E = O(\rho^m)$. Consider a new metric $g^{(m)}$ given by $g^{(m+1)}_{ij} = g^{(m)}_{ij} + \psi_{ij}$, where $\psi_{ij} = O(\rho^m)$. Then, by Lemma 6.5, the difference $\delta E = E^{(m+1)} - E^{(m)}$ modulo $O(\rho^{m+1})$ is

$$\delta E_{\infty\infty} \equiv -\tfrac{1}{2}m(m-4)\psi_{00} - m(m-2)\psi_\alpha{}^\alpha, \tag{6.22a}$$

$$\delta E_{\infty 0} \equiv 0, \tag{6.22b}$$

$$\delta E_{\infty\alpha} \equiv -\tfrac{i}{2}(m+1)\psi_{0\alpha}, \tag{6.22c}$$

$$\delta E_{00} \equiv -\tfrac{1}{8}(m^2 - (2n+4)m - 4n)\psi_{00} + \tfrac{1}{2}(m-2)\psi_\alpha{}^\alpha, \tag{6.22d}$$

$$\delta E_{0\alpha} \equiv -\tfrac{1}{8}(m+1)(m-2n-3)\psi_{0\alpha}, \tag{6.22e}$$

(6.22f) $\delta E_{\alpha\overline{\beta}} \equiv -\frac{1}{8}(m^2-(2n+2)m-8)\psi_{\alpha\overline{\beta}} + \frac{1}{8}h_{\alpha\overline{\beta}}(m-4)\psi_{00} + \frac{1}{4}h_{\alpha\overline{\beta}}m\psi_\gamma{}^\gamma,$

(6.22g) $\delta E_{\alpha\beta} \equiv -\frac{1}{8}m(m-2n-2)\psi_{\alpha\beta}.$

By taking the trace and the trace-free part of (6.22f), we obtain

$$\delta E_\gamma{}^\gamma \equiv -\tfrac{1}{8}(m^2-(4n+2)m-8)\psi_\gamma{}^\gamma + \tfrac{1}{8}n(m-4)\psi_{00}, \tag{6.23a}$$

$$\operatorname{tf}(\delta E_{\alpha\overline{\beta}}) \equiv -\tfrac{1}{8}(m^2-(2n+2)m-8)\operatorname{tf}(\psi_{\alpha\overline{\beta}}). \tag{6.23b}$$

Look at (6.22e), (6.22g), and (6.23b). Since the coefficients appearing in these equalities are nonzero for $m \le 2n+1$, we can uniquely determine $\psi_{0\alpha}$, $\psi_{\alpha\beta}$, and $\operatorname{tf}(\psi_{\alpha\overline{\beta}})$ modulo $O(\rho^{m+1})$ so that $E^{(m+1)}_{0\alpha}$, $E^{(m+1)}_{\alpha\beta}$, and $\operatorname{tf}(\delta E_{\alpha\overline{\beta}})$ are $O(\rho^{m+1})$. Next we regard (6.22d) and (6.23a) as a system of linear equations for ψ_{00} and $\psi_\alpha{}^\alpha$. The determinant of the coefficients

$$D_m := \det\begin{pmatrix} -\frac{1}{8}(m^2-(2n+4)m-4n) & \frac{1}{2}(m-2) \\ \frac{1}{8}n(m-4) & -\frac{1}{8}(m^2-(4n+2)m-8) \end{pmatrix}$$

is actually

$$D_m = \frac{1}{64}m(m+2)(m-2n-4)(m-4n-4), \tag{6.24}$$

which shows that this system is nondegenerate for $m \le 2n+1$. Hence we can determine ψ_{00} and $\psi_\alpha{}^\alpha$, both modulo $O(\rho^{m+1})$, so that $E^{(m+1)}_{00}$ and $E^{(m+1)}{}_\gamma{}^\gamma$ are $O(\rho^{m+1})$. Thus we have attained $E^{(m+1)}_{ij} = O(\rho^{m+1})$, and if $g^{(m)}_{ij}$ are determined unique up to $O(\rho^m)$ at the beginning of step m, then $g^{(m+1)}_{ij}$ are unique modulo $O(\rho^{m+1})$.

To go to the next step, we have to check that $E^{(m+1)}_{\infty\infty}$, $E^{(m+1)}_{\infty 0}$, and $E^{(m+1)}_{\infty\alpha}$ are also $O(\rho^{m+1})$. This can be seen from Lemma 6.6. In fact, since $E^{(m+1)}_{ij} = O(\rho^{m+1})$ is already achieved, (6.17) shows that $(m-4n-4)E^{(m+1)}_{\infty\infty}$, $(m-2n-4)E^{(m+1)}_{\infty 0}$, and $(m-2n-2)E^{(m+1)}_{\infty\alpha}$ are $O(\rho^{m+1})$. Therefore, for $m \le 2n+1$, it follows that $E^{(m+1)}_{\infty\infty}$, $E^{(m+1)}_{\infty 0}$, and $E^{(m+1)}_{\infty\alpha}$ are $O(\rho^{m+1})$ and hence so is the whole $E^{(m+1)}$. Hence the induction is complete and the first part of Theorem 6.1 is proved.

To obtain the second part, now we consider g with components $g_{ij} = g^{(2n+2)}_{ij} + \psi_{ij}$, $\psi_{ij} = O(\rho^{2n+2})$. Equations (6.22a), (6.22d), and (6.23a) imply

$$\begin{aligned} \operatorname{tr}_g E = \operatorname{Scal} + (n+1)(n+2) &= \tfrac{1}{4}\delta E_{\infty\infty} + \delta E_{00} + 2\,\delta E_\gamma{}^\gamma + O(\rho^{2n+3}) \\ &= \tfrac{1}{2}(n+2)(\psi_{00} + 2\psi_\gamma{}^\gamma) + O(\rho^{2n+3}). \end{aligned}$$

Therefore we can take ψ so that $\operatorname{tr}_g E = O(\rho^{2n+3})$. Suppose g and g_1 are chosen in this way and let $(g_1)_{ij} = g_{ij} + \psi_{ij}$. Then $\psi_{00} + 2\psi_\gamma{}^\gamma = \operatorname{tr}_g(g_1 - g) + O(\rho^{2n+3})$, so $\operatorname{tr}_g(g_1 - g)$ should be $O(\rho^{2n+3})$.

6.7. Evenness. Here we introduce the evenness condition to C^∞-smooth ACH metrics. To define the evenness, it is more appropriate to see an ACH metric g as a usual Riemannian metric on X rather than a Θ-metric. Let us start with the case where g is a normal-form ACH metric for $(M, T^{1,0}M, \theta)$. As a Riemannian metric, it is defined on $M \times (0, \infty)$, and can be written as

$$g = \frac{4d\rho^2 + h_\rho}{\rho^2}, \tag{6.25}$$

where h_ρ is a family of Riemannian metrics with parameter ρ. The family h_ρ is divergent in the direction of the Reeb vector field when ρ tends to 0. If we consider $\rho^2 h_\rho$, then it is convergent and admits an asymptotic expansion at $\rho = 0$ in the (nonnegative) powers of ρ with coefficients in the space of symmetric 2-tensors on M.

Definition 6.7. A C^∞-smooth normal-form ACH metric g is *even* if the asymptotic expansion of $\rho^2 h_\rho$ at $\rho = 0$ contains even-degree terms only. An arbitrary C^∞-smooth ACH metric is *even* if its normalization is even.

The well-definedness of the evenness for the general case is due to the remark made in the last paragraph of [**GS**, §3.2].

Proposition 6.8. *For any Θ-manifold and a compatible partially integrable CR structure $T^{1,0}M$ on the boundary, one can always take a C^∞-smooth ACH metric g that induces $T^{1,0}M$ so that it satisfies* (2.1), (2.2) *and is even.*

Proof. Take a normal-form ACH metric $g = \rho^{-2}(4\,d\rho^2 + h_\rho)$ satisfying (2.1) and (2.2). Although h_ρ is defined for $\rho > 0$, we can smoothly extend $\rho^2 h_\rho$ to $-\varepsilon < \rho < \varepsilon$. Making ε smaller if necessary, we can define a Riemannian metric g_- on $M \times (-\varepsilon, 0)$ by setting $g_- := \rho^{-2}(4\,d\rho^2 + h_\rho)$. Then the tensor $E_- := \operatorname{Ric}(g_-) + \frac{1}{2}(n+2)g_-$ also satisfies (2.1) and (2.2). If we consider the pullback of g_- by the inversion $\iota\colon (x, \rho) \longmapsto (x, -\rho)$, then the metric $g' = \iota^* g_-$ on X again fulfills (2.1) and (2.2). If we write $g' = \rho^{-2}(4\,d\rho^2 + h'_\rho)$, then the odd-degree terms of the expansions of h_ρ and h'_ρ have opposite signs, while the even-degree ones are the same. Hence, by the uniqueness result stated in Theorem 6.1, the determined odd-degree coefficients in the expansion of h_ρ must be zero. Therefore, putting the undetermined terms as zero for example, we obtain an even normal-form ACH metric satisfying (2.1) and (2.2). □

Let C^∞_{even} be the space of C^∞-smooth functions on $M \times [0, \infty)$. Since the vector field $\rho\partial_\rho$ preserves, as an operator on smooth functions, preserves C^∞_{even}, the components of the Riemann curvature tensor of an even normal-form ACH metric with respect to $\{\, \rho\partial_\rho, T, Z_\alpha, Z_{\overline{\alpha}} \,\}$ are all belong to C^∞_{even}. Therefore, if g is an even normal-form ACH metric that satisfies (2.1), then it is moreover true that

$$E_{\infty\alpha} = O(\rho^{2n+3}), \qquad E_{0\alpha} = O(\rho^{2n+3}) \qquad \text{and} \qquad \operatorname{tr}_g E = O(\rho^{2n+4}),$$

where the components of $E = \operatorname{Ric} + \frac{1}{2}(n+2)g$ are now with respect to $\{\, \boldsymbol{Z}_I \,\}$. Thus we obtain Theorem 2.3. Furthermore, we can go ahead with the induction in the proof of Theorem 6.1 a bit more: we can determine all the ρ^{2n+2}-coefficients of g_{00} and $g_{\alpha\overline{\beta}}$ so that the ρ^{2n+2} coefficients of $E_{\infty\infty}$, E_{00}, and $E_{\alpha\overline{\beta}}$ vanishes. By (6.17b), $E_{\infty 0} = O(\rho^{2n+3})$. By the evenness,

$$\begin{aligned} E_{\infty\infty} &= O(\rho^{2n+4}), & E_{\infty 0} &= O(\rho^{2n+4}), \\ E_{00} &= O(\rho^{2n+4}), & E_{\alpha\overline{\beta}} &= O(\rho^{2n+4}). \end{aligned}$$

Let us summarize the result—this will be used in Subsection 7.1.

Proposition 6.9. *For any $(M, T^{1,0}M, \theta)$, one can take an even C^∞-smooth normal-form ACH metric g so that $E = \mathrm{Ric} + \frac{1}{2}(n+2)g$ satisfies*

$$E_{IJ} = O(\rho^{2n+2+a(I,J)}), \tag{6.26}$$

where

$$a(I,J) := \begin{cases} 2, & (i,j) = (\infty,\infty),\ (\infty,0),\ (0,0),\ (\alpha,\overline{\beta}), \\ 1, & (i,j) = (\infty,\alpha),\ (0,\alpha), \\ 0, & (i,j) = (\alpha,\beta). \end{cases} \tag{6.27}$$

The components g_{ij} are uniquely determined modulo $O(\rho^{2n+2+a(i,j)})$.

7. CR obstruction tensor

7.1. CR obstruction tensor. Recall the proof of Theorem 6.1 given in Subsection 6.6. In spite of the success of the inductive determination of the coefficients of g_{ij} up to the $(2n+1)^{\mathrm{st}}$ order, the next step cannot be executed. This is because of (6.22g)—although the metric g in Theorem 6.1 has $O(\rho^{2n+2})$ freedom, no matter how we determine g, there is no effect on the ρ^{2n+2}-coefficient of $\mathrm{Ein}_{\alpha\beta}$. So we get the well-defined tensor

$$\mathcal{O}_{\alpha\beta} := \left(\rho^{-2n-2}E_{\alpha\beta}\right)\big|_{\rho=0}. \tag{7.1}$$

This is the CR obstruction tensor $\mathcal{O}_{\alpha\beta}$ of Definition 2.6. It follows that $\mathcal{O}_{\alpha\beta}$ is given by a universal formula in terms of the Levi form and its dual, the Nijenhuis tensor, the pseudohermitian torsion tensor, the pseudohermitian Ricci tensor, and their covariant derivatives. To see this, we take $g^{(1)}$ so that $g^{(1)}_{ij}$ are constant in ρ. Then, by Lemma 6.5

$$\begin{aligned} E^{(1)}_{\infty\infty} &= E^{(1)}_{\infty 0} = E^{(1)}_{\infty\alpha} = E^{(1)}_{00} = 0, \\ E^{(1)}_{0\alpha} &= \rho^3 A_{\alpha\beta,}{}^{\beta} + \rho^3 N_{\alpha}{}^{\overline{\beta}\overline{\gamma}} A_{\overline{\beta}\overline{\gamma}}, \\ E^{(1)}_{\alpha\overline{\beta}} &= \rho^2 R_{\alpha\overline{\beta}} - 2\rho^2 N_{\alpha}{}^{\overline{\gamma}}{}_{\rho} N_{\overline{\beta}}{}^{\rho}{}_{\overline{\gamma}}, \\ E^{(1)}_{\alpha\beta} &= in\rho^2 A_{\alpha\beta} + 2\rho^2 \nabla^{\gamma} N_{\gamma(\alpha\beta)} - \rho^4 A_{\alpha\beta,0}. \end{aligned}$$

Hence, if $g^{(2)}$ solves $E^{(2)} = O(\rho^2)$ then it must be equal to $g^{(1)}$ modulo $O(\rho^2)$, and $g^{(3)}$ solving $E^{(3)} = O(\rho^3)$ should be taken as follows:

$$g_{00} = 1 + O(\rho^3), \tag{7.3a}$$

$$g_{0\alpha} = O(\rho^3), \tag{7.3b}$$

$$g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + \rho^2 \Phi_{\alpha\overline{\beta}} + O(\rho^3), \tag{7.3c}$$

$$g_{\alpha\beta} = \rho^2 \Phi_{\alpha\beta} + O(\rho^3), \tag{7.3d}$$

where

$$\Phi_{\alpha\overline{\beta}} := -\frac{2}{n+2}\left(R_{\alpha\overline{\beta}} - 2N_{\alpha\sigma\tau}N_{\overline{\beta}}{}^{\tau\sigma} - \frac{1}{2(n+1)}(R - 2N_{\gamma\sigma\tau}N^{\gamma\tau\sigma})h_{\alpha\overline{\beta}}\right), \tag{7.4a}$$

$$\Phi_{\alpha\beta} := -\frac{2}{n}(iA_{\alpha\beta} + 2\nabla^\gamma N_{\gamma(\alpha\beta)}). \tag{7.4b}$$

If, in each step in the induction, we construct $g^{(m+1)}$ by taking ψ_{ij} for which $\rho^{-m}\psi_{ij}$ is constant in ρ, then it is obvious that the components of ψ_{ij}. As a result, $\mathcal{O}_{\alpha\beta}$ is also given by such a formula. Thus we have proved Theorem 2.5.

Proof of Theorem 2.10. Let g be a normal-form ACH metric satisfying the condition of Proposition 6.9. We would like to compute the right-hand side of (6.18) modulo $O(\rho^{2n+5})$ for $K = 0$ and modulo $O(\rho^{2n+4})$ for $K = \alpha$.

Let first $K = \alpha$. Since $\varphi_{ij} = O(\rho^2)$ by the evenness, it is immediate that the right-hand side of (6.19) is actually $O(\rho^{2n+4})$. Since $\overline{T}^L{}_{\infty J} = 0$ and $\overline{\nabla}_\alpha E_{\infty\infty}$, $\overline{\nabla}_\alpha E_{00}$, $\overline{\nabla}_0 E_{0\alpha}$, $\overline{\nabla}_\alpha E_{\beta\overline{\gamma}}$, and $\overline{\nabla}_\beta E_{\overline{\gamma}\alpha}$ are all $O(\rho^{2n+4})$,

$$\begin{aligned} O(\rho^{2n+4}) &= \tfrac{1}{4}(-2\overline{\nabla}_\infty E_{\infty\alpha} + 2D^L{}_{\infty\infty}E_{\alpha L}) + (2D^L{}_{00}E_{L\alpha} - 2\overline{T}^L{}_{0\alpha}E_{0L}) \\ &\quad + 2h^{\beta\overline{\gamma}}(-\overline{\nabla}_{\overline{\gamma}}E_{\beta\alpha} + (D^L{}_{\beta\overline{\gamma}} + D^L{}_{\overline{\gamma}\beta})E_{L\alpha} - \overline{T}^L{}_{\beta\alpha}E_{\overline{\gamma}L} - \overline{T}^L{}_{\overline{\gamma}\alpha}E_{\beta L}). \end{aligned}$$

By Table 6.4, in the current situation (6.20) actually holds in $M \times [0, \infty)$ modulo $O(\rho^2)$. If we also recall (6.3), then the equality above simplifies to

$$\begin{aligned} O(\rho^{2n+4}) &= \tfrac{1}{4}(-2\overline{\nabla}_\infty E_{\infty\alpha} - 2E_{\infty\alpha}) + E_{\infty\alpha} \\ &\qquad + 2h^{\beta\overline{\gamma}}(-\overline{\nabla}_{\overline{\gamma}}E_{\beta\alpha} + \tfrac{1}{2}h_{\beta\overline{\gamma}}E_{\infty\alpha} + \rho N_{\beta\alpha}{}^{\overline{\sigma}}E_{\overline{\gamma}\overline{\sigma}} + ih_{\alpha\overline{\gamma}}E_{0\beta}) \\ &= -\tfrac{1}{2}(\rho\partial_\rho - 2n - 3)E_{\infty\alpha} - 2\rho\nabla^\beta E_{\alpha\beta} - 2\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}E_{\overline{\beta}\overline{\gamma}} + 2iE_{0\alpha} \\ &= -2\rho\nabla^\beta E_{\alpha\beta} - 2\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}E_{\overline{\beta}\overline{\gamma}} + 2iE_{0\alpha} + O(\rho^{2n+4}). \end{aligned}$$

Therefore we obtain

$$E_{0\alpha} = -i\rho^{2n+3}(\nabla^\beta \mathcal{O}_{\alpha\beta} + N_\alpha{}^{\overline{\beta}\overline{\gamma}}\mathcal{O}_{\overline{\beta}\overline{\gamma}}) + O(\rho^{2n+4}). \tag{7.5}$$

Let us again recall (6.18), and next we consider the case $K = 0$. This time note that $E_{IJ} = O(\rho^{2n+3})$ if $0 \in \{I, J\}$, and also that $\overline{\nabla}_L E_{IJ} = O(\rho^{2n+3})$ if $0 \in \{I, J, L\}$ is 0. Moreover, $\overline{T}^L{}_{I0}$ is at least $O(\rho^2)$ whatever I and L are. By these facts and $\varphi_{ij} = O(\rho^2)$, we conclude

$$\begin{aligned}O(\rho^{2n+5}) = \tfrac{1}{4}&(\overline{\nabla}_0 E_{\infty\infty} - 2\overline{\nabla}_\infty E_{\infty 0} + 2D^L{}_{\infty\infty} E_{0L} - 2\overline{T}^L{}_{\infty 0} E_{\infty L}) \\ &+ 1\cdot(\overline{\nabla}_0 E_{00} - 2\overline{\nabla}_0 E_{00} + 2D^L{}_{00} E_{0L} - 2\overline{T}^L{}_{00} E_{0L}) \\ &+ 2h^{\beta\overline{\gamma}}(\overline{\nabla}_0 E_{\beta\overline{\gamma}} - \overline{\nabla}_\beta E_{0\overline{\gamma}} - \overline{\nabla}_{\overline{\gamma}} E_{0\beta} + (D^L{}_{\beta\overline{\gamma}} + D^L{}_{\overline{\gamma}\beta}) E_{0L} \\ &\qquad - \overline{T}^L{}_{\beta 0} E_{\overline{\gamma} L} - \overline{T}^L{}_{\overline{\gamma}0} E_{\beta L}).\end{aligned}$$

Since $\overline{T}^L{}_{\infty 0} = \overline{T}^L{}_{00} = 0$ and $\overline{\nabla}_0 E_{\infty\infty}$, $\overline{\nabla}_0 E_{00}$, $\overline{\nabla}_0 E_{\beta\overline{\gamma}}$ are $O(\rho^{2n+5})$ (in fact $O(\rho^{2n+6})$),

$$\begin{aligned}O(\rho^{2n+5}) = \tfrac{1}{4}&(-2\overline{\nabla}_\infty E_{\infty 0} + 2D^L{}_{\infty\infty} E_{0L}) + 2D^L{}_{00} E_{0L} \\ &+ 2h^{\beta\overline{\gamma}}(-\overline{\nabla}_\beta E_{0\overline{\gamma}} - \overline{\nabla}_{\overline{\gamma}} E_{0\beta} + (D^L{}_{\beta\overline{\gamma}} + D^L{}_{\overline{\gamma}\beta}) E_{0L} \\ &\qquad - \overline{T}^L{}_{\beta 0} E_{\overline{\gamma} L} - \overline{T}^L{}_{\overline{\gamma}0} E_{\beta L}).\end{aligned}$$

This implies, because of (6.3) and the fact that (6.20) holds in $M \times [0,\infty)$ modulo $O(\rho^2)$,

$$\begin{aligned}O(\rho^{2n+5}) &= \tfrac{1}{4}(-2\overline{\nabla}_\infty E_{\infty 0} - 2E_{\infty 0}) + E_{\infty 0} \\ &\quad + 2h^{\beta\overline{\gamma}}(-\overline{\nabla}_\beta E_{0\overline{\gamma}} - \overline{\nabla}_{\overline{\gamma}} E_{0\beta} + \tfrac{1}{2} h_{\beta\overline{\gamma}} E_{\infty 0} - \rho^2 A_\beta{}^{\overline{\sigma}} E_{\overline{\gamma}\overline{\sigma}} - \rho^2 A_{\overline{\gamma}}{}^{\sigma} E_{\beta\sigma}) \\ &= -\tfrac{1}{2}(\rho\partial_\rho - 2n - 4) E_{\infty 0} - 2\rho(\nabla^\alpha E_{0\alpha} + \nabla^{\overline{\alpha}} E_{0\overline{\alpha}}) \\ &\quad - 2\rho^2(A^{\alpha\beta} E_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}} E_{\overline{\alpha}\overline{\beta}}) \\ &= -2\rho(\nabla^\alpha E_{0\alpha} + \nabla^{\overline{\alpha}} E_{0\overline{\alpha}}) - 2\rho^2(A^{\alpha\beta} E_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}} E_{\overline{\alpha}\overline{\beta}}) + O(\rho^{2n+5}).\end{aligned}$$

Therefore,

$$\nabla^\alpha E_{0\alpha} + \nabla^{\overline{\alpha}} E_{0\overline{\alpha}} = -\rho^{2n+3}(A^{\alpha\beta}\mathcal{O}_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}\mathcal{O}_{\overline{\alpha}\overline{\beta}}) + O(\rho^{2n+4}). \tag{7.6}$$

Combining (7.5) and (7.6), we obtain

$$D^{\alpha\beta}\mathcal{O}_{\alpha\beta} - D^{\overline{\alpha}\overline{\beta}}\mathcal{O}_{\overline{\alpha}\overline{\beta}} = 0,$$

where $D^{\alpha\beta} = \nabla^\alpha\nabla^\beta - iA^{\alpha\beta} - N^{\gamma\alpha\beta}\nabla_\gamma - (\nabla_\gamma N^{\gamma\alpha\beta})$. □

7.2. First variation of CR obstruction tensor. In this subsection, we compute a part of the first-order term of the obstruction tensor with respect to a variation on the Heisenberg group $M = \mathcal{H}$ from the standard CR structure $T^{1,0}M$. Let $\varphi_t \colon T^{1,0}M \longrightarrow \overline{T^{1,0}M}$ be a $\mathbb{C}$-homomorphism representing a smooth 1-parameter family $\hat{T}^{1,0}_t$ based at $T^{1,0}M$. Let $\psi_{\alpha\beta} \in \mathcal{E}_{(\alpha\beta)}(1,1)$ be its derivative at $t = 0$. Let $\{ Z_\alpha \}$ be the standard frame.

Let $g = g_t$ be a smooth normal-form ACH-Einstein metric that satisfies (2.1) for each $(M, \hat{T}^{1,0}_t, \theta)$. This can be taken smoothly with respect to t. We set

$$g^\bullet_{00} = \Psi_{00}, \qquad g^\bullet_{0\alpha} = \Psi_{0\alpha}, \qquad g^\bullet_{\alpha\overline{\beta}} = \Psi_{\alpha\overline{\beta}}, \qquad g^\bullet_{\alpha\beta} = \Psi_{\alpha\beta},$$

where $(g_t)_{ij}$s are the components with respect to $\{\hat{Z}_\alpha = Z_\alpha + \varphi_t(Z_\alpha)\}$. Then, g_{ij}s are uniquely determined modulo $O(\rho^{2n+2})$, and so are Ψ_{ij}s. Let $\Psi_{ij}^{(m)}$ be the uniquely determined coefficients. Because of (3.27), we can write

$$\Psi_{ij} = \sum_{m=1}^{2n+1} \rho^m \Psi_{ij}^{(m)} + O(\rho^{2n+2}).$$

Since g_t can be taken so that it is even, $\Psi_{00}^{(m)}$, $\Psi_{\alpha\overline{\beta}}^{(m)}$, and $\Psi_{\alpha\beta}^{(m)}$ are zero for m odd, and $\Psi_{0\alpha}^{(m)}$ is zero for m even. Since each $(g_t)_{ij}$ admits a universal expression as a local pseudohermitian invariant, by (3.28) and (3.29), each $\Psi_{ij}^{(m)}$ is given as a linear combination of covariant derivatives of ψ_{ij} (trivialized by θ).

If we consider a new contact form $\hat{\theta} = e^{2\Upsilon}\theta$ for constant Υ, then the formula that gives $\Psi_{ij}^{(m)}$ as a linear combination will not change because (3.28) and (3.29) remain valid for such $\hat{\theta}$. On the other hand, $\hat{\rho}$ used for the normalization with respect to $\hat{\theta}$ is $e^\Upsilon \rho$, and hence

$$\hat{\rho}^2 \hat{T} = \rho T, \qquad \hat{\rho} Z_\alpha = e^\Upsilon \rho Z_\alpha.$$

Therefore,

$$\begin{aligned} \hat{\rho}^m \hat{\Psi}_{00}^{(m)} &= \rho^m \Psi_{00}^{(m)}, & \hat{\rho}^m \hat{\Psi}_{0\alpha}^{(m)} &= e^\Upsilon \rho^m \Psi_{0\alpha}^{(m)}, \\ \hat{\rho}^m \hat{\Psi}_{\alpha\overline{\beta}}^{(m)} &= e^{2\Upsilon} \rho^m \Psi_{\alpha\overline{\beta}}^{(m)}, & \hat{\rho}^m \hat{\Psi}_{\alpha\beta}^{(m)} &= e^{2\Upsilon} \rho^m \Psi_{\alpha\beta}^{(m)}. \end{aligned}$$

Consequently, $\Psi_{ij}^{(m)}$ must satisfy

$$\begin{aligned} \hat{\Psi}_{00}^{(m)} &= e^{-m\Upsilon} \Psi_{00}^{(m)}, & \hat{\Psi}_{0\alpha}^{(m)} &= e^{-(m-1)\Upsilon} \Psi_{0\alpha}^{(m)}, \\ \hat{\Psi}_{\alpha\overline{\beta}}^{(m)} &= e^{-(m-2)\Upsilon} \Psi_{\alpha\overline{\beta}}^{(m)}, & \hat{\Psi}_{\alpha\beta}^{(m)} &= e^{-(m-2)\Upsilon} \Psi_{\alpha\beta}^{(m)}. \end{aligned}$$

Thus we conclude all the possible terms in $\Psi_{ij}^{(m)}$ are as shown in Table 7.1. Similarly, the variation $\mathcal{O}^\bullet_{\alpha\beta}$ of the CR obstruction tensor should be a linear combination of the following terms:

$$\begin{aligned} &\Delta_b^k \nabla_0^{n+1-k} \psi_{\alpha\beta}, \qquad \Delta_b^k \nabla_0^{n-k} \nabla_{(\alpha} \nabla^\sigma \psi_{\beta)\sigma}, \\ &\Delta_b^k \nabla_0^{n-1-k} \nabla_\alpha \nabla_\beta \nabla^\sigma \nabla^\tau \psi_{\sigma\tau} \qquad \text{and} \qquad \Delta_b^k \nabla_0^{n-1-k} \nabla_\alpha \nabla_\beta \nabla^{\overline{\sigma}} \nabla^{\overline{\tau}} \psi_{\overline{\sigma}\overline{\tau}}. \end{aligned} \tag{7.7}$$

These terms are linearly independent if $n \geq 2$.

We determine the coefficient of $\Delta_b^{n+1} \psi_{\alpha\beta}$ in $\mathcal{O}^\bullet_{\alpha\beta}$. For this purpose, it suffices to compute $\Psi_{ij}^{(m)}$ neglecting $\nabla_0 \psi_{\alpha\beta}$, $\nabla^\beta \psi_{\alpha\beta}$, $\nabla^{\overline{\beta}} \psi_{\overline{\alpha}\overline{\beta}}$ and their covariant derivatives, because these terms cannot contribute to the final $(\Delta_b^{n+1}\psi_{\alpha\beta})$-coefficient. This means, as can be seen from Table 7.1. that we can compute as if $\Psi_{00}^{(m)}$, $\Psi_{0\alpha}^{(m)}$, and $\Psi_{\alpha\overline{\beta}}^{(m)}$ are zero.

By (7.4b), modulo these terms

$$\Psi_{\alpha\beta}^{(2)} \equiv \frac{2}{n} \Delta_b \psi_{\alpha\beta}. \tag{7.8}$$

Type	Terms
$\Psi^{(2l)}_{00}$	$\Delta_b^k\nabla_0^{l-1-k}\nabla^\alpha\nabla^\beta\psi_{\alpha\beta}$, $\Delta_b^k\nabla_0^{l-1-k}\nabla^{\overline{\alpha}}\nabla^{\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}}$
$\Psi^{(2l+1)}_{0\alpha}$	$\Delta_b^k\nabla_0^{l-k}\nabla^\beta\psi_{\alpha\beta}$, $\Delta_b^k\nabla_0^{l-1-k}\nabla_\alpha\nabla^\sigma\nabla^\tau\psi_{\sigma\tau}$, $\Delta_b^k\nabla_0^{l-1-k}\nabla_\alpha\nabla^{\overline{\sigma}}\nabla^{\overline{\tau}}\psi_{\overline{\sigma}\overline{\tau}}$
$\Psi^{(2l)}_{\alpha\overline{\beta}}$	$\Delta_b^k\nabla_0^{l-1-k}\nabla_\alpha\nabla^{\overline{\sigma}}\psi_{\overline{\beta}\overline{\sigma}}$, $\Delta_b^k\nabla_0^{l-1-k}\nabla_{\overline{\beta}}\nabla^\sigma\psi_{\alpha\sigma}$, $\Delta_b^k\nabla_0^{l-2-k}\nabla_\alpha\nabla_{\overline{\beta}}\nabla^\sigma\nabla^\tau\psi_{\sigma\tau}$, $\Delta_b^k\nabla_0^{l-2-k}\nabla_{\overline{\beta}}\nabla_\alpha\nabla^{\overline{\sigma}}\nabla^{\overline{\tau}}\psi_{\overline{\sigma}\overline{\tau}}$, $h_{\alpha\overline{\beta}}\Delta_b^k\nabla_0^{l-1-k}\nabla^\sigma\nabla^\tau\psi_{\sigma\tau}$, $h_{\alpha\overline{\beta}}\Delta_b^k\nabla_0^{l-1-k}\nabla^{\overline{\sigma}}\nabla^{\overline{\tau}}\psi_{\overline{\sigma}\overline{\tau}}$
$\Psi^{(2l)}_{\alpha\beta}$	$\Delta_b^k\nabla_0^{l-k}\psi_{\alpha\beta}$, $\Delta_b^k\nabla_0^{l-1-k}\nabla_{(\alpha}\nabla^\sigma\psi_{\beta)\sigma}$, $\Delta_b^k\nabla_0^{l-2-k}\nabla_\alpha\nabla_\beta\nabla^\sigma\nabla^\tau\psi_{\sigma\tau}$, $\Delta_b^k\nabla_0^{l-2-k}\nabla_\alpha\nabla_\beta\nabla^{\overline{\sigma}}\nabla^{\overline{\tau}}\psi_{\overline{\sigma}\overline{\tau}}$

TABLE 7.1. Terms that can appear in $\Psi^{(m)}_{ij}$

Moreover, by computing $E = \mathrm{Ric} + \frac{1}{2}(n+2)g$ again using this technique, one can prove that $\Psi^{(2l)}_{\alpha\beta}$ satisfy

$$0 \equiv l(l-n-1)\Psi^{(2l)}_{\alpha\beta} - \Delta_b\Psi^{(2l-2)}_{\alpha\beta} \quad \text{for } l \geq 2,$$

and hence, for $1 \leq l \leq n$,

$$\varphi^{(2l)}_{\alpha\beta} \equiv \frac{2\cdot(-1)^{l+1}}{l!\cdot n!/(n-l)!}\Delta_b^l\mu_{\alpha\beta}.$$

Then, by a similar computation, one can derive

$$\mathcal{O}^\bullet_{\alpha\beta} \equiv \frac{1}{2}\Delta_b\Psi^{(2n)}_{\alpha\beta} \equiv \frac{(-1)^{n+1}}{(n!)^2}\Delta_b^{n+1}\psi_{\alpha\beta}.$$

Thus we have shown the following.

PROPOSITION 7.1. *Let $n \geq 2$. If we express $\mathcal{O}^\bullet_{\alpha\beta}$ as a linear combination of the terms in* (7.7)*, then the coefficient of $\Delta_b^{n+1}\psi_{\alpha\beta}$ is equal to $(-1)^{n+1}/(n!)^2$.*

COROLLARY 7.2. *Let $n \geq 2$. Then there is a partially integrable CR structure on the $(2n+1)$-dimensional Heisenberg group, arbitrarily close to the standard one, for which the obstruction tensor does not vanish.*

8. Formal solutions involving logarithmic singularities

8.1. ACH metrics with logarithmic singularities. In this section, we are going to prove Theorems 2.11 and 2.12. Let $(\overline{X}, [\Theta])$ be a Θ-manifold. Recall that a continuous Θ-tensor S belongs to $\mathcal{A}(\overline{X})$ if S admits an expansion of the form (2.9). We write $S \in \mathcal{A}^m$ if $S^{(q)} = O(\rho^m)$ for all $q \geq 0$, and set $\mathcal{A}^\infty := \bigcap_{m=0}^\infty \mathcal{A}^m$. The symbol $\mathcal{A}^m$ will be used similarly to $O(\rho^m)$—for example, $f = f_0 + \mathcal{A}^m$ means that $f - f_0 \in \mathcal{A}^m$. Moreover, the symbol $\mathcal{A}^m$ is used also for the respective components (we make this agreement so that equation (8.5), for example, makes sense).

Again in this section, we set $\overline{X} = M \times [0,\infty)$, where $(M, T^{1,0}M, \theta)$ is a nondegenerate partially integrable CR manifold with fixed contact form. We consider

ACH metrics of the form (4.9) with g_{ij} satisfying (4.10), which we call *normal-form ACH metrics with logarithmic singularities* for $(M, T^{1,0}M, \theta)$. All the computations regarding the Ricci tensor go in the same way as in Section 6 except that, while $\rho\partial_\rho$ acts on the space of smooth $O(\rho^m)$ functions as a mere "m times" operator modulo $O(\rho^{m+1})$, it is no longer the case when $O(\rho^m)$ and $O(\rho^{m+1})$ are replaced by $\mathcal{A}^m$ and $\mathcal{A}^{m+1}$.

Let us consider a normal-form ACH metric given in Proposition 6.9. For specificity, let $\overline{g}$ be such a metric with the property that each component $\overline{g}_{ij}$ is (at least near $M = M \times \{0\}$) given by a polynomial of ρ. Then the tensor $\overline{E} = \overline{\mathrm{Ric}} + \frac{1}{2}(n+2)\overline{g}$ is unambiguously defined by $(M, T^{1,0}M, \theta)$. We set

$$\overline{E}_{\infty\infty} = \rho^{2n+4} F_{\infty\infty} + O(\rho^{2n+5}), \tag{8.1a}$$

$$\overline{E}_{\infty 0} = \rho^{2n+3} F_{\infty 0} + O(\rho^{2n+4}), \tag{8.1b}$$

$$\overline{E}_{\infty\alpha} = \rho^{2n+4} F_{\infty\alpha} + O(\rho^{2n+5}), \tag{8.1c}$$

$$\overline{E}_{00} = \rho^{2n+4} F_{00} + O(\rho^{2n+5}), \tag{8.1d}$$

$$\overline{E}_{0\alpha} = \rho^{2n+3} F_{0\alpha} + O(\rho^{2n+4}), \tag{8.1e}$$

$$\overline{E}_{\alpha\overline{\beta}} = \rho^{2n+4} F_{\alpha\overline{\beta}} + O(\rho^{2n+5}), \tag{8.1f}$$

$$\overline{E}_{\alpha\beta} = \rho^{2n+2} F_{\alpha\beta} + O(\rho^{2n+3}), \tag{8.1g}$$

where F_{IJ} is constant in the ρ-direction. We already know that $F_{\alpha\beta} = \mathcal{O}_{\alpha\beta}$ and $F_{0\alpha} = -i(\nabla^\beta \mathcal{O}_{\alpha\beta} + N_\alpha{}^{\overline{\beta}\overline{\gamma}} \mathcal{O}_{\overline{\beta}\overline{\gamma}})$. We define

$$u := -\frac{1}{n+1}(F_{\infty 0} - i\nabla^\alpha F_{\infty\alpha} + i\nabla^{\overline{\alpha}} F_{\infty\overline{\alpha}}). \tag{8.2}$$

THEOREM 8.1. *Let κ be any smooth function and $\lambda_{\alpha\beta}$ a smooth symmetric 2-tensor satisfying*

$$D^{\alpha\beta}\lambda_{\alpha\beta} - D^{\overline{\alpha}\overline{\beta}}\lambda_{\overline{\alpha}\overline{\beta}} = iu. \tag{8.3}$$

Then there is a normal-form ACH metric with logarithmic singularity g satisfying $E_{IJ} = \mathcal{A}^\infty$ and

$$\frac{1}{(2n+4)!}\left(\partial_\rho^{2n+4} g_{00}^{(0)}\right)\Big|_M = \kappa, \qquad \frac{1}{(2n+2)!}\left(\partial_\rho^{2n+2} g_{\alpha\beta}^{(0)}\right)\Big|_M = \lambda_{\alpha\beta}, \tag{8.4}$$

where $g_{ij} \sim \sum_{q=0}^\infty g_{ij}^{(q)}(\log\rho)^q$ is the asymptotic expansion of g_{ij}. The components g_{ij} are uniquely determined modulo $\mathcal{A}^\infty$ by the condition above.

As is clear from the proof below, Theorem 8.1 also holds in the following formal sense. Let $p \in M$, κ a smooth function and $\lambda_{\alpha\beta}$ a tensor satisfying (8.3) to the infinite order at p. Then there exists a normal-form ACH metric with logarithmic singularity g satisfying (8.4) and $E_{IJ} = \mathcal{A}^\infty$ to the infinite order at p, and the Taylor expansions of $g_{ij}^{(q)}$ at p are uniquely determined by those of κ and $\lambda_{\alpha\beta}$. On the other hand, there is a formal power series solution to (8.3) by the Cauchy–Kovalevskaya Theorem. Hence, by Borel's Lemma, we have $\lambda_{\alpha\beta}$ solving (8.3) to the infinite order at p. Thus we obtain Theorem 2.11.

REMARK 8.2. The appearance of a formally undetermined term $\lambda_{\alpha\beta}$ at the $(2n+2)^{\mathrm{nd}}$ order generalizes a result of Biquard–Herzlich [**BH1**, Corollary 5.4] in the case $n=1$.

8.2. Non-logarithmic part. The following result can be obtained by following the argument in Section 6 again.

PROPOSITION 8.3. *There exists a normal-form ACH metric with logarithmic singularity g for which $E = \operatorname{Ric} + \frac{1}{2}(n+2)g$ satisfies*

$$E_{IJ} = \mathcal{A}^{2n+2+a(I,J)}, \tag{8.5}$$

where $a(I,J)$ is defined by (6.27). The components g_{ij} are uniquely determined modulo $\mathcal{A}^{2n+2+a(i,j)}$, and do not contain logarithmic terms up to this order.

PROOF. Let g be a normal-form ACH metric with logarithmic singularity. If we define φ_{ij} by (6.2), then the Ricci tensor of g can be computed as we did in Subsections 6.3 and 6.4, and as a result, Lemma 6.5 is again valid for g, where the omitted terms are now those of the form φ_{ij} times $\mathcal{A}^1$. In particular, the contributions of $\mathcal{A}^{2n+2}$-terms in φ_{ij} to E is $\mathcal{A}^{2n+2}$. Take a large N so that φ_{ij} and E_{IJ} for given g are of the form

$$\varphi_{ij} = \sum_{q=0}^{N} \varphi_{ij}^{(q)} (\log\rho)^q + \mathcal{A}^{2n+2}, \qquad \varphi_{ij}^{(q)} \in C^\infty(\overline{X}),$$

and

$$E_{IJ} = \sum_{q=0}^{N} E_{IJ}^{(q)} (\log\rho)^q + \mathcal{A}^{2n+2}, \qquad E_{IJ}^{(q)} \in C^\infty(\overline{X}).$$

Our intermediate goal is proving that it is necessary for $E_{IJ} = \mathcal{A}^{2n+2}$ that $\varphi_{ij}^{(q)} = 0$ (modulo $O(\rho^{2n+2})$) for $q \ge 1$. Then $\varphi_{ij}^{(0)}$s are determined modulo $O(\rho^{2n+2})$ by Theorem 6.1.

We obtain from Lemma 6.5 that, if we set ${g'}_{ij} = g_{ij} + \psi_{ij}$ with $\psi_{ij} = \mathcal{A}^m$ and $\delta E = E' - E$, then the following equalities hold modulo $\mathcal{A}^{m+1}$:

$$\delta E_{\infty\infty} \equiv -\tfrac{1}{2}\rho\partial_\rho(\rho\partial_\rho - 4)\psi_{00} - \rho\partial_\rho(\rho\partial_\rho - 2){\psi_\alpha}^\alpha, \tag{8.6a}$$

$$\delta E_{\infty 0} \equiv 0, \tag{8.6b}$$

$$\delta E_{\infty\alpha} \equiv -\tfrac{i}{2}(\rho\partial_\rho + 1)\psi_{0\alpha}, \tag{8.6c}$$

$$\delta E_{00} \equiv -\tfrac{1}{8}((\rho\partial_\rho)^2 - (2n+4)\rho\partial_\rho - 4n)\psi_{00} + \tfrac{1}{2}(\rho\partial_\rho - 2){\psi_\alpha}^\alpha, \tag{8.6d}$$

$$\delta E_{0\alpha} \equiv -\tfrac{1}{8}(\rho\partial_\rho + 1)(\rho\partial_\rho - 2n - 3)\psi_{0\alpha}, \tag{8.6e}$$

$$\delta {E_\gamma}^\gamma \equiv -\tfrac{1}{8}((\rho\partial_\rho)^2 - (4n+2)\rho\partial_\rho - 8){\psi_\gamma}^\gamma + \tfrac{1}{8}n(\rho\partial_\rho - 4)\psi_{00}, \tag{8.6f}$$

$$\operatorname{tf}(\delta E_{\alpha\overline{\beta}}) \equiv -\tfrac{1}{8}((\rho\partial_\rho)^2 - (2n+2)\rho\partial_\rho - 8)\operatorname{tf}(\psi_{\alpha\overline{\beta}}), \tag{8.6g}$$

$$\delta E_{\alpha\beta} \equiv -\tfrac{1}{8}\rho\partial_\rho(\rho\partial_\rho - 2n - 2)\psi_{\alpha\beta}. \tag{8.6h}$$

Therefore, if $\psi_{ij} = \sum_{q=0}^{N} \psi_{ij}^{(q)}(\log\rho)^q + \mathcal{A}^{m+1}$, then the following holds for $q = N$ modulo $O(\rho^{m+1})$:

$$\delta E_{\infty\infty}^{(q)} \equiv -\tfrac{1}{2}m(m-4)\psi_{00}^{(q)} - m(m-2){\psi^{(q)}}_\alpha{}^\alpha, \tag{8.7a}$$

$$\delta E_{\infty 0}^{(q)} \equiv 0, \tag{8.7b}$$

$$\delta E_{\infty\alpha}^{(q)} \equiv -\tfrac{i}{2}(m+1)\psi_{0\alpha}^{(q)}, \tag{8.7c}$$

$$\delta E_{00}^{(q)} \equiv -\tfrac{1}{8}(m^2 - (2n+4)m - 4n)\psi_{00}^{(q)} + \tfrac{1}{2}(m-2){\psi^{(q)}}_\alpha{}^\alpha, \tag{8.7d}$$

$$\delta E_{0\alpha}^{(q)} \equiv -\tfrac{1}{8}(m+1)(m-2n-3)\psi_{0\alpha}^{(q)}, \tag{8.7e}$$

$$\delta {E^{(q)}}_\gamma{}^\gamma \equiv -\tfrac{1}{8}(m^2 - (4n+2)m - 8){\psi^{(q)}}_\gamma{}^\gamma + \tfrac{1}{8}n(m-4)\psi_{00}^{(q)}, \tag{8.7f}$$

$$\operatorname{tf}(\delta E_{\alpha\overline{\beta}}^{(q)}) \equiv -\tfrac{1}{8}(m^2 - (2n+2)m - 8)\operatorname{tf}(\psi_{\alpha\overline{\beta}}^{(q)}), \tag{8.7g}$$

$$\delta E_{\alpha\beta}^{(q)} \equiv -\tfrac{1}{8}m(m-2n-2)\psi_{\alpha\beta}^{(q)}. \tag{8.7h}$$

In particular, consider the case where g_{ij} does not contain logarithmic terms at all. In this case E does not contain logarithmic terms either. Hence, if $1 \le m \le 2n+1$, (8.7) implies that $(E')^{(N)}$ necessarily has a nonzero component at m^{th} order as soon as some component of $\psi_{ij}^{(N)}$ becomes nonzero modulo $O(\rho^{m+1})$. This shows that we cannot introduce $\psi_{ij}^{(N)}$ that is not $O(\rho^{2n+2})$ while keeping logarithmic terms away from E. The same argument is also possible for $q = N-1$, $N-2$, …, 1, so we conclude that g cannot have logarithmic terms modulo $\mathcal{A}^{2n+2}$.

Moreover, setting $m = 2n+3$ we similarly obtain that there are no logarithmic terms in g_{00}, $g_{0\alpha}$, and $g_{\alpha\overline{\beta}}$ modulo $\mathcal{A}^{2n+3}$, and by moving forward to the case $m = 2n+4$, g_{00} and $g_{\alpha\overline{\beta}}$ cannot have logarithmic terms modulo $\mathcal{A}^{2n+4}$. □

8.3. Computation on Ricci tensor. For the discussion that follows, we need a formula for the behavior of the Ricci tensor that is different from (8.6). Let g be a fixed normal-form ACH metric with logarithmic singularity such that $E = \operatorname{Ric} + \frac{1}{2}(n+2)g$ is at least $\mathcal{A}^3$. The proof of Proposition 8.3 shows that such g cannot contain logarithmic terms modulo $\mathcal{A}^3$. So its components are determined as (7.3). For an integer $m \ge 3$, we consider another metric g' given as follows:

$$g'_{ij} = g_{ij} + \psi_{ij}, \qquad \psi_{ij} = \mathcal{A}^{m+a(i,j)}. \tag{8.8}$$

Let $\delta E := E' - E$.

Lemma 8.4. *Let g be a normal-form ACH metric with logarithmic singularity such that $E = \mathcal{A}^3$. For the perturbation (8.8) of g, the tensor δE is given by*

$$\begin{aligned}\delta E_{\infty\infty} &= -\tfrac{1}{2}\rho\partial_\rho(\rho\partial_\rho - 4)\psi_{00} - \rho\partial_\rho(\rho\partial_\rho - 2)\psi_\gamma{}^\gamma \\ &\quad + \tfrac{1}{2}\rho^2(\rho\partial_\rho)^2(\Phi^{\alpha\beta}\psi_{\alpha\beta} + \Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}}) + \mathcal{A}^{m+3},\end{aligned} \tag{8.9a}$$

(8.9b)
$$\begin{aligned}\delta E_{\infty 0} &= \tfrac{1}{2}\rho(\rho\partial_\rho+1)(\nabla^\alpha\psi_{0\alpha}+\nabla^{\overline{\alpha}}\psi_{0\overline{\alpha}})\\ &\quad-\tfrac{1}{2}\rho^2\cdot\rho\partial_\rho(A^{\alpha\beta}\psi_{\alpha\beta}+A^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})+\mathcal{A}^{m+3},\end{aligned}$$

(8.9c)
$$\delta E_{\infty\alpha} = -\tfrac{i}{2}(\rho\partial_\rho+1)\psi_{0\alpha}+\tfrac{1}{2}\rho\cdot\rho\partial_\rho\nabla^\beta\psi_{\alpha\beta}+\tfrac{1}{2}\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}\rho\partial_\rho\psi_{\overline{\beta}\overline{\gamma}}+\mathcal{A}^{m+2},$$

(8.9d)
$$\begin{aligned}\delta E_{00} &= -\tfrac{1}{8}((\rho\partial_\rho)^2-(2n+4)\rho\partial_\rho-4n)\psi_{00}+\tfrac{1}{2}(\rho\partial_\rho-2)\psi_\alpha{}^\alpha\\ &\quad+i\rho(\nabla^\alpha\psi_{0\alpha}-\nabla^{\overline{\alpha}}\psi_{0\overline{\alpha}})-\tfrac{1}{4}\rho^2\cdot\rho\partial_\rho(\Phi^{\alpha\beta}\psi_{\alpha\beta}+\Phi^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})+\mathcal{A}^{m+3},\end{aligned}$$

(8.9e)
$$\delta E_{0\alpha} = -\tfrac{1}{8}(\rho\partial_\rho+2)(\rho\partial_\rho-2n-2)\psi_{0\alpha}+\mathcal{A}^{m+2},$$

(8.9f)
$$\begin{aligned}\delta E_\alpha{}^\alpha &= \tfrac{1}{8}n(\rho\partial_\rho-2)\psi_{00}-\tfrac{1}{8}((\rho\partial_\rho)^2-(4n-2)\rho\partial_\rho-8n-8)\psi_\alpha{}^\alpha\\ &\quad+(\mathcal{A}^{m+2}\ \textit{terms depending on}\ \psi_{0\alpha}\ \textit{and}\ \psi_{\alpha\beta})+\mathcal{A}^{m+3},\end{aligned}$$

(8.9g)
$$\begin{aligned}\operatorname{tf}(\delta E_{\alpha\overline{\beta}}) &= -\tfrac{1}{8}((\rho\partial_\rho)^2-2n\rho\partial_\rho-2n-9)\operatorname{tf}(\psi_{\alpha\overline{\beta}})\\ &\quad+(\mathcal{A}^{m+2}\ \textit{terms depending on}\ \psi_{0\alpha}\ \textit{and}\ \psi_{\alpha\beta})+\mathcal{A}^{m+3},\end{aligned}$$

(8.9h)
$$\delta E_{\alpha\beta} = -\tfrac{1}{8}\rho\partial_\rho(\rho\partial_\rho-2n-2)\psi_{\alpha\beta}+\mathcal{A}^{m+1},$$

where we define $\Phi_{\alpha\beta}$ *by* (7.4b).

Lemma 8.4 is proved by the following idea. In Subsection 6.4, we used (6.8) to compute the Ricci tensor itself. However, if we try to show Lemma 8.4 again on this plan, then it will demand too much complicated computation. So we take the difference from the start: (6.8) implies that

(8.10)
$$\begin{aligned}\delta\operatorname{Ric}_{IJ} &= \overline{\nabla}_K(\delta D)^K{}_{IJ}-\overline{\nabla}_I(\delta D)^K{}_{KJ}-\delta D^L{}_{KI}\cdot D^K{}_{LJ}-D^L{}_{KI}\cdot\delta D^K{}_{LJ}\\ &\quad+\delta D^L{}_{IJ}\cdot D^K{}_{KL}+D^L{}_{IJ}\cdot\delta D^K{}_{KL}+\mathcal{A}^{m+3}.\end{aligned}$$

We omitted the quadratic terms of δD. This is allowed because any $\delta D^K{}_{IJ}$ is $\mathcal{A}^m$ and we assumed that $m\geq 3$. To compute the remaining part modulo $\mathcal{A}^{m+3}$, it suffices to compute $D^K{}_{IJ}$ modulo $\mathcal{A}^3$ and $\delta D^K{}_{IJ}$ modulo $\mathcal{A}^{m+3}$.

LEMMA 8.5. *If g is a normal-form ACH metric with logarithmic singularity for which $E=\operatorname{Ric}+\frac{1}{2}(n+2)g$ is $\mathcal{A}^3$, then $D^K{}_{IJ}$ modulo $\mathcal{A}^3$ is given by Table 8.1.*

PROOF. Since $\varphi_{ij}=\mathcal{A}^2$, this follows by Lemma 6.5 if we modify it for metrics with logarithmic singularities. □

LEMMA 8.6. *If g is a normal-form ACH metric with logarithmic singularity for which $E=\operatorname{Ric}+\frac{1}{2}(n+2)g$ is $\mathcal{A}^3$, then for the perturbation* (8.8) *of the metric, $\delta D^K{}_{IJ}$ modulo $\mathcal{A}^{m+3}$ is given by Table 8.2.*

PROOF. Since we can read $\delta D^K{}_{IJ}$ off from Table 6.1 only modulo $\mathcal{A}^{m+1}$, we have to take a roundabout route as follows. We first compute δD_{KIJ} modulo $\mathcal{A}^{m+3}$ by using (6.9). For this we need $\delta(\overline{\nabla}_K g_{IJ})=\overline{\nabla}_K\psi_{IJ}$, which is zero if $\infty\in\{I,J\}$, and $\delta\overline{T}_{KIJ}=\overline{T}^L{}_{IJ}\psi_{KL}$, which is zero if $\infty\in\{I,J,K\}$. By (6.4), the components

Type	*Value*
$D^{\infty}{}_{\infty\infty}$	-1
$D^{\infty}{}_{\infty 0}$	0
$D^{\infty}{}_{\infty\alpha}$	0
$D^{\infty}{}_{00}$	$\frac{1}{2}$
$D^{\infty}{}_{0\alpha}$	0
$D^{\infty}{}_{\alpha\overline{\beta}}$	$\frac{1}{4}h_{\alpha\overline{\beta}}$
$D^{\infty}{}_{\alpha\beta}$	0

Type	*Value*
$D^{0}{}_{\infty\infty}$	0
$D^{0}{}_{\infty 0}$	-2
$D^{0}{}_{\infty\alpha}$	0
$D^{0}{}_{00}$	0
$D^{0}{}_{0\alpha}$	0
$D^{0}{}_{\alpha\overline{\beta}}$	$-\frac{i}{2}h_{\alpha\overline{\beta}}$
$D^{0}{}_{\alpha\beta}$	$-\rho^2 A_{\alpha\beta}$

Type	*Value*
$D^{\gamma}{}_{\infty\infty}$	0
$D^{\gamma}{}_{\infty 0}$	0
$D^{\gamma}{}_{\infty\alpha}$	$-\delta_\alpha{}^\gamma + \rho^2\Phi_\alpha{}^\gamma$
$D^{\gamma}{}_{\infty\overline{\alpha}}$	$\rho^2\Phi_{\overline{\alpha}}{}^\gamma$
$D^{\gamma}{}_{00}$	0
$D^{\gamma}{}_{0\alpha}$	$\frac{i}{2}\delta_\alpha{}^\gamma$
$D^{\gamma}{}_{0\overline{\alpha}}$	$i\rho^2\Phi_{\overline{\alpha}}{}^\gamma$

Type	*Value*
$D^{\gamma}{}_{\alpha\beta}$	0
$D^{\gamma}{}_{\alpha\overline{\beta}}$	0
$D^{\gamma}{}_{\overline{\alpha}\overline{\beta}}$	$\rho N^{\gamma}{}_{\overline{\beta}\overline{\alpha}}$

TABLE 8.1. $D^K{}_{IJ}$ modulo $\mathcal{A}^3$ (Lemma 8.5)

of the first tensor is as given in Table 8.3. On the other hand, by (6.3), those of the second one is as in Table 8.4. As a result,

$$\delta D_{KIJ} = \tfrac{1}{2}(\delta(\overline{\nabla}_I g_{JK}) + \delta(\overline{\nabla}_J g_{IK}) - \delta(\overline{\nabla}_K g_{IJ}) + \delta\overline{T}_{IJK} + \delta\overline{T}_{JIK} - \delta\overline{T}_{KIJ})$$

is given by Table 8.3.

Then we compute $\delta D^K{}_{IJ}$ by the formula

$$\delta D^K{}_{IJ} = g^{KL}\cdot\delta D_{LIJ} + \delta g^{KL}\cdot D_{LIJ} + \mathcal{A}^{m+3}. \tag{8.11}$$

Here we need

$$g^{00} = 1 + O(\rho^3), \qquad g^{0\alpha} = O(\rho^3),$$
$$g^{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} - \rho^2\Phi^{\alpha\overline{\beta}} + O(\rho^3), \qquad g^{\alpha\beta} = -\rho^2\Phi^{\alpha\overline{\beta}} + O(\rho^3)$$

and

$$\delta g^{00} = -\psi_{00} + \mathcal{A}^{m+3},$$
$$\delta g^{0\alpha} = -\psi_0{}^\alpha + \mathcal{A}^{m+3},$$
$$\delta g^{\alpha\overline{\beta}} = -\psi^{\alpha\overline{\beta}} + \rho^2(\Phi^\alpha{}_{\overline{\gamma}}\psi^{\overline{\beta}\overline{\gamma}} + \Phi^{\overline{\beta}}{}_\gamma\psi^{\alpha\gamma}) + \mathcal{A}^{m+3},$$
$$\delta g^{\alpha\beta} = -\psi^{\alpha\beta} + \rho^2(\Phi^\alpha{}_\gamma\psi^{\beta\gamma} + \Phi^\beta{}_\gamma\psi^{\alpha\gamma}) + \mathcal{A}^{m+3}.$$

Moreover, D_{LIJ} modulo $\mathcal{A}^3$ is read off from Table 6.1 as Table 8.6. We put all these into (8.11). The details are omitted. □

We remark one more thing here: for metrics with logarithmic singularities such that $E_{IJ} = \mathcal{A}^{m+a(I,J)}$, we need a version of Lemma 6.6. A similar use of the contracted Bianchi identity now leads to the following lemma, whose proof is again omitted.

LEMMA 8.7. *Let $m \geq 1$ be a positive integer. Suppose that g is a normal-form ACH metric with logarithmic singularity for which $E := \operatorname{Ric} + \frac{1}{2}(n+2)g$ satisfies*

$$E_{IJ} = \mathcal{A}^{m+a(I,J)}.$$

Then we have

$$\begin{aligned}\mathcal{A}^{m+3} &= (\rho\partial_\rho - 4n - 4)E_{\infty\infty} - 4(\rho\partial_\rho - 4)E_{00} - 8(\rho\partial_\rho - 2)E_\alpha{}^\alpha \\ &\quad + 16\rho \operatorname{Re}(\nabla^\alpha E_{\infty\alpha}) + 8\rho^2(\rho\partial_\rho - 2)\operatorname{Re}(\Phi^{\alpha\beta}E_{\alpha\beta}),\end{aligned} \tag{8.12a}$$

$$\mathcal{A}^{m+3} = (\rho\partial_\rho - 2n - 4)E_{\infty 0} + 8\rho \operatorname{Re}(\nabla^\alpha E_{0\alpha}) + 8\rho^2 \operatorname{Re}(A^{\alpha\beta}E_{\alpha\beta}), \tag{8.12b}$$

$$\mathcal{A}^{m+2} = (\rho\partial_\rho - 2n - 3)E_{\infty\alpha} - 4iE_{0\alpha} + 4\rho\nabla^\beta E_{\alpha\beta} + 4\rho N_\alpha{}^{\overline{\beta}\overline{\gamma}}E_{\overline{\beta}\overline{\gamma}}. \tag{8.12c}$$

8.4. Construction of the full expansion. To construct expansions of g_{ij} that solves $E = \mathcal{A}^\infty$, the first point where a logarithmic term comes into is the $(2n+2)^{\mathrm{nd}}$ order. This is due to (8.9h). Moreover, (6.17) and (6.24) indicate that the following orders also need some special care: $2n+3$, $2n+4$, and $4n+4$. We first discuss the $(2n+2)^{\mathrm{nd}}$ and two more orders that follow.

LEMMA 8.8. *Let u be as in* (8.2) *and assume that there is a solution $\lambda_{\alpha\beta}$ to the differential equation* (8.3)*. Then there exists a normal-form ACH metric with logarithmic singularity g for which $E = \operatorname{Ric} + \frac{1}{2}(n+2)g$ satisfies*

$$E = \mathcal{A}^{2n+5}.$$

Such a metric g is necessarily of the form

$$g_{00} = \overline{g}_{00} + \mathcal{A}^{2n+4}, \tag{8.13a}$$

$$g_{0\alpha} = \overline{g}_{0\alpha} + \mathcal{A}^{2n+3}, \tag{8.13b}$$

$$g_{\alpha\overline{\beta}} = \overline{g}_{\alpha\overline{\beta}} + \mathcal{A}^{2n+4}, \tag{8.13c}$$

$$g_{\alpha\beta} = \overline{g}_{\alpha\beta} + \mathcal{A}^{2n+2}, \tag{8.13d}$$

where $\overline{g}$ is the metric described in Subsection 8.1. For a particular choice of $\lambda_{\alpha\beta}$ that solves (8.3) *and any smooth real-valued function κ on M, we can take g so that it satisfies* (8.4)*. Such g is unique modulo $\mathcal{A}^{2n+5}$.*

PROOF. We set

$$g_{ij} = \overline{g}_{ij} + \sum_{a=0}^{2}\sum_{q=0}^{N}(\chi_a)_{ij}^{(q)}\rho^{2n+2+a}(\log\rho)^q + \mathcal{A}^{2n+5}, \tag{8.14}$$

where $(\chi_a)_{ij}^{(q)}$s are tensors on M, and we shall determine when g satisfies the condition $E = \mathcal{A}^{2n+5}$. The symbol δE will be used in various ways in this proof—this always denotes some change of the tensor E, but we will consider many different perturbations of the metric. The situations will be made clear every time when δE is used.

We first determine the terms with $a = 0$ so that $E = \mathcal{A}^{2n+3}$ is satisfied. Since $\overline{g}$ is already chosen so that $\overline{E}_{IJ} = \mathcal{A}^{2n+2+a(I,J)}$, an easy argument shows that $(\chi_0)^{(q)}_{00}$, $(\chi_0)^{(q)}_{0\alpha}$, and $(\chi_0)^{(q)}_{\alpha\overline{\beta}}$ must be zero for all q. For $(\chi_0)^{(q)}_{\alpha\beta}$, we need the following formula of δE when these coefficients are introduced, which follows from (8.9h):

$$\begin{aligned} -8\,\delta E_{\alpha\beta} &= (q+1)(2n+2)\sum_{q=0}^{N-1}(\chi_0)^{(q+1)}_{\alpha\beta}\rho^{2n+2}(\log\rho)^q \\ &\quad + (q+1)(q+2)\sum_{q=0}^{N-2}(\chi_0)^{(q+2)}_{\alpha\beta}\rho^{2n+2}(\log\rho)^q + \mathcal{A}^{2n+3}. \end{aligned}$$

Since $\overline{E}_{\alpha\beta}$ has no logarithmic term, $(\chi_0)^{(q)}_{\alpha\beta}$ must be zero for $q \geq 2$. The next coefficient $(\chi_0)^{(1)}_{\alpha\beta}$ is determined so that the ρ^{2n+2}-coefficient of $E_{\alpha\beta}$ vanishes as a result. Namely, we set

$$(\chi_0)^{(1)}_{\alpha\beta} = \frac{4}{n+1}F_{\alpha\beta} = \frac{4}{n+1}\mathcal{O}_{\alpha\beta}. \tag{8.15}$$

By Lemma 8.7, $E_{\infty I}$ are automatically $\mathcal{A}^{2n+3}$. The remaining coefficient $(\chi_0)^{(0)}_{\alpha\beta}$ does not controlled by any restriction so far.

Next we determine the terms with $a = 1$ so that $E = \mathcal{A}^{2n+4}$ holds. We have introduced the (potentially) nonzero logarithmic coefficient $(\chi_0)^{(1)}_{\alpha\beta}$, and this is now reflected in the tensor E. However, by applying Lemma 8.4 with $m = 2n+2$, there is no effect of this term in E_{00}, $E_{\alpha\overline{\beta}}$ modulo $\mathcal{A}^{2n+4}$. So E_{00} and $E_{\alpha\overline{\beta}}$ are already $\mathcal{A}^{2n+4}$, as $\overline{E}_{00}$ and $\overline{E}_{\alpha\overline{\beta}}$ are from the beginning. Hence $(\chi_1)^{(q)}_{00}$ and $(\chi_1)^{(q)}_{\alpha\overline{\beta}}$ must be zero for all q. Lemma 8.7 implies that $E_{\infty\infty} = \mathcal{A}^{2n+4}$. Moreover, since we already achieved $E = \mathcal{A}^{2n+3}$, the same lemma implies that $E_{\infty 0} = \mathcal{A}^{2n+4}$. The remaining components of E to be considered are $E_{\infty\alpha}$, $E_{0\alpha}$, and $E_{\alpha\beta}$. Equation (8.9e) shows that $E_{0\alpha}$ has no logarithmic term modulo $\mathcal{A}^{2n+4}$, so $(\chi_1)^{(q)}_{0\alpha}$ should be zero for $q \geq 2$, because (8.9e) implies

$$\begin{aligned} -8\,\delta E_{\alpha\beta} &= (q+1)(2n+4)\sum_{q=0}^{N-1}(\chi_1)^{(q+1)}_{0\alpha}\rho^{2n+3}(\log\rho)^q \\ &\quad + (q+1)(q+2)\sum_{q=0}^{N-2}(\chi_1)^{(q+2)}_{0\alpha}\rho^{2n+3}(\log\rho)^q + \mathcal{A}^{2n+4}. \end{aligned}$$

We introduce $(\chi_1)^{(1)}_{0\alpha}$ to kill $E_{0\alpha}$:

$$(\chi_1)^{(1)}_{0\alpha} = \frac{4}{n+2}F_{0\alpha} = -\frac{4i}{n+2}(\nabla^\beta\mathcal{O}_{\alpha\beta} + N_\alpha{}^{\overline{\beta}\overline{\gamma}}\mathcal{O}_{\overline{\beta}\overline{\gamma}}). \tag{8.16}$$

Since (8.12c) cannot be used to prove that $E_{\infty\alpha} = \mathcal{A}^{2n+4}$, we need another means to kill $E_{\infty\alpha}$ in this order. Note that, since $F_{0\alpha} = -i(\nabla^\beta F_{\alpha\beta} + N_\alpha{}^{\overline{\beta}\overline{\gamma}}F_{\overline{\beta}\overline{\gamma}})$, (8.15), (8.16), and (8.9c) imply that $E_{\infty\alpha}$ has no logarithmic term modulo $\mathcal{A}^{2n+4}$ so far.

By (8.9c) again, if we can take $(\chi_1)^{(0)}_{0\alpha}$ and $(\chi_0)^{(0)}_{\alpha\beta}$ so that the following holds, then $E_{\infty\alpha}$ becomes $\mathcal{A}^{2n+4}$:

$$
\begin{aligned}
&- i(n+2)(\chi_1)^{(0)}_{0\alpha} + (n+1)\nabla^{\beta}(\chi_0)^{(0)}_{\alpha\beta} + (n+1)N_{\alpha}{}^{\overline{\beta}\overline{\gamma}}(\chi_0)^{(0)}_{\overline{\alpha\beta}} \\
&= -F_{\infty\alpha} + \frac{2i}{n+2}F_{0\alpha} - \frac{2}{n+1}(\nabla^{\beta}F_{\alpha\beta} + N_{\alpha}{}^{\overline{\beta}\overline{\gamma}}F_{\overline{\beta\gamma}}).
\end{aligned}
\tag{8.17}
$$

Now suppose (8.17) can be satisfied. As for $E_{\alpha\beta}$, Lemma 8.4 shows that, because of the nonzero coefficient $(\chi_0)^{(1)}_{\alpha\beta}$, $E_{\alpha\beta}$ can be written as follows by using tensors $\varepsilon^{(0)}_{\alpha\beta}$ and $\varepsilon^{(1)}_{\alpha\beta}$ on M:

$$
E_{\alpha\beta} = \varepsilon^{(0)}_{\alpha\beta}\rho^{2n+3} + \varepsilon^{(1)}_{\alpha\beta}\rho^{2n+3}\log\rho + \mathcal{A}^{2n+4}.
$$

So we conclude, by using (8.6h) for $m = 2n+3$, that $(\chi_1)^{(q)}_{\alpha\beta}$ must be zero for $q \geq 2$, and $(\chi_1)^{(1)}_{\alpha\beta}$, $(\chi_1)^{(0)}_{\alpha\beta}$ are uniquely determined by the condition $E_{\alpha\beta} = \mathcal{A}^{2n+4}$.

Finally we consider $a = 2$ to achieve $E = \mathcal{A}^{2n+5}$. The effect of the existence of $(\chi_1)^{(1)}_{\alpha\beta}$ and $(\chi_0)^{(1)}_{\alpha\beta}$ may appear in E_{00} and $E_{\alpha\overline{\beta}}$, so we write

$$
\begin{aligned}
E_{00} &= \varepsilon^{(0)}_{00}\rho^{2n+4} + \varepsilon^{(1)}_{00}\rho^{2n+4}\log\rho + \mathcal{A}^{2n+5}, \\
E_{\alpha\overline{\beta}} &= \varepsilon^{(0)}_{\alpha\overline{\beta}}\rho^{2n+4} + \varepsilon^{(1)}_{\alpha\overline{\beta}}\rho^{2n+4}\log\rho + \mathcal{A}^{2n+5}.
\end{aligned}
$$

It follows from (8.9g) that $\mathrm{tf}((\chi_2)^{(q)}_{\alpha\overline{\beta}})$ must be zero for $q \geq 2$, and $\mathrm{tf}((\chi_2)^{(1)}_{\alpha\overline{\beta}})$, $\mathrm{tf}((\chi_2)^{(0)}_{\alpha\overline{\beta}})$ are uniquely determined by requiring $\mathrm{tf}(E_{\alpha\overline{\beta}}) = \mathcal{A}^{2n+5}$. For the determination of $(\chi_2)^{(q)}_{00}$ and $(\chi_2)^{(q)}{}_{\gamma}{}^{\gamma}$, the following equality for the perturbation (8.8), which follows from (8.9d) and (8.9f), is important:

$$
\begin{aligned}
n\,\delta E_{00} - 2\,\delta E_{\gamma}{}^{\gamma} &= -\tfrac{1}{8}(\rho\partial_\rho + 2)(\rho\partial_\rho - 2n - 4)(n\psi_{00} - 2\psi_{\gamma}{}^{\gamma}) \\
&\quad + (\mathcal{A}^{m+2} \text{ terms depending on } \psi_{0\alpha} \text{ and } \psi_{\alpha\beta}) + \mathcal{A}^{m+3}.
\end{aligned}
$$

Therefore, in the current situation, the effect of $(\chi_2)^{(q)}_{00}$ and $(\chi_2)^{(q)}{}_{\gamma}{}^{\gamma}$ on $nE_{00} - 2E_{\gamma}{}^{\gamma}$ is

$$
\begin{aligned}
&- 8(n\,\delta E_{00} - 2\,\delta E_{\gamma}{}^{\gamma}) \\
&= (q+1)(2n+6)\sum_{q=0}^{N-1}(n(\chi_2)^{(q+1)}_{00} - 2(\chi_2)^{(q+1)}{}_{\gamma}{}^{\gamma})\rho^{2n+4}(\log\rho)^q \\
&\quad + (q+1)(q+2)\sum_{q=0}^{N-2}(n(\chi_2)^{(q+2)}_{00} - 2(\chi_2)^{(q+2)}{}_{\gamma}{}^{\gamma})\rho^{2n+4}(\log\rho)^q + \mathcal{A}^{2n+5}.
\end{aligned}
$$

Hence $n(\chi_2)^{(q)}_{00} - 2(\chi_2)^{(q)}{}_{\gamma}{}^{\gamma}$ must be zero for $q \geq 3$, and we uniquely determine $n(\chi_2)^{(q)}_{00} - 2(\chi_2)^{(q)}{}_{\gamma}{}^{\gamma}$, for $q = 2$, 1 to kill $nE_{00} - 2E_{\gamma}{}^{\gamma}$ modulo $\mathcal{A}^{2n+5}$. Another linear combination of $(\chi_2)^{(q)}_{00}$ and $(\chi_2)^{(q)}{}_{\gamma}{}^{\gamma}$ is determined by considering another combination of E_{00} and $E_{\gamma}{}^{\gamma}$. For example, E_{00} itself will do. By equation (8.9d), we see that the following holds for $q = N$:

$$\delta E_{00}^{(q)} = \rho^{2n+4}\left(\frac{1}{2}n(\chi_2)_{00}^{(q)} + (n+1)(\chi_2)^{(q)}{}_{\alpha}{}^{\alpha}\right) + O(\rho^{2n+5}).$$

Therefore, $\frac{1}{2}n(\chi_2)_{00}^{(q)} + (n+1)(\chi_2)^{(q)}{}_{\gamma}{}^{\gamma}$ should be zero for $q = N$. The same thing is true for $q = N-1, \ldots, 2$. Then $\frac{1}{2}n(\chi_2)_{00}^{(q)} + (n+1)(\chi_2)^{(q)}{}_{\gamma}{}^{\gamma}$ for $q = 1, 0$ are set so that E_{00} becomes $\mathcal{A}^{2n+5}$. Thus $(\chi_2)_{00}^{(q)}$ and $(\chi_2)^{(q)}{}_{\gamma}{}^{\gamma}$ are all determined except one linear combination: $n(\chi_2)_{00}^{(0)} - 2(\chi_2)^{(0)}{}_{\gamma}{}^{\gamma}$. By (8.12a), $E_{\infty\infty}$ is automatically $\mathcal{A}^{2n+5}$.

It remains to consider $E_{\infty 0}$, $E_{\infty\alpha}$, $E_{0\alpha}$, and $E_{\alpha\beta}$. It follows, by using (8.9e) and (8.9h), that we can determine $(\chi_2)_{0\alpha}^{(q)}$ and $(\chi_2)_{\alpha\beta}^{(q)}$ for all q so that $E_{0\alpha}$ and $E_{\alpha\beta}$ are $\mathcal{A}^{2n+5}$. Then, by (8.12c), $E_{\infty\alpha}$ is $\mathcal{A}^{2n+5}$. We try to control $E_{\infty 0}$ by using $(\chi_1)_{0\alpha}^{(0)}$ and $(\chi_0)_{\alpha\beta}^{(0)}$, which are already subject to (8.17). What we need is the following relation:

$$\begin{aligned}(n+2)(\nabla^{\alpha}(\chi_1)_{0\alpha}^{(0)} + \nabla^{\overline{\alpha}}(\chi_1)_{0\overline{\alpha}}^{(0)}) - (n+1)(A^{\alpha\beta}(\chi_0)_{\alpha\beta}^{(0)} + A^{\overline{\alpha}\overline{\beta}}(\chi_0)_{\overline{\alpha}\overline{\beta}}^{(0)})\\ = -F_{\infty 0} - \frac{2}{n+2}(\nabla^{\alpha}F_{0\alpha} + \nabla^{\overline{\alpha}}F_{0\overline{\alpha}}) + \frac{2}{n+1}(A^{\alpha\beta}F_{\alpha\beta} + A^{\overline{\alpha}\overline{\beta}}F_{\overline{\alpha}\overline{\beta}}).\end{aligned} \tag{8.18}$$

If this is satisfied, then we get $E_{\infty 0} = \mathcal{A}^{2n+5}$.

Under the condition (8.18), (8.17) reduces to the following equation on $(\chi_0)_{\alpha\beta}^{(0)}$:

$$D^{\alpha\beta}(\chi_0)_{\alpha\beta}^{(0)} - D^{\overline{\alpha}\overline{\beta}}(\chi_0)_{\overline{\alpha}\overline{\beta}}^{(0)} = iu,$$

where u is given by (8.2). If we have a solution $\lambda_{\alpha\beta}$ to (8.3), then we set $(\chi_0)_{\alpha\beta}^{(0)} = \lambda_{\alpha\beta}$ and suitably determine $(\chi_1)_{0\alpha}^{(0)}$ by (8.18).

We still have one undetermined real-valued function $n(\chi_2)_{00}^{(0)} - 2(\chi_2)^{(0)}{}_{\gamma}{}^{\gamma}$, which can be arbitrarily prescribed. Since there is another linear combination of $(\chi_2)_{00}^{(0)}$ and $(\chi_2)^{(0)}{}_{\gamma}{}^{\gamma}$ that is determined, this is still equivalent if we state as (8.4). □

Now we finish the proof of Theorem 8.1. The only point that we need extra attention is where we determine the ρ^{4n+4}-coefficient of g_{ij}.

Proof of Theorem 8.1. Let $m \geq 2n+5$ and suppose that g is a normal-form ACH metric with logarithmic singularity for which $E = \mathcal{A}^m$. We set

$$g'_{ij} = g_{ij} + \sum_{q=0}^{N} \psi_{ij}^{(q)}(\log\rho)^q, \qquad \psi_{ij}^{(q)} = O(\rho^m).$$

and shall prove that $\psi_{ij}^{(q)}$ modulo $O(\rho^{m+1})$ may be uniquely determined so that $E' = \mathcal{A}^{m+1}$ holds. Then the induction works and we obtain the theorem.

By (8.6), we can express $\delta E = E' - E$ as

$$\delta E_{IJ} = \sum_{q=0}^{N} \delta E_{IJ}^{(q)}(\log\rho)^q + \mathcal{A}^{m+1}.$$

Modulo $O(\rho^{m+1})$, (8.7) holds for $q = N$. It follows from (6.24) that if $m \neq 4n+4$ then $\psi_{ij}^{(N)}$ is uniquely determined modulo $O(\rho^{m+1})$, and inductively, $\psi_{ij}^{(q)}$ for $0 \leq q \leq N-1$ are also determined modulo $O(\rho^{m+1})$ by the condition $E'_{ij} = \mathcal{A}^{m+1}$. By Lemma 6.6 modified to the case of metrics with logarithmic singularities, $E'_{\infty I}$s are also $\mathcal{A}^{m+1}$.

If $m = 4n+4$, instead of the pair of (8.6d) and (8.6f), we use that of (8.6a) and (8.6d) to determine $\psi_{00}^{(q)}$ and $\psi^{(q)}{}_{\gamma}{}^{\gamma}$ modulo $O(\rho^{4n+5})$. Then we can determine $\psi_{ij}^{(N)}$, $\psi_{ij}^{(N-1)}$, $\ldots$, $\psi_{ij}^{(0)}$ inductively so that $E'_{\infty\infty}$, E'_{00}, $E'_{0\alpha}$, $\operatorname{tf}(E'_{\alpha\overline{\beta}})$ and $E'_{\alpha\beta}$ are all $\mathcal{A}^{4n+5}$. By Lemma 6.6, we obtain that $E'_{\alpha}{}^{\alpha}$, $E'_{\infty 0}$, and $E'_{\infty\alpha}$ are also $\mathcal{A}^{4n+5}$. □

8.5. Logarithmic-free solutions. Finally we discuss the construction of a completely logarithmic-free solution when $\mathcal{O}_{\alpha\beta} = 0$. We set

$$
\begin{aligned}
v := -F_{00} + \frac{2}{n}F_{\alpha}{}^{\alpha} - \frac{1}{n}(\nabla^{\alpha}F_{\infty\alpha} + \nabla^{\overline{\alpha}}F_{\infty\overline{\alpha}}) + \frac{2}{n(n+2)}i(\nabla^{\alpha}F_{0\alpha} - \nabla^{\overline{\alpha}}F_{0\overline{\alpha}}) \\
- \frac{2}{n(n+1)}(\nabla^{\alpha}\nabla^{\beta}F_{\alpha\beta} + \nabla^{\overline{\alpha}}\nabla^{\overline{\beta}}F_{\overline{\alpha}\overline{\beta}} + N^{\gamma\alpha\beta}\nabla_{\gamma}F_{\alpha\beta} + N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}\nabla_{\overline{\gamma}}F_{\overline{\alpha}\overline{\beta}} \\
+ (\nabla_{\gamma}N^{\gamma\alpha\beta})F_{\alpha\beta} + (\nabla_{\overline{\gamma}}N^{\overline{\gamma}\overline{\alpha}\overline{\beta}})F_{\overline{\alpha}\overline{\beta}})
\end{aligned}
$$

and define the differential operator $D' = (D')^{\alpha\beta}$ by

$$
(D')^{\alpha\beta} := D^{\alpha\beta} + 2N^{\gamma\alpha\beta}\nabla_{\gamma} + 2\left(1 + \frac{1}{n}\right)(\nabla_{\gamma}N^{\gamma\alpha\beta}).
$$

THEOREM 8.9. *Suppose that $\mathcal{O}_{\alpha\beta} = 0$. Let κ be a smooth function and $\lambda_{\alpha\beta}$ a smooth symmetric 2-tensor satisfying*

$$
\begin{cases} D^{\alpha\beta}\lambda_{\alpha\beta} - D^{\overline{\alpha}\overline{\beta}}\lambda_{\overline{\alpha}\overline{\beta}} = iu, \\ (D')^{\alpha\beta}\lambda_{\alpha\beta} + (D')^{\overline{\alpha}\overline{\beta}}\lambda_{\overline{\alpha}\overline{\beta}} = v. \end{cases} \tag{8.19}
$$

Then there is a normal-form ACH metric g, which is free of logarithmic terms, satisfying $\operatorname{Ein}_{IJ} = O(\rho^{\infty})$ *and*

$$
\frac{1}{(2n+4)!}\left(\partial_{\rho}^{2n+4}g_{00}\right)\big|_{M} = \kappa, \qquad \frac{1}{(2n+2)!}\left(\partial_{\rho}^{2n+2}g_{\alpha\beta}\right)\big|_{M} = \lambda_{\alpha\beta}. \tag{8.20}
$$

The components g_{ij} are unique modulo $O(\rho^{\infty})$.

Again this theorem also holds in the formal sense. Since the principal parts of $D^{\alpha\beta}$ and $(D')^{\alpha\beta}$ agree, the system (8.19) is formally solvable at any given point; in fact, if one arbitrarily prescribes the components of $\lambda_{\alpha\beta}$ except λ_{11}, for example, and writes $\lambda_{11} = \mu + i\nu$ where μ and ν are real-valued, then (8.19) can be regarded as a system of PDEs for μ and ν and the Cauchy–Kovalevskaya theorem can be applied to this system. Thus we can show the second statement of Theorem 2.11.

Proof. If $\mathcal{O}_{\alpha\beta} = 0$, then a (potentially) singular normal-form ACH metric g satisfying the conditions in the statement of Lemma 8.8 is of the form (8.14). By the proof of Lemma 8.8, $(\chi_0)_{ij}^{(q)}$ and $(\chi_1)_{ij}^{(q)}$ for $q \geq 1$ are zero if $\mathcal{O}_{\alpha\beta} = 0$. The remaining potential log-term coefficients appears in $n(\chi_2)_{00}^{(1)} - 2{(\chi_2)^{(1)}}_{\gamma}{}^{\gamma}$, so let us look at the dependence of $nE_{00}^{(0)} - 2{E^{(0)}}_{\alpha}{}^{\alpha}$ on $(\chi_0)_{ij}^{(0)}$ and $(\chi_1)_{ij}^{(0)}$. Using (8.9d) and (8.9f), we obtain

$$\begin{aligned}
&\rho^{-2n-4}(n\,\delta E_{00}^{(0)} - 2\,\delta {E^{(0)}}_{\alpha}{}^{\alpha}) \\
&= -\tfrac{1}{2}(n+2)(n(\chi_2)_{00}^{(1)} - 2{(\chi_2)^{(1)}}_{\alpha}{}^{\alpha}) \\
&\quad + i(n+2)(\nabla^{\alpha}(\chi_1)_{0\alpha}^{(0)} - \nabla^{\overline{\alpha}}(\chi_1)_{0\overline{\alpha}}^{(0)}) - \tfrac{1}{2}n(\Phi^{\alpha\beta}(\chi_0)_{\alpha\beta}^{(0)} + \Phi^{\overline{\alpha}\overline{\beta}}(\chi_0)_{\overline{\alpha}\overline{\beta}}^{(0)}) \\
&\quad - (\nabla^{\alpha}\nabla^{\beta}(\chi_0)_{\alpha\beta}^{(0)} + \nabla^{\overline{\alpha}}\nabla^{\overline{\beta}}(\chi_0)_{\overline{\alpha}\overline{\beta}}^{(0)} + N^{\gamma\alpha\beta}\nabla_{\gamma}(\chi_0)_{\alpha\beta}^{(0)} + N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}\nabla_{\overline{\gamma}}(\chi_0)_{\overline{\alpha}\overline{\beta}}^{(0)} \\
&\qquad + (\nabla_{\gamma}N^{\gamma\alpha\beta})(\chi_0)_{\alpha\beta}^{(0)} + (\nabla_{\overline{\gamma}}N^{\overline{\gamma}\overline{\alpha}\overline{\beta}})(\chi_0)_{\overline{\alpha}\overline{\beta}}^{(0)}) + O(\rho).
\end{aligned}$$

So we impose another equation to $(\chi_0)_{\alpha\beta}^{(0)}$ and $(\chi_1)_{0\alpha}^{(0)}$ to kill $n(\chi_2)_{00}^{(1)} - 2{(\chi_2)^{(1)}}_{\gamma}{}^{\gamma}$. If the following is satisfied, then we do not have to introduce the logarithmic coefficient $n(\chi_2)_{00}^{(1)} - 2{(\chi_2)^{(1)}}_{\gamma}{}^{\gamma}$:

$$\begin{aligned}
&i(n+2)(\nabla^{\alpha}(\chi_1)_{0\alpha}^{(0)} - \nabla^{\overline{\alpha}}(\chi_1)_{0\overline{\alpha}}^{(0)}) - \tfrac{1}{2}n(\Phi^{\alpha\beta}(\chi_0)_{\alpha\beta}^{(0)} + \Phi^{\overline{\alpha}\overline{\beta}}(\chi_0)_{\overline{\alpha}\overline{\beta}}^{(0)}) \\
&- (\nabla^{\alpha}\nabla^{\beta}(\chi_0)_{\alpha\beta}^{(0)} + \nabla^{\overline{\alpha}}\nabla^{\overline{\beta}}(\chi_0)_{\overline{\alpha}\overline{\beta}}^{(0)} + N^{\gamma\alpha\beta}\nabla_{\gamma}(\chi_0)_{\alpha\beta}^{(0)} + N^{\overline{\gamma}\overline{\alpha}\overline{\beta}}\nabla_{\overline{\gamma}}(\chi_0)_{\overline{\alpha}\overline{\beta}}^{(0)} \\
&\qquad + (\nabla_{\gamma}N^{\gamma\alpha\beta})(\chi_0)_{\alpha\beta}^{(0)} + (\nabla_{\overline{\gamma}}N^{\overline{\gamma}\overline{\alpha}\overline{\beta}})(\chi_0)_{\overline{\alpha}\overline{\beta}}^{(0)}) \\
&= -nF_{00} + 2{F_{\alpha}}^{\alpha}.
\end{aligned}$$

Combined with (8.18), the equation above is equivalent to

$$(D')^{\alpha\beta}(\chi_0)_{\alpha\beta}^{(0)} + (D')^{\overline{\alpha}\overline{\beta}}(\chi_0)_{\overline{\alpha}\overline{\beta}}^{(0)} = v.$$

So we set $(\chi_0)_{\alpha\beta}^{(0)} = \lambda_{\alpha\beta}$ and determine $(\chi_1)_{0\alpha}^{(0)}$ by (8.18). Thus we obtain an ACH metric g without logarithmic singularity that satisfies $E = \mathcal{A}^{2n+5}$ in a unique way.

No logarithmic terms occur in the remaining process of constructing a metric for which $E = \mathcal{A}^{\infty}$. $\square$

Type	*Value*
$\delta D^{\infty}{}_{\infty\infty}$	0
$\delta D^{\infty}{}_{\infty 0}$	0
$\delta D^{\infty}{}_{\infty\alpha}$	0
$\delta D^{\infty}{}_{00}$	$-\frac{1}{8}(\rho\partial_\rho - 4)\psi_{00}$
$\delta D^{\infty}{}_{0\alpha}$	$-\frac{1}{8}(\rho\partial_\rho - 3)\psi_{0\alpha}$
$\delta D^{\infty}{}_{\alpha\overline{\beta}}$	$-\frac{1}{8}(\rho\partial_\rho - 2)\psi_{\alpha\overline{\beta}}$
$\delta D^{\infty}{}_{\alpha\beta}$	$-\frac{1}{8}(\rho\partial_\rho - 2)\psi_{\alpha\beta}$
$\delta D^{0}{}_{\infty\infty}$	0
$\delta D^{0}{}_{\infty 0}$	$\frac{1}{2}\rho\partial_\rho\psi_{00}$
$\delta D^{0}{}_{\infty\alpha}$	$\frac{1}{2}(\rho\partial_\rho - 1)\psi_{0\alpha}$
$\delta D^{0}{}_{00}$	0
$\delta D^{0}{}_{0\alpha}$	$-\frac{i}{2}\psi_{0\alpha}$
$\delta D^{0}{}_{\alpha\overline{\beta}}$	$\frac{1}{2}\rho(\nabla_\alpha\psi_{0\overline{\beta}} + \nabla_{\overline{\beta}}\psi_{0\alpha}) - \frac{1}{2}\rho^2(A_\alpha{}^{\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}} + A_{\overline{\beta}}{}^{\gamma}\psi_{\alpha\gamma})$
$\delta D^{0}{}_{\alpha\beta}$	$\frac{1}{2}\rho(\nabla_\alpha\psi_{0\beta} + \nabla_\beta\psi_{0\alpha}) - \frac{1}{2}\rho(N^{\overline{\gamma}}{}_{\alpha\beta} + N^{\overline{\gamma}}{}_{\beta\alpha})\psi_{0\overline{\gamma}} - \frac{1}{2}\rho^2\nabla_0\psi_{\alpha\beta}$
$\delta D^{\gamma}{}_{\infty\infty}$	0
$\delta D^{\gamma}{}_{\infty 0}$	$\frac{1}{2}(\rho\partial_\rho + 1)\psi_0{}^{\gamma}$
$\delta D^{\gamma}{}_{\infty\alpha}$	$\frac{1}{2}\rho\partial_\rho\psi_\alpha{}^{\gamma} - \frac{1}{2}\rho^2\cdot\rho\partial_\rho\Phi^{\gamma\sigma}\psi_{\alpha\sigma} - \rho^2\Phi_{\alpha\sigma}\psi^{\gamma\sigma}$
$\delta D^{\gamma}{}_{\infty\overline{\alpha}}$	$\frac{1}{2}\rho\partial_\rho\psi_{\overline{\alpha}}{}^{\gamma} - \frac{1}{2}\rho^2\cdot\rho\partial_\rho\Phi^{\gamma\overline{\sigma}}\psi_{\overline{\alpha}\overline{\sigma}} - \rho^2\Phi_{\sigma\overline{\alpha}}\psi^{\gamma\sigma}$
$\delta D^{\gamma}{}_{00}$	0
$\delta D^{\gamma}{}_{0\alpha}$	$\frac{i}{2}\delta_\alpha{}^{\gamma}\psi_{00} - \frac{i}{2}\psi_\alpha{}^{\gamma} + \frac{1}{2}\rho(\nabla_\alpha\psi_0{}^{\gamma} - \nabla^{\gamma}\psi_{0\alpha}) - \frac{1}{2}\rho^2(A_{\alpha\sigma}\psi^{\gamma\sigma} - A^{\gamma\sigma}\psi_{\alpha\sigma}) + \frac{i}{2}\rho^2(\Phi^{\gamma\sigma}\psi_{\alpha\sigma} + \Phi_{\alpha\sigma}\psi^{\gamma\sigma})$
$\delta D^{\gamma}{}_{0\overline{\alpha}}$	$\frac{i}{2}\psi_{\overline{\alpha}}{}^{\gamma} + \frac{1}{2}\rho(\nabla_{\overline{\alpha}}\psi_0{}^{\gamma} - \nabla^{\gamma}\psi_{0\overline{\alpha}} - N_{\overline{\alpha}}{}^{\gamma\sigma}\psi_{0\sigma}) + \frac{1}{2}\rho^2\nabla_0\psi_{\overline{\alpha}}{}^{\gamma} - \frac{i}{2}\rho^2(\Phi^{\gamma\overline{\sigma}}\psi_{\overline{\alpha}\overline{\sigma}} + \Phi_{\overline{\alpha}\sigma}\psi^{\gamma\sigma})$
$\delta D^{\gamma}{}_{\alpha\beta}$	$\frac{i}{2}\delta_\alpha{}^{\gamma}\psi_{0\beta} + \frac{i}{2}\delta_\beta{}^{\gamma}\psi_{0\alpha} - \frac{1}{2}\rho\nabla^{\gamma}\psi_{\alpha\beta} - \frac{1}{2}\rho(N^{\overline{\sigma}}{}_{\alpha\beta} + N^{\overline{\sigma}}{}_{\beta\alpha})\psi_{\overline{\sigma}}{}^{\gamma}$
$\delta D^{\gamma}{}_{\alpha\overline{\beta}}$	$\frac{i}{2}\delta_\alpha{}^{\gamma}\psi_{0\overline{\beta}} + \frac{1}{2}\rho(\nabla_\alpha\psi_{\overline{\beta}}{}^{\gamma} - N_{\overline{\beta}}{}^{\gamma\sigma}\psi_{\alpha\sigma})$
$\delta D^{\gamma}{}_{\overline{\alpha}\overline{\beta}}$	$\frac{1}{2}\rho(\nabla_{\overline{\alpha}}\psi_{\overline{\beta}}{}^{\gamma} + \nabla_{\overline{\beta}}\psi_{\overline{\alpha}}{}^{\gamma} - \nabla^{\gamma}\psi_{\overline{\alpha}\overline{\beta}})$

TABLE 8.2. $\delta D^K{}_{IJ}$ modulo $\mathcal{A}^{m+3}$ for perturbation (8.8) (Lemma 8.6)

Type	Value
$\overline{\nabla}_\infty g_{00}$	$(\rho\partial_\rho - 4)\psi_{00}$
$\overline{\nabla}_\infty g_{0\alpha}$	$(\rho\partial_\rho - 3)\psi_{0\alpha}$
$\overline{\nabla}_\infty g_{\alpha\overline{\beta}}$	$(\rho\partial_\rho - 2)\psi_{\alpha\overline{\beta}}$
$\overline{\nabla}_\infty g_{\alpha\beta}$	$(\rho\partial_\rho - 2)\psi_{\alpha\beta}$

Type	Value
$\delta(\overline{\nabla}_0 g_{00})$	0
$\delta(\overline{\nabla}_0 g_{0\alpha})$	0
$\delta(\overline{\nabla}_0 g_{\alpha\overline{\beta}})$	0
$\delta(\overline{\nabla}_0 g_{\alpha\beta})$	$\rho^2\nabla_0\psi_{\alpha\beta}$

Type	Value
$\delta(\overline{\nabla}_{\overline{\gamma}} D_{00})$	0
$\delta(\overline{\nabla}_{\overline{\gamma}} D_{0\alpha})$	$\rho\nabla_{\overline{\gamma}}\psi_{0\alpha}$
$\delta(\overline{\nabla}_{\overline{\gamma}} D_{0\overline{\alpha}})$	$\rho\nabla_{\overline{\gamma}}\psi_{0\alpha}$

Type	Value
$\delta(\overline{\nabla}_{\overline{\gamma}} D_{\alpha\beta})$	$\rho\nabla_{\overline{\gamma}}\psi_{\alpha\beta}$
$\delta(\overline{\nabla}_{\overline{\gamma}} D_{\alpha\overline{\beta}})$	0
$\delta(\overline{\nabla}_{\overline{\gamma}} D_{\overline{\alpha}\overline{\beta}})$	$\rho\nabla_{\overline{\gamma}}\psi_{\overline{\alpha}\overline{\beta}}$

Table 8.3. $\delta(\overline{\nabla}_K g_{IJ})$ modulo $\mathcal{A}^{m+3}$ (Lemma 8.6)

Type	Value
$\overline{T}_{000}$	0
$\overline{T}_{00\alpha}$	$\rho^2 A_\alpha{}^{\overline{\gamma}}\psi_{0\overline{\gamma}}$
$\overline{T}_{0\alpha\overline{\beta}}$	$ih_{\alpha\overline{\beta}}\psi_{00}$
$\overline{T}_{0\alpha\beta}$	$-\rho N_{\alpha\beta}{}^{\overline{\gamma}}\psi_{0\overline{\gamma}}$

Type	Value
$\overline{T}_{\overline{\gamma}00}$	0
$\overline{T}_{\overline{\gamma}0\alpha}$	$\rho^2 A_\alpha{}^{\overline{\sigma}}\psi_{\overline{\gamma}\overline{\sigma}}$
$\overline{T}_{\overline{\gamma}0\overline{\alpha}}$	0

Type	Value
$\overline{T}_{\overline{\gamma}\alpha\beta}$	$-\rho N_{\alpha\beta}{}^{\overline{\sigma}}\psi_{\overline{\gamma}\overline{\sigma}}$
$\overline{T}_{\overline{\gamma}\alpha\overline{\beta}}$	$ih_{\alpha\overline{\beta}}\psi_{0\overline{\gamma}}$
$\overline{T}_{\overline{\gamma}\overline{\alpha}\overline{\beta}}$	0

Table 8.4. $\overline{T}_{KIJ}$ modulo $\mathcal{A}^3$ (Lemma 8.6)

Type	Value
$\delta D_{\infty\infty\infty}$	0
$\delta D_{\infty\infty 0}$	0
$\delta D_{\infty\infty\alpha}$	0
$\delta D_{\infty 00}$	$-\frac{1}{2}(\rho\partial_\rho - 4)\psi_{00}$
$\delta D_{\infty 0\alpha}$	$-\frac{1}{2}(\rho\partial_\rho - 3)\psi_{0\alpha}$
$\delta D_{\infty\alpha\overline{\beta}}$	$-\frac{1}{2}(\rho\partial_\rho - 2)\psi_{\alpha\overline{\beta}}$
$\delta D_{\infty\alpha\beta}$	$-\frac{1}{2}(\rho\partial_\rho - 2)\psi_{\alpha\beta}$
$\delta D_{0\infty\infty}$	0
$\delta D_{0\infty 0}$	$\frac{1}{2}(\rho\partial_\rho - 4)\psi_{00}$
$\delta D_{0\infty\alpha}$	$\frac{1}{2}(\rho\partial_\rho - 3)\psi_{0\alpha}$
δD_{000}	0
$\delta D_{00\alpha}$	0
$\delta D_{0\alpha\overline{\beta}}$	$\frac{i}{2}h_{\alpha\overline{\beta}}\psi_{00} + \frac{1}{2}\rho(\nabla_\alpha\psi_{0\overline{\beta}} + \nabla_{\overline{\beta}}\psi_{0\alpha}) - \frac{1}{2}\rho^2(A_\alpha{}^{\overline{\gamma}}\psi_{\overline{\beta}\overline{\gamma}} + A_{\overline{\beta}}{}^{\gamma}\psi_{\alpha\gamma})$
$\delta D_{0\alpha\beta}$	$\frac{1}{2}\rho(\nabla_\alpha\psi_{0\beta} + \nabla_\beta\psi_{0\alpha} - N_{\alpha\beta}{}^{\overline{\gamma}}\psi_{0\overline{\gamma}}) - \frac{1}{2}\rho^2\nabla_0\psi_{\alpha\beta}$
$\delta D_{\overline{\gamma}\infty\infty}$	0
$\delta D_{\overline{\gamma}\infty 0}$	$\frac{1}{2}(\rho\partial_\rho - 3)\psi_{0\overline{\gamma}}$
$\delta D_{\overline{\gamma}\infty\alpha}$	$\frac{1}{2}(\rho\partial_\rho - 2)\psi_{\alpha\overline{\gamma}}$
$\delta D_{\overline{\gamma}\infty\overline{\alpha}}$	$\frac{1}{2}(\rho\partial_\rho - 2)\psi_{\overline{\alpha}\overline{\gamma}}$
$\delta D_{\overline{\gamma}00}$	0
$\delta D_{\overline{\gamma}0\alpha}$	$\frac{i}{2}h_{\alpha\overline{\gamma}}\psi_{00} + \frac{1}{2}\rho(\nabla_\alpha\psi_{0\overline{\gamma}} - \nabla_{\overline{\gamma}}\psi_{0\alpha}) - \frac{1}{2}\rho^2(A_\alpha{}^{\overline{\sigma}}\psi_{\overline{\gamma}\overline{\sigma}} + A_{\overline{\gamma}\overline{\sigma}}\psi_\alpha{}^{\overline{\sigma}})$
$\delta D_{\overline{\gamma}0\overline{\alpha}}$	$\frac{1}{2}\rho(\nabla_{\overline{\alpha}}\psi_{0\overline{\gamma}} - \nabla_{\overline{\gamma}}\psi_{0\overline{\alpha}} - N_{\overline{\alpha}\overline{\gamma}}{}^{\sigma}\psi_{0\sigma}) + \frac{1}{2}\rho^2\nabla_0\psi_{\overline{\alpha}\overline{\gamma}}$
$\delta D_{\overline{\gamma}\alpha\beta}$	$\frac{i}{2}h_{\alpha\overline{\gamma}}\psi_{0\beta} + \frac{i}{2}h_{\beta\overline{\gamma}}\psi_{0\alpha} - \frac{1}{2}\rho(\nabla_{\overline{\gamma}}\psi_{\alpha\beta} + N_{\alpha\beta}{}^{\overline{\sigma}}\psi_{\overline{\gamma}\overline{\sigma}})$
$\delta D_{\overline{\gamma}\alpha\overline{\beta}}$	$\frac{i}{2}\delta_{\alpha\overline{\gamma}}\psi_{0\overline{\beta}} + \frac{i}{2}\delta_{\alpha\overline{\gamma}}\psi_{0\overline{\beta}} + \frac{1}{2}\rho(\nabla_\alpha\psi_{\overline{\beta}\overline{\gamma}} - N_{\overline{\beta}\overline{\gamma}}{}^{\sigma}\psi_{\alpha\sigma})$
$\delta D_{\overline{\gamma}\overline{\alpha}\overline{\beta}}$	$\frac{1}{2}\rho(\nabla_{\overline{\alpha}}\psi_{\overline{\beta}\overline{\gamma}} + \nabla_{\overline{\beta}}\psi_{\overline{\alpha}\overline{\gamma}} - \nabla_{\overline{\gamma}}\psi_{\overline{\alpha}\overline{\beta}})$

TABLE 8.5. δD_{KIJ} modulo $\mathcal{A}^{m+3}$ for perturbation (8.8) (Lemma 8.6)

Type	Value	Type	Value	Type	Value	Type	Value
$D_{\infty\infty\infty}$	-4	$D_{0\infty\infty}$	0	$D_{\overline{\gamma}\infty\infty}$	0	$D_{\overline{\gamma}\alpha\beta}$	0
$D_{\infty\infty 0}$	0	$D_{0\infty 0}$	-2	$D_{\overline{\gamma}\infty 0}$	0	$D_{\overline{\gamma}\alpha\overline{\beta}}$	0
$D_{\infty\infty\alpha}$	0	$D_{0\infty\alpha}$	0	$D_{\overline{\gamma}\infty\alpha}$	$-h_{\alpha\overline{\gamma}}$	$D_{\overline{\gamma}\overline{\alpha}\overline{\beta}}$	$\rho N_{\overline{\gamma}\overline{\beta}\overline{\alpha}}$
$D_{\infty 00}$	2	D_{000}	0	$D_{\overline{\gamma}\infty\overline{\alpha}}$	0		
$D_{\infty 0\alpha}$	0	$D_{00\alpha}$	0	$D_{\overline{\gamma}00}$	0		
$D_{\infty\alpha\overline{\beta}}$	$h_{\alpha\overline{\beta}}$	$D_{0\alpha\overline{\beta}}$	$-\frac{i}{2}h_{\alpha\overline{\beta}}$	$D_{\overline{\gamma}0\alpha}$	$\frac{i}{2}h_{\alpha\overline{\gamma}}$		
$D_{\infty\alpha\beta}$	0	$D_{0\alpha\beta}$	$-\rho^2 A_{\alpha\beta}$	$D_{\overline{\gamma}0\overline{\alpha}}$	0		

TABLE 8.6. D_{KIJ} modulo $\mathcal{A}^3$ (Lemma 8.6)

CHAPTER 4

CR Q-curvature

9. Dirichlet problems and volume expansion

9.1. Laplacian. Let g be a $(2n+2)$-dimensional C^∞-smooth ACH metric on a Θ-manifold $(\overline{X}, [\Theta])$. Then, as we saw in Proposition 4.11, for any choice of a contact form θ, we can identify g with a normal-form ACH metric on $M \times [0, \infty)$ near the boundary. In this sense, by abusing the notation we write

$$g = \frac{4d\rho^2 + h_\rho}{\rho^2}.$$

Here, h_ρ is a 1-parameter family of Riemannian metrics on M defined for $\rho \in (0, \infty)$. Let T be the Reeb vector field and h the Levi form for θ, both extended constantly in the direction of $[0, \infty)$. We define the *parabolic dilation* $M_\rho \colon TM = H \oplus \mathbb{R}T \to TM$ by

$$M_\rho T = \rho^2 T, \qquad M_\rho Y = \rho Y \quad (Y \in H),$$

and set

$$k_\rho = \rho^{-2} M_\rho^* h_\rho. \tag{9.1}$$

Then, the ACH condition described in Proposition 4.12 can also be stated as follows: k_ρ should extend to $\rho = 0$ and the restriction to the boundary, which we write k, must be equal to $\theta^2 + h$. Recall that any boundary defining function ρ that can be used for such a normalization is called a model boundary defining function. Note that the following holds:

$$(\rho^4 g)|_{TM} = \theta^2. \tag{9.2}$$

If g is an even metric, then it means that $\rho^2 h_\rho$ admits an asymptotic expansion in the even powers of ρ with coefficients in the space of symmetric 2-tensors on M. It is convenient to define $C^\infty_{\mathrm{even}}(M \times [0, \infty))$ to be the space of smooth functions on $M \times [0, \infty)$ with even expansions.

Since $\det k_\rho = \rho^2 \det h_\rho$, the volume forms satisfy $dV_{k_\rho} = \rho \, dV_{h_\rho}$. If the metric is even, then $\det h_\rho$ also has even expansion, and so does $\det k_\rho$. Therefore dV_{k_ρ} has expansion of the form

$$dV_{k_\rho} \sim dV_k \cdot (1 + v^{(2)}\rho^2 + v^{(4)}\rho^4 + \cdots), \tag{9.3}$$

where $v^{(2)}$, $v^{(4)}$, $\ldots$ are some smooth functions on M.

As stated in [**GS**, Equation (5.1)], the Laplacian of a normal-form ACH metric is given by the formula

$$\Delta_g = -\frac{1}{4}(\rho\partial_\rho)^2 + \frac{n+1}{2}\rho\partial_\rho + \rho^2\Delta_{h_\rho} - \frac{1}{8}\rho\partial_\rho(\log|\det k_\rho|)\rho\partial_\rho. \tag{9.4}$$

We want to express the Laplacian $\Delta_{h_\rho} = (\nabla_{h_\rho})^*\nabla_{h_\rho}$ of h_ρ in terms of $(h_\rho)_{ij}$, $(h_\rho^{-1})^{ij}$, the Nijenhuis tensor, the Tanaka–Webster connection ∇ the associated pseudohermitian torsion and curvature tensors. Let $\{ Z_i \} = \{ T, Z_\alpha, Z_{\overline{\alpha}} \}$ be a local frame, where $Z_0 = T$. We define the tensor $K = K_\rho$ by

$$(\nabla_{h_\rho})_i Z_j = \nabla_i Z_j + K^k{}_{ij} Z_k. \tag{9.5}$$

Then we obtain, for a function f,

$$\Delta_{h_\rho} f = -(h_\rho^{-1})^{ij}\nabla_i\nabla_j f - (h_\rho^{-1})^{ij}(h_\rho^{-1})^{kl}K_{kij}\nabla_l f, \tag{9.6}$$

where the upper index of K is lowered using h_ρ.

Take any $p \in M$. Then, by Lemma 3.4, one can take $\{ Z_\alpha \}$ so that the Tanaka–Webster connection forms are zero at p. For such a frame, $K^k{}_{ij}$ is nothing but the Christoffel symbol $\Gamma^k{}_{ij}$ of ∇_{h_ρ}. Therefore

$$K_{kij} = \frac{1}{2}(Z_i(h_\rho)_{jk} + Z_j(h_\rho)_{ik} - Z_k(h_\rho)_{ij}),$$

and because of our choice of the frame,

$$K_{kij} = \frac{1}{2}(\nabla_i(h_\rho)_{jk} + \nabla_j(h_\rho)_{ik} - \nabla_k(h_\rho)_{ij}). \tag{9.7}$$

Since all the terms in (9.7) are tensorial, this is a frame-independent formula.

LEMMA 9.1. *Let g be an even normal-form ACH metric. Then the tensor K smoothly extends to $\rho = 0$. Moreover, the components of K with respect to the local frame $\{ Z_i \} = \{ T, Z_\alpha, Z_{\overline{\alpha}} \}$ are even, and $K_{kij} = O(\rho^2)$ if $0 \notin \{ i, j, k \}$.*

PROOF. If h_ρ is even, then (9.7) shows that K_{kij} is even. Since the Tanaka–Webster connection ∇ annihilates θ, the ρ^{-2}-term of $(h_\rho)_{00}$ does not contribute to K. Similarly, there is no contribution from the ρ^0-term of $(h_\rho)_{\alpha\overline{\beta}}$ because ∇ annihilates the Levi form. Therefore we conclude as stated. □

PROPOSITION 9.2. *The Laplacian of an even normal-form ACH metric g is*

$$\Delta_g = -\frac{1}{4}(\rho\partial_\rho)^2 + \frac{n+1}{2}\rho\partial_\rho - \frac{1}{8}\rho\partial_\rho(\log|\det k_\rho|)\rho\partial_\rho + \rho^2\Delta_b + \rho^4\Psi, \tag{9.8}$$

where

$$\Delta_b = -h^{\alpha\overline{\beta}}(\nabla_\alpha\nabla_{\overline{\beta}} + \nabla_{\overline{\beta}}\nabla_\alpha)$$

and Ψ is a differential operator of order at most two given by a polynomial of ∇_0, ∇_α, $\nabla_{\overline{\alpha}}$ with coefficients in $C^\infty_{\mathrm{even}}(M \times [0,\infty))$ and with no zeroth-order term. In particular, $C^\infty_{\mathrm{even}}(M \times [0,\infty))$ is closed under the action of Δ_g.

PROOF. Since $(h_\rho^{-1})^{ij} = O(1)$, and if at least one of i, j is 0 then $(h_\rho^{-1})^{ij} = O(\rho^2)$, by Lemma 9.1 we conclude $(h_\rho^{-1})^{ij}(h_\rho^{-1})^{kl}K_{kij} = O(\rho^2)$ whatever l is. Hence it follows from (9.6) that Δ_{h_ρ} is of the form $\Delta_b + \rho^2\Psi$. By (9.4), we obtain (9.8). $\square$

9.2. Dirichlet problems. In this subsection, we prove two theorems that are given in Subsection 2.4 at once.

PROOF OF THEOREMS 2.13 AND 2.14. We may assume that our metric g is an even normal-form ACH metric for some choice of a contact form θ on $(M, T^{1,0}M)$, and ρ is the associated model boundary defining function. Let

$$\Delta_{g,s} := \Delta_g - s(n+1-s),$$

where $s \in \mathbb{R}$ is arbitrary. We set

$$F \sim \sum_{j=0}^{\infty} \rho^j f_j, \qquad f_j \in C^\infty(M),$$

and try to solve $\Delta_{g,s}(\rho^{2(n+1-s)}F) = O(\rho^\infty)$ by determining f_js. Proposition 9.2 implies

$$\Delta_{g,s}(\rho^{2(n+1-s)+j}f_j) \sim \rho^{2(n+1-s)+j} \cdot \left(-\frac{1}{4}j(j-4s+2n+2)f_j + \rho^2 D_{j,s}f_j\right),$$

where $D_{j,s}$ is an even formal power series in ρ with coefficients in second-order linear differential operators on M. This is used to determine the expansion of F. First, if we set $F_0 = f_0 = f$, then

$$\Delta_{g,s}(\rho^{2(n+1-s)}F_0) = \rho^{2(n+1-s)}(0 + \rho^2 D_{0,s}f_0) = O(\rho^{2(n+1-s)+2}).$$

We inductively define f_j, as far as $j - 4s + 2n + 2 \neq 0$, by

$$\frac{1}{4}j(j-4s+2n+2)f_j = (\text{the } \rho^{2(n+1-s)+j}\text{-coefficient of } \Delta_{g,s}(\rho^{2(n+1-s)}F_{j-1})),$$

and set $F_j := F_{j-1} + \rho^j f_j$. Indeed, only even powers of ρ can appear in F_j, and the coefficients is written as follows using linear differential operators $p_{l,s}$ on M, where $[j/2]$ is a largest integer not greater than $j/2$:

$$F_j = f + \rho^2 p_{1,s}f + \rho^4 p_{2,s}f + \cdots + \rho^{2[j/2]}p_{[j/2],s}f. \tag{9.9}$$

If we furthermore set $p_{0,s} := 1$, then $p_{l,s}$ is recursively given by

$$p_{l,s} := -\frac{1}{l(2s-n-1-l)}\sum_{\nu=0}^{l-1}(\text{the } \rho^{2l-2\nu-2}\text{-coefficient of } D_{2\nu,s})p_{\nu,s}. \tag{9.10}$$

In view of this fact, we put

$$c_{0,s} := 1, \qquad c_{l,s} := (-1)^l \prod_{i=1}^{l} \frac{1}{i(2s-n-1-i)}, \tag{9.11}$$

and inductively define $P_{l,s}$ by $P_{0,s} := 1$ and

$$P_{l,s} := \sum_{\nu=0}^{l-1} d_{l,s,\nu}(\text{the } \rho^{2l-2\nu-2}\text{-coefficient of } D_{2\nu,s})P_{\nu,s}, \tag{9.12a}$$

$$d_{l,s,\nu} := \frac{c_{\nu,s}}{c_{l-1,s}} = (-1)^{l-\nu-1} \prod_{i=\nu+1}^{l-1} i(2s-n-1-i). \tag{9.12b}$$

Then $P_{l,s}$ is polynomial in s (note that this is not the case for $p_{l,s}$). Since the ρ^0-coefficient for any $D_{j,s}$ is Δ_b plus some zeroth-order term, the principal part of $P_{l,s}$ is actually Δ_b^l. Furthermore, comparing (9.10) and (9.12a), we obtain

$$p_{l,s} = c_{l,s}P_{l,s}.$$

So we can rewrite (9.9) as

$$F_j = f + \rho^2 c_{1,s}P_{1,s}f + \rho^4 c_{2,s}P_{2,s}f + \cdots + \rho^{2[j/2]}c_{[j/2],s}P_{[j/2],s}f. \tag{9.13}$$

When $2s-n-1$ is not a positive integer, this construction does not stop and we get a formally unique solution to $\Delta_{g,s}(\rho^{2(n+1-s)}F) = O(\rho^\infty)$. Thus we have shown Theorem 2.14.

If $2s-n-1 = k \in \mathbb{Z}_+$, i.e., $s = (n+1+k)/2$, then the construction goes through until j reaches $4s-2n-2 = 2k$. As a result, F_{2k-1} is determined so that $\Delta_{g,s}(\rho^{n+1-k}F_{2k-1}) = O(\rho^{(n+1-k)+2k}) = O(\rho^{n+1+k})$ is satisfied. Even if we move on to the next step by setting $F_{2k} = F_{2k-1} + \rho^{2k}f_{2k}$, in general it cannot solve $\Delta_{g,s}(\rho^{n+1-k}F_{2k}) = O(\rho^{n+1+k+1})$. Here we need the first logarithmic term. Since for $g_j \in C^\infty(M)$ we have

$$\begin{aligned}
&\Delta_{g,(n+1+k)/2}(\rho^{n+1+k+j}\log\rho\cdot g_j)\\
&= \rho^{n+1+k+j}\log\rho\cdot\left(-\frac{1}{4}j(j+2k)g_j + \rho^2 D_{2k+j,(n+1+k)/2}g_j\right)\\
&\quad + \rho^{n+1+k+j}\cdot\left(-\frac{1}{2}(j+k)g_j - \frac{1}{8}\rho\partial_\rho(\log|\det k_\rho|)g_j\right),
\end{aligned}$$

we set

$$g_0 := \frac{2}{k}c_{k-1,(n+1+k)/2}P_{k,(n+1+k)/2}f = c_k P_{k,(n+1+k)/2}f. \tag{9.14}$$

Here reflected is the fact that the ρ^{n+1+k}-coefficient of $\Delta_{g,(n+1+k)/2}(\rho^{n+1-k}F_{2k-1})$ is equal to $c_{k-1,(n+1+k)/2}P_{k,(n+1+k)/2}f$. We arbitrarily choose $f_{2k} \in C^\infty(M)$. By setting $F_{2k} := F_{2k-1} + \rho^{2k}f_{2k}$ and $G_0 := g_0$, we obtain $\Delta_{g,s}(\rho^{n+1-k}F_{2k} + \rho^{n+1+k}g_0) = O(\rho^{n+1+k})$. More precisely, we can write

$$\Delta_{g,s}(\rho^{n+1-k}F_{2k} + \rho^{n+1+k}G_0) = \rho^{n+1+k+1}R_0 + \rho^{n+1+k+1}\log\rho\cdot S_0,$$

where $R_0, S_0 \in C^\infty_{\text{even}}(M\times[0,\infty))$. What we want to do in the sequel is defining F_{2k+j} and G_j, which are polynomials in ρ of degrees $2k+j$ and j, respectively, by adding higher-order terms to F_{2k} and G_0 so that

$$\Delta_{g,s}(\rho^{n+1-k}F_{2k+j} + \rho^{n+1+k}G_j) = \rho^{n+1+k}R_j + \rho^{n+1+k}\log\rho\cdot S_j$$

with some $R_j,\ S_j \in C^\infty_{\mathrm{even}}(M\times[0,\infty))$ such that $R_j,\ S_j = O(\rho^j)$. This is uniquely achieved by the following choice: we set $F_{2k+j} := F_{2k+j-1} + \rho^{2k+j}f_{2k+j}$ and $G_j := G_{j-1} + \rho^j g_j$, where

$$\begin{aligned} g_j &:= \frac{4}{j(j+2k)}(\text{the } \rho^{j-1}\text{-coefficient of } S_{j-1}),\\ f_{2k+j} &:= \frac{4}{j(j+2k)}\left((\text{the } \rho^{j-1}\text{-coefficient of } R_{j-1}) - \frac{1}{2}(j+k)g_j\right). \end{aligned}$$

Thus we obtain F_∞ and G_∞ such that

$$\Delta_{g,s}(\rho^{n+1-k}F_\infty + \rho^{n+1+k}G_\infty) = O(\rho^\infty).$$

The ambiguity lives only in where we determine f_{2k}, which is the ρ^{2k}-coefficient of F_∞. Therefore the statement of Theorem 2.13 is true if we set $P_k = P_{k,(n+1+k)/2}$, as is clear by (9.14). $\square$

In the proof, it is also observed that the operator $P_{l,s}$ is a polynomial with respect to s. So we can construct the quantity Q as in Subsection 2.4.

9.3. Volume expansion. Let g be an even ACH metric on a compact Θ-manifold $(\overline{X},[\Theta])$. From the expression $g = (4d\rho^2 + h_\rho)/\rho^2$, we obtain

$$dV_g = 2\rho^{-2n-2}dV_{h_\rho}d\rho = 2\rho^{-2n-3}dV_{k_\rho}d\rho.$$

Therefore, (9.3) implies, for some arbitrarily fixed ε_0, that the volume of the subset $\{\,\varepsilon\le\rho\le\varepsilon_0\,\}\subset X$ has the following asymptotic behavior when $\varepsilon\to 0$:

$$\mathrm{Vol}_g(\{\,\varepsilon\le\rho\le\varepsilon_0\,\}) = c_0\varepsilon^{-2n-2} + c_2\varepsilon^{-2n} + \cdots + c_{2n}\varepsilon^{-2} + L\log(1/\varepsilon) + O(1). \tag{9.15}$$

The purpose of this subsection is proving the proposition below, which shows that the logarithmic coefficient L is related to Q.

Proposition 9.3. *Let g be an even normal-form ACH metric for $(M,T^{1,0}M,\theta)$, where M is compact. Then the coefficient L in (9.15) is given by*

$$L = \frac{(-1)^{n+1}}{n!^2(n+1)!}\overline{Q}, \tag{9.16}$$

where $\overline{Q}$ is the integral of Q with respect to the volume form $\theta\wedge(d\theta)^n$:

$$\overline{Q} := \int_M Q\,\theta\wedge(d\theta)^n = n!\int_M Q\,dV_k.$$

We prove this fact by rewriting the definition of Q as follows. This is the CR version of the discussion given in [**FG2**].

LEMMA 9.4. *Let g be an even ACH metric on $(\overline{X}, [\Theta])$ and $T^{1,0}M$ the induced nondegenerate partially integrable CR structure. If θ is any contact form on $(M, T^{1,0}M)$ and $\rho \in \mathcal{F}_\theta$, then there exists $U \in C^\infty(\overline{X})$ of the form*

$$U = 2\log\rho + A + B\rho^{2n+2}\log\rho, \qquad A, B \in C^\infty_{\mathrm{even}}(\overline{X}), \qquad A|_M = 0 \tag{9.17}$$

such that

$$\Delta_g U = n+1 \mod O(\rho^{2n+4}\log\rho). \tag{9.18}$$

Moreover, A and B are unique modulo $O(\rho^{2n+2})$ and $O(\rho^2)$, respectively, and

$$B|_M = -c_{n+1}Q, \tag{9.19}$$

where c_{n+1} is the constant defined by (2.12).

PROOF. We may assume that g is an even normal-form ACH metric. By Proposition 9.2, $\Delta_g(2\log\rho)$ is smooth up to the boundary and $\Delta_g(2\log\rho)|_M = n+1$. Moreover, if $f \in C^\infty_{\mathrm{even}}(\overline{X})$, then again by Proposition 9.2,

$$\Delta_g(\rho^{2l}f) = -l(l-n-1)\rho^{2l}f + O(\rho^{2l+2}) \tag{9.20}$$

and

$$\begin{aligned}\Delta_g(\rho^{2l}\log\rho\cdot f) = &-\frac{2l-n-1}{2}\rho^{2l}f + O(\rho^{2l+2})\\ &-\log\rho\cdot(l(l-n-1)\rho^{2l}f + O(\rho^{2l+2})),\end{aligned} \tag{9.21}$$

where the terms denoted by $O(\rho^{2l+2})$ are all smooth and even. By using (9.20) inductively, we can show that there is a unique finite expansion

$$U_1 = 2\log\rho + \sum_{l=1}^{n}\rho^{2l}f_{(l)}, \qquad f_{(l)} \in C^\infty(M)$$

such that $\Delta_g U_1$ is equal to $n+1$ modulo $O(\rho^{2n+2})$. The next thing to do is to introduce a ($\rho^{2n+2}\log\rho$)-term so that Δ_g applied to it kills the ρ^{2n+2}-coefficient of $\Delta_g U_1 - n - 1$. This is possible in view of (9.21) because $2l-n-1$ is nonzero for $l = n+1$. Thus we obtain

$$U_2 = 2\log\rho + \sum_{l=1}^{n}\rho^{2l}f_{(l)} + \rho^{2n+2}\log\rho\cdot g_{(0)}, \qquad f_{(l)},\, g_{(0)} \in C^\infty(M)$$

with $\Delta_g U_2 = n+1 + O(\rho^{2n+4}\log\rho)$.

To prove (9.19), let u be a solution of (2.14) to the Dirichlet datum $f \equiv 1$. Recall the asymptotic expansion of u is given by (2.15). The differential operator $P_{l,s}$ is polynomial in s, and its zeroth-order term can be factored by $n+1-s$. We decompose as $P_{l,s}1 = \frac{1}{2}(n+1-s)Q_{l,s}$. So the expansion of u has the following expression:

$$u \sim \rho^{2(n+1-s)}(1 + \hat{c}_{1,s}\rho^2 Q_{1,s} + \hat{c}_{2,s}\rho^4 Q_{2,s} + \cdots), \tag{9.22}$$

where we have written $\hat{c}_{l,s} = \frac{1}{2}(n+1-s)c_{l,s}$. By (9.11), $\hat{c}_{l,s}$ has no pole at $s = n+1$, and hence $\hat{c}_{1,n+1}, \cdots, \hat{c}_{n,n+1}$ are zero. Note that

$$\hat{c}_{n+1,n+1} = \frac{1}{4}c_{n+1}. \tag{9.23}$$

Now we differentiate the equation $\Delta_{g,s}u = 0$ by s, and put $s = n+1$ into the result. Then we get

$$\Delta_g \left(\left.\frac{\partial u}{\partial s}\right|_{s=n+1} \right) + (n+1)u|_{s=n+1} = 0. \tag{9.24}$$

Here, (9.22) implies

$$\begin{aligned} \left.\frac{\partial u}{\partial s}\right|_{s=n+1} \sim -2\log\rho \cdot \left(1 + \sum_{j=0}^{\infty} \hat{c}_{n+1+j,n+1}\rho^{2(n+1+j)}Q_{n+1+j,n+1} \right) \\ + \text{(non-logarithmic terms)}, \end{aligned} \tag{9.25}$$

and hence if we choose U' so that $-U'$ expands as the right-hand side of (9.25), then by (9.24),

$$\begin{aligned} \Delta_g U' &= (n+1)u|_{s=n+1} \\ &= n+1+(n+1)(\hat{c}_{n+1,s}|_{s=n+1})\rho^{2n+2}(Q_{n+1,s}|_{s=n+1}) + O(\rho^{2n+4}) \\ &= n+1 - \frac{n+1}{4}c_{n+1}\rho^{2n+2}Q + O(\rho^{2n+4}). \end{aligned}$$

For this we have computed as follows:

$$\begin{aligned} \hat{c}_{n+1,s}|_{s=n+1} &= \left.\left(\frac{\frac{1}{2}(n+1-s)\cdot(-1)^{n+1}}{(n+1)!(2s-n-2)(2s-n-3)\cdots(2s-2n-2)} \right)\right|_{s=n+1} \\ &= -\frac{(-1)^{n+1}}{4n!(n+1)!} = -\frac{1}{4}c_{n+1}. \end{aligned}$$

Therefore, if we put $U := U' - \frac{1}{2}c_{n+1}Q\rho^{2n+2}\log\rho$, then this solves $\Delta_g U = n+1+O(\rho^{2n+4}\log\rho)$. Thus we have constructed the solution to (9.18) in another way. Since $-\frac{1}{2}c_{n+1}Q$ is the $(\rho^{2n+2}\log\rho)$-coefficient of U' by (9.23) and (9.25), we obtain $B|_M = -c_{n+1}Q$. $\square$

Proof of Proposition 9.3. We use Green's formula. The outward unit normal vector field along $\{\, \rho = \varepsilon \,\}$ is $-\frac{1}{2}\varepsilon\partial_\rho$, and so for any $U \in C^2(\overline{X})$

$$\begin{aligned} \int_{\varepsilon\le\rho\le\varepsilon_0} \Delta_g U\, dV_g &= \int_{\rho=\varepsilon} \frac{1}{2}\varepsilon\partial_\rho U \cdot \varepsilon^{-2n-1} dV_{h_\varepsilon} + O(1) \\ &= \frac{1}{2}\varepsilon^{-2n} \int_{\rho=\varepsilon} \partial_\rho U\, dV_{h_\varepsilon} + O(1) \\ &= \frac{1}{2}\varepsilon^{-2n-1} \int_{\rho=\varepsilon} \partial_\rho U\, dV_{k_\varepsilon} + O(1). \end{aligned}$$

Now let U be a solution for (9.18). Then since $\Delta_g U = n+1+O(\rho^{2n+4}\log\rho)$, the equality above reads

$$(n+1)\operatorname{Vol}_g(\{\,\varepsilon \le \rho \le \varepsilon_0\,\}) = \frac{1}{2}\varepsilon^{-2n-1}\int_{\rho=\varepsilon} \partial_\rho U\, dV_{k_\varepsilon} + O(1).$$

Comparing the coefficients of $\log(1/\varepsilon)$ we can see

$$(n+1)L = -(n+1)\int_M (B|_M) dV_k = \frac{(-1)^{n+1}}{n!^3}\int_M Q\,\theta\wedge(d\theta)^n.$$

Thus we obtain (9.16). □

10. CR Q-curvature of partially integrable CR manifolds

10.1. Invariance. Let $(M, T^{1,0}M)$ be a nondegenerate partially integrable CR manifold and θ a contact form. We will apply the results of the previous section to ACH metrics with infinity $(M, T^{1,0}M)$ that are approximately Einstein to get CR-invariant objects. To carry out this idea, the dependence of the Laplacian on the ambiguity of g may be problematic. So we first discuss this point. Suppose g is an even ACH metric described in Theorem 2.3. If we moreover assume that this is normalized, then the approximate Einstein condition implies that the components of k_ρ is determined modulo the orders that are shown below:

$$\begin{aligned} &(k_\rho)_{00} \mod O(\rho^{2n+2}), \qquad (k_\rho)_{0\alpha} \mod O(\rho^{2n+3}),\\ &(k_\rho)_{\alpha\overline{\beta}} \mod O(\rho^{2n+2}), \qquad (k_\rho)_{\alpha\beta} \mod O(\rho^{2n+2}); \end{aligned} \tag{10.1}$$

moreover, the trace $\operatorname{tr}_k k_\rho$ is determined modulo $O(\rho^{2n+4})$. If we express in terms of h_ρ, the followings are determined as well as $\operatorname{tr}_h h_\rho$ modulo $O(\rho^{2n+2})$:

$$\begin{aligned} &(h_\rho)_{00} \mod O(\rho^{2n}), \qquad (h_\rho)_{0\alpha} \mod O(\rho^{2n+2}),\\ &(h_\rho)_{\alpha\overline{\beta}} \mod O(\rho^{2n+2}), \qquad (h_\rho)_{\alpha\beta} \mod O(\rho^{2n+2}). \end{aligned} \tag{10.2}$$

LEMMA 10.1. *Let g be an even ACH metric inducing $(M, T^{1,0}M)$ satisfying the conditions in Theorem 2.3. If $f \in C^\infty(\overline{X})$, then $\Delta_g f$ is determined only by $(M, T^{1,0}M)$ modulo $O(\rho^{2n+4})$. Moreover, $\Delta_g(\log\rho)$ is determined only by $(M, T^{1,0}M)$ modulo $O(\rho^{2n+4}\log\rho)$.*

PROOF. We may work in the normal form relative to θ. Then, by (9.7) and (10.2), K_{kij} is uniquely determined modulo $O(\rho^{2n+2})$ if the indices i, j, k are all different from 0, and modulo $O(\rho^{2n})$ otherwise. So $(h_\rho^{-1})^{ij}(h_\rho^{-1})^{kl}K_{kij}\nabla_l f$ is determined modulo $O(\rho^{2n+2})$. On the other hand, since the components of the cofactor matrix of $(h_\rho)_{ij}$ is determined modulo $O(\rho^{2n})$, the inverse $(h_\rho^{-1})^{ij}$ is determined modulo $O(\rho^{2n+2})$. Hence (9.6) implies that $\Delta_{h_\rho} f$ is determined up to the error of $O(\rho^{2n+2})$. Furthermore, (10.1) and the trace condition implies that $|\det k_\rho|$ is determined modulo $O(\rho^{2n+4})$, and hence so is $\rho\partial_\rho(\log|\det k_\rho|)$. Therefore, by (9.4), $\Delta_g f$ is determined modulo $O(\rho^{2n+4})$. The second statement is shown similarly. □

PROOF OF THEOREM 2.16. Recall the problem in Theorem 2.13. Lemma 10.1 shows that, if $k \le n+1$, then the correspondence of $F|_{\partial X} = f$ and $G|_M$ is invariant as far as we impose (2.1) and (2.3). Therefore, the operators P_k, $k \le n+1$, are defined only by $(M, T^{1,0}M)$.

Likewise, the operators $P_{l,s}$ that appear in relation with Theorem 2.14 are invariantly defined by $(M, T^{1,0}M, \theta)$ if $l \le n+1$, so Q is also well-defined. $\square$

PROPOSITION 10.2. *Let $T^{1,0}M$ be the integrable CR structure induced by some embedding $M \hookrightarrow \mathbb{C}^{n+1}$, and θ the associated invariant contact form. Then the CR Q-curvature vanishes for $(M, T^{1,0}M, \theta)$.*

PROOF. Let φ be a Fefferman approximate solution to (5.6). Then, by Proposition 5.5, the Bergman-type metric g with a Kähler potential $4\log(1/\varphi)$ is an even smooth approximate ACH-Einstein metric that induces $T^{1,0}M$. If we take $\rho = (\varphi/2)^{1/2}$, then $\rho \in \mathcal{F}_\theta$ as we saw in Proposition 5.2. Since $\Delta_g = -2g^{i\overline{j}}\partial_i\partial_{\overline{j}}$, we obtain $\Delta_g(\log\varphi) = n+1$. Therefore,

$$\Delta_g(2\log\rho) = 2\log\varphi = n+1.$$

This shows that $2\log\rho$ is a solution to (9.18). Hence $Q = 0$ by Lemma 9.4. $\square$

10.2. First variation of total CR Q-curvature. We finally prove the first variational formula of the total CR Q-curvature. This is based on Proposition (9.3), which implies the equality between the total Q-curvature and the logarithmic coefficient of the volume expansion of an even smooth approximate ACH-Einstein metric. We assume that the metric is normalized for some choice of a contact form θ, and moreover, that it is taken so that the tensor E satisfies (6.26).

The key to the proof is introducing the first logarithmic term to our metric. Let $\rho^{-2}(4d\rho^2 + h_\rho^{\mathrm{sm}})$ be our metric ("sm" is for "smooth"), and $\mathcal{O}_{\alpha\beta}$ the CR obstruction tensor that is trivialized by θ. We consider the new ACH metric g on $M \times [0,\infty)$ given by

$$g := \frac{4d\rho^2 + h_\rho^{\mathrm{sm}} + 4(n+1)^{-1}\mathcal{O}\cdot\rho^{2n+2}\log\rho}{\rho^2}. \tag{10.3}$$

Then the component of the tensor E *with respect to the frame* $\{T, Z_\alpha, Z_{\overline{\alpha}}\}$ satisfies the following:

$$E_{\infty\infty} = O(\rho^{2n+2}), \qquad E_{\infty 0} = O(\rho^{2n+1}), \qquad E_{\infty\alpha} = O(\rho^{2n+1}), \tag{10.4a}$$

$$E_{00} = O(\rho^{2n}), \qquad E_{0\alpha} = O(\rho^{2n}), \tag{10.4b}$$

$$E_{\alpha\overline{\beta}} = O(\rho^{2n+2}), \qquad E_{\alpha\beta} = O(\rho^{2n+2}\log\rho). \tag{10.4c}$$

Note that if we are given a smooth family of partially integrable CR structures $(T^{1,0}M)_t$, then we can take a family g_t of metrics as above so that the coefficients of each components of g_t smoothly depend on the parameter t. This is clear from the construction of g.

Proof of Theorem 2.18. Let $\hat{T}_t^{1,0}$ be a smooth 1-parameter family of partially integrable CR structures that is tangent to $\psi_{\alpha\beta} \in \mathcal{E}_{(\alpha\beta)}(1,1)$. We choose a smooth 1-parameter family of normal-form ACH metrics g_t such that each g_t is of the form (10.3) with infinity $(M, \hat{T}_t^{1,0})$. If we compute with a frame $\{ T, Z_\alpha, Z_{\overline{\alpha}} \}$, where $\{ Z_\alpha \}$ is a frame of the original partially integrable CR structure $\hat{T}_0^{1,0} = T^{1,0}M$, then the derivative of the metric h_ρ, which we write $h_\rho^\bullet$, satisfies

$$\begin{aligned} (h_\rho^\bullet)_{00} &= O(1), & (h_\rho^\bullet)_{0\alpha} &= O(1), \\ (h_\rho^\bullet)_{\alpha\overline{\beta}} &= O(\rho^2), & (h_\rho^\bullet)_{\alpha\beta} &= -2\psi_{\alpha\beta} + O(\rho^2), \end{aligned}$$

and hence,

$$\begin{aligned} g^\bullet_{00} &= O(\rho^{-2}), & g^\bullet_{0\alpha} &= O(\rho^{-2}), \\ g^\bullet_{\alpha\overline{\beta}} &= O(1), & g^\bullet_{\alpha\beta} &= \rho^{-2}\psi_{\alpha\beta} + O(1), \end{aligned}$$

where $\psi_{\alpha\beta}$ is trivialized by θ.

By (10.4), there is a uniform estimate on the scalar curvature R_t:

$$R_t = -(n+1)(n+2) + O(\rho^{2n+4}).$$

From this we see that

$$\int_{\varepsilon \le \rho \le \varepsilon_0} R^\bullet dV_g = O(1) \qquad \text{as } \varepsilon \to 0.$$

On the other hand, the well-known formula of the first variation of the scalar curvature implies

$$\begin{aligned} R_g^\bullet &= {g^\bullet}_{IJ,}{}^{IJ} - {g^\bullet}_I{}^I{}_{,J}{}^J - \mathrm{Ric}^{IJ} g^\bullet_{IJ} \\ &= {g^\bullet}_{IJ,}{}^{IJ} - {g^\bullet}_I{}^I{}_{,J}{}^J + \frac{1}{2}(n+2) g^{IJ} g^\bullet_{IJ} + O(\rho^{2n+4}\log\rho). \end{aligned} \tag{10.5}$$

Since $dV_g = O(\rho^{-2n-3})$ and $dV_g^\bullet = \frac{1}{2} g^{IJ} g^\bullet_{IJ} dV_g$, (10.5) integrates to

$$\int_{\varepsilon \le \rho \le \varepsilon_0} ({g^\bullet}_{IJ,}{}^{IJ} - {g^\bullet}_I{}^I{}_{,J}{}^J) dV_g + (n+2) \int_{\varepsilon \le \rho \le \varepsilon_0} dV_g^\bullet = O(1). \tag{10.6}$$

Let ν $(= -\frac{1}{2}\varepsilon\partial_\rho)$ be the unit outward normal vector for g along $\{ \rho = \varepsilon \}$. Then

$$\begin{aligned} &(n+2)\,\mathrm{Vol}_g(\{ \varepsilon \le \rho \le \varepsilon_0 \})^\bullet \\ &= -\int_{\rho=\varepsilon} ({g^\bullet}_{IJ,}{}^I - {g^\bullet}_I{}^I{}_{,J}) \nu^J d\sigma + O(1) \\ &= \int_M ({g^\bullet}_{IJ,}{}^I - {g^\bullet}_I{}^I{}_{,J}) \cdot \frac{1}{2}\varepsilon \delta_\infty{}^J \cdot \varepsilon^{-2n-2} dV_{k_\varepsilon} + O(1) \\ &= \frac{1}{2}\varepsilon^{-2n-1} \int_M ({g^\bullet}_{I\infty,}{}^I - {g^\bullet}_I{}^I{}_{,\infty}) dV_{k_\varepsilon} + O(1). \end{aligned} \tag{10.7}$$

We compare the $\log(1/\varepsilon)$-terms of the both sides of (10.7). That of the left-hand side is obviously $(n+2)L^\bullet$. As for the right-hand side, we use

$$
\begin{aligned}
&g^\bullet{}_{I\infty,}{}^{I} - g^\bullet{}_{I}{}^{I}{}_{,\infty}\\
&= -\frac{1}{2}(h_\rho^{-1})^{ij}(h_\rho^{-1})^{kl}(h'_\rho)_{jl}(h^\bullet_\rho)_{ik} + \rho^{-1}(h_\rho^{-1})^{ij}(h^\bullet_\rho)_{ij} - ((h_\rho^{-1})^{ij}(h^\bullet_\rho)_{ij})',
\end{aligned}
$$

where the primes denotes differentiations in ρ. Since $(\det h_\rho)^\bullet$ is equal to $\det h_\rho$ times $(h_\rho^{-1})^{ij}(h^\bullet_\rho)_{ij}$, we conclude that $(h_\rho^{-1})^{ij}(h^\bullet_\rho)_{ij}$ contains no $(\rho^{2n+2}\log\rho)$-term, which implies that the potential contribution to the $\log(1/\varepsilon)$-term in the right-hand side of (10.7) may come only from $-\frac{1}{2}(h_\rho^{-1})^{ij}(h_\rho^{-1})^{kl}(h'_\rho)_{jl}(h^\bullet_\rho)_{ik}$.

The logarithmic term that appears in the expansion of $(h'_\rho)_{jl}$ is $8\mathcal{O}\cdot\rho^{2n+1}\log\rho$, and hence, modulo smooth terms,

$$
\begin{aligned}
&-\frac{1}{2}(h_\rho^{-1})^{ij}(h_\rho^{-1})^{kl}(h'_\rho)_{jl}(h^\bullet_\rho)_{ik}\\
&\equiv -4\rho^{2n+1}\log\rho\cdot(h_\rho^{-1})^{\overline{\alpha}\beta}(h_\rho^{-1})^{\overline{\gamma}\sigma}\mathcal{O}_{\beta\sigma}(h^\bullet_\rho)_{\overline{\alpha}\overline{\gamma}} + (\text{the complex conjugate})\\
&\quad + O(\rho^{2n+3}\log\rho)\\
&\equiv -4\rho^{2n+1}\log\rho\cdot(\mathcal{O}^{\alpha\beta}(h^\bullet_\rho)_{\alpha\beta} + \mathcal{O}^{\overline{\alpha}\overline{\beta}}(h^\bullet_\rho)_{\overline{\alpha}\overline{\beta}}) + O(\rho^{2n+3}\log\rho).
\end{aligned}
$$

Moreover, $(h^\bullet_\rho)_{\alpha\beta} = -2\psi_{\alpha\beta} + O(\rho^2)$. Therefore the $\log(1/\varepsilon)$-coefficient of the right-hand side of (10.7) is

$$
-4\int_M(\mathcal{O}^{\alpha\beta}\psi_{\alpha\beta} + \mathcal{O}^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})dV_k.
$$

We conclude that

$$
\begin{aligned}
L^\bullet &= -\frac{4}{n+2}\int_M(\mathcal{O}^{\alpha\beta}\psi_{\alpha\beta} + \mathcal{O}^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})dV_k\\
&= -\frac{4}{(n+2)\cdot n!}\int_M(\mathcal{O}^{\alpha\beta}\psi_{\alpha\beta} + \mathcal{O}^{\overline{\alpha}\overline{\beta}}\psi_{\overline{\alpha}\overline{\beta}})\theta\wedge(d\theta)^n.
\end{aligned}
$$

By combining this with Proposition 9.3, we obtain (2.22). $\square$

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