Heat flow

and

Ricci curvature.

§1 Heat flow on $\boxed{\text{metric measure space}}$

m. m. sp.

Goal

$\boxed{\text{Define Laplacian and heat flow on m. m. sp.}}$

Def. 1.1 (m. m. sp.)

(X, d, m) : m. m. sp.

$\overset{\text{def.}}{\Leftrightarrow}$ $\begin{cases} \text{① } (X, d) : \text{cplt sep. metric sp.} \\ \text{② } m : \text{Borel meas. on } X \\ \quad (\text{with } \operatorname{supp} m = X) \end{cases}$ ⌟

Example 1.2

$\{\cdot\ (M^n, g_{M^n})$: cplt Riem.

$\Big\{ \cdot\ f \in C^\infty(M^n)$

vol_f (i.e. $\mathrm{vol}_f A = \int_A e^{-f} d\mathrm{vol}$)

$//$

$(M^n,\ d_{g_{M^n}},\ e^{-f} d\mathrm{vol})$: m. m. sp.

" C^∞- m. m. sp. "

$\boxed{\text{Fix}}$

(X, d, m) : m. m. sp.

<u>Def. 1.3</u> (Lip f)

$f \in \underline{LIP_{loc}(X, d)}$

The set of all locally Lip. fcts on X.

$$\underset{\text{slope}}{\underline{\mathrm{Lip}\, f}}(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y, x)}$$

<u>Example 1.4</u>

$\forall f \in C^1(M^n)$

$$\mathrm{Lip}\, f(x) = |\nabla f|(x) \quad \forall x \in M^n \lrcorner$$

Def. 1.5 (Sobolev sp. via Cheeger energy)

(1) $\mathrm{Ch} : \mathcal{L}^2(X, m) \to [0, \infty]$

is defined by

$$\mathrm{Ch}(f)$$

$$:= \inf_{\{f_n\}_n} \left\{ \liminf_{n\to\infty} \frac{1}{2} \int_X (\mathrm{Lip} f_n)^2 \, dm \;\middle|\; \begin{array}{l} f_n \in \underbrace{\mathrm{LIP}_b(X, d)}_{\text{The set of all bdd Lip fcts on } X} \cap L^2(X, m) \\ f_n \to f \text{ in } \mathcal{L}^2(X, m) \end{array} \right\}$$

$\Rightarrow$ by def, Ch is convex, lower semi cont. ect.

(2) $H^{1,2}(X, d, m)$

$$:= \{ f \in \mathcal{L}^2(X, m) \mid \mathrm{Ch}(f) < \infty \}$$

$\Rightarrow$ $(H^{1,2}, \|\cdot\|_{H^{1,2}})$ is a Banach sp, where

$$\|f\|_{H^{1,2}} := \sqrt{\|f\|_{L^2}^2 + 2\mathrm{Ch}(f)}$$

Rem 1.6

① In general, $H^{1,2}$ is not Hilbert.

c.f.

$$H^{1,2}(\mathbb{R}^n, d_{L^p}, \mathcal{L}^n) \qquad (p \neq 2)$$

is a Banach sp, not Hilbert.,

where $d_{L^p}(x,y) := \left(\sum_i |x_i - y_i|^p\right)^{\frac{1}{p}}$ ⌟

② If we consider "L^p" instead of $\mathcal{L}^2$ in Def 1.5, then the corresponding sp is $\begin{cases} H^{1,p} & \text{if } p>1 \\ BV & \text{if } p=1 \end{cases}$ ⌟

Example 1.7

$H^{1,2}(M^n, d_{g_{M^n}}, vol_g)$: Hilbert. ⌟

In order to define

$$\underbrace{|\nabla f|}_{\text{"metric notion in some sense"}} \in \mathcal{L}^2(X, m)$$

for $f \in H^{1,2}(X, d, m)$, we prepare;

Def. 1.8 ("$g \geq |\nabla f|$")

Let $f \in H^{1,2}(X, d, m)$

$RS(f)$ $\left(\text{"} = \{ g \in \mathcal{L}^2(X,m) \mid g \geq |\nabla f| \} \text{"} \right)$

relaxed slope

$$\overset{\text{def}}{:=} \left\{ g \in \mathcal{L}^2(X, m) \;\middle|\; \begin{array}{l} \exists h \in \mathcal{L}^2(X, m) \quad \exists f_i \in \mathrm{Lip}_b(X,d) \cap L^2(X,m) \\ \text{s.t.} \quad f_i \longrightarrow f \quad \text{in } L^2(X, m) \\ \qquad \mathrm{Lip} f_i \xrightarrow{\text{weak}} h \quad \text{in } L^2(X, m) \\ \qquad g \geq h \quad m\text{-a.e} \end{array} \right\}$$

$$\left(\boxed{\begin{array}{l} \text{recall :} \quad (H, \langle \cdot, \cdot \rangle) \text{ : Hilbert sp} \\ \qquad v_i \in H \text{ with } \sup_i \|v_i\| < \infty \\ \qquad \Rightarrow \exists i(j) \ \exists v \in H \text{ s.t. } v_{i(j)} \xrightarrow{\text{weak}} v \text{ in } H \end{array}} \right)$$

$\neq \phi$

$\lrcorner$

<u>Prop 1.9</u> (Property on $RS(f)$)

$RS(f)$ is closed convex subset in $\mathcal{L}^2(X, m)$ $\lrcorner$

Pf. By Mazur's lemma and convexity

$$\left(\begin{array}{l} \left\{\begin{array}{l} \cdot\ (H, \langle\cdot,\cdot\rangle) : \text{Hilbert} \\ \cdot\ V \subset H : \text{sub sp.} \end{array}\right. \\ \Rightarrow \text{The closure of } V \text{ w.r.t str.top} \\ = \quad\quad '' \quad\quad \text{weak } '' \end{array}\right) \text{ of Ch.}$$ //

Recall.

$$\left\{\begin{array}{l} \cdot\ (H, \langle\cdot,\cdot\rangle) : \text{Hilbert sp.} \\ \cdot\ V \subset H : \text{closed convex subset.} \end{array}\right.$$

$\neq^{\divideontimes}$

$\Rightarrow \ \exists^1 v \in V$ s.t.

$$\|v\| = \min_{w \in V} \|w\|$$

┘

relaxed

Def - Prop 1.10 (The minimal slope $|\nabla f|$)

(1) $\forall f \in H^{1,2}(X, d, m)$

$\exists^1 |\nabla f| \in \mathcal{L}^2(X, m)$ s.t.

$$\| |\nabla f| \|_{L^2} = \min_{g \in RS(f)} \|g\|_{L^2}$$

$$\Rightarrow \quad Ch(f) = \frac{1}{2}\int_X |\nabla f|^2 \, dm$$

~~(in general, $|\nabla f| \neq \mathrm{lip}\, f$ $\quad f \in LIP(X, d)$)~~

(2) If $H^{1,2}(X, d, m)$ is Hilbert sp,

then $f \mapsto |\nabla f|^2$ is quadratic

m-a.e sense, that is,

$$\forall f, \forall g \in H^{1,2}(X, d, m)$$

$$|\nabla(f+g)|^2 + |\nabla(f-g)|^2 = 2|\nabla f|^2 + 2|\nabla g|^2$$

m-a.e

$$\Rightarrow \quad {}^{\exists}\langle \nabla f, \nabla g \rangle \quad : \text{m-a.e.}$$

"Riemannian metric on (X, d, m)"

canonical

Additional assumptions

ⓐ $H^{1,2}(X, d, m)$ is Hilbert sp

ⓑ $H^{1,2}$ is dense in L^2

(M^n, d_{g_M}, vol_g) satisfies these.

Remark 1.11

① In general

$$|\nabla f| \neq \mathrm{Lip}\, f \qquad f \in LIP(X, d).$$

$$\left(\begin{array}{l} \text{c.f.} \quad {}^{\exists} m \ : \text{ a Borel meas on } [0,1] \\ \quad \text{s.t. } {}^{\forall} f : [0,1] \to \mathbb{R} : \text{Lip.} \\ \qquad |\nabla f|_{([0,1],\, d_{[0,1]},\, m)} = 0 \quad m\text{-a.e.} \end{array} \right)$$

see Ambrosio-Gigli-Savaré Invent.

② Under assuming "nice geometric conditions"
(doubling & Poincaré)

$${}^{\forall} f \in LIP_{loc}(X, d) \cap H^{1,2}(X, d, m)$$

$$\mathrm{Lip}\, f = |\nabla f| \qquad m\text{- a.e.}$$

(c.f. Cheeger GAFA) ⌟

Def 1.~~#~~ 12 (Laplacian Δ)

Let $f \in \mathcal{L}^2(X, m)$

$f \in \mathcal{D}(\Delta)$

$\overset{\text{def}}{\Longleftrightarrow}$ $f \in H^{1,2}(X, d, m)$ $\exists g \in \mathcal{L}^2(X, m)$ s.t.

$$\int_X \langle \nabla f, \nabla h \rangle \, dm = \int_X g h \, dm$$

$\forall h \in H^{1,2}(X, d, m)$

Since g is unique, we denote it by
(by ⓑ)

$$\Delta f.$$

Example 1.~~12~~ 13 (Witten Lap. ~~and~~ v.s. Lap.)

- $(M^n, d_{g_{M^n}}, \mathrm{vol}_f)$ with $\sup|f| < \infty$.

Then.

$$\begin{cases} ① \quad h \in \mathcal{L}^2(M^n, vol_f) \\ \quad (H^{1,2}(M^n, d_{g_{M^n}}, vol_f),\ \mathcal{D}(\Delta^{vol_f}) \\ \quad \text{resp.}) \\ \Leftrightarrow h \in \mathcal{L}^2(M^n, vol) \\ \quad (H^{1,2}(M^n, d_{g_{M^n}}, vol),\ \mathcal{D}(\Delta^{vol}) \\ \quad \text{resp.}) \\ ② \quad \Delta^{vol_f} h = \Delta h + \langle \nabla f, \nabla h \rangle \\ \quad {}^{\forall} h \in \mathcal{D}(\Delta^{vol_f}) = \mathcal{D}(\Delta^{vol}) \end{cases}$$

$\lrcorner$

Def. ~~1.13~~ 1.14 (Heat flow) — Prop.

${}^{\forall} f \in \mathcal{L}^2(X, m)$

$$\exists 1 \quad \begin{array}{ccc} (0, \infty) & \longrightarrow & \mathcal{D}(\Delta) \subset \mathcal{L}^2(X, m) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ t & \longmapsto & h_t f \end{array}$$

: loc. abs. cont.

s.t. $\begin{cases} ① \ h_t f \to f \text{ in } \mathcal{L}^2(X, m) \text{ as } t \downarrow 0 \\ ② \ \frac{d}{dt} h_t f = -\Delta h_t f, \ \mathcal{L}^1\text{-a.e } t \in (0, \infty) \end{cases}$

Pf. By ~~convexity~~ lower semicontinuity of Ch.

(c.f. H. Brezis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.)

"Brezis-Komura theory of gradient flows of lower semi cont. functionals in Hilbert sp"

//

Prop 1.~~14~~ 15 (General properties on heat flow) without ass. on curv.

① $h_t : \mathcal{L}^p(X, m) \cap \mathcal{L}^2(X, m) \to \mathcal{L}^p(X, m)$

is well-defined $\forall p \in [1, \infty)$

② $$\int_X |\nabla h_t f|^2 \, dm \leq \frac{\|f\|_{L^2}^2}{t} \qquad \forall t>0$$

$$\int_X |\Delta h_t f|^2 \, dm \leq \frac{\|f\|_{L^2}^2}{t^2} \qquad \forall t>0$$

$$\int_X h_t f \, dm = \int_X f \, dm. \qquad \forall t>0$$

③ If $f \in H^{1,2}$, then

$$h_t f \to f \quad \text{in } H^{1,2} \text{ as } t \downarrow 0$$

┘

Rem. 1.~~15~~ 16

Let

- $\cancel{f}\, G = (V, E)$: (locally finite) graph
- $X = V$ with the length distance d (~~i.e~~ where ~~each~~ the length of each edge is 1)
- $m = \sum_{v \in V} \delta_v$

$$\left\{ \begin{array}{l} \cdot \ \mathrm{Lip}^a f(x) = \sqrt{\sum_{y \sim x} |f(y) - f(x)|^2} \\ \text{for } \forall f : X \to \mathbb{R} \end{array} \right.$$

($\Longleftarrow$ it is automatically loc. Lip)

Then the corresponding Laplacian

which is given by arguments above via $\mathrm{Lip}^a f$

(instead of using $\mathrm{Lip} f$) is "graph Laplacian". $\lrcorner$

§2 Bakry-Émery theory on Ricci curvature.

Thm 2.1 (Bakry-Émery est.)

Let $\begin{cases} \cdot\ (M^n, d_{g_{M^n}}, \mathrm{vol}_f). \\ \cdot\ K \in \mathbb{R}. \end{cases}$

$\mathrm{Ric}^f_{M^n} \geq K.$

Then $\boxed{\underbrace{\boxed{\mathrm{Ric}_{M^n} + \mathrm{Hess}_f}}_{\mathrm{Ric}^f_{M^n}} \geq K g_{M^n}}$

$\Leftrightarrow$ $\begin{cases} \text{① (Volume estimate)} \\ \quad {}^{\exists} c > 0,\ {}^{\exists} x_0 \in M^n \text{ s.t.} \\ \quad \displaystyle\int_{M^n} \exp(-c\, d(x_0, x)^2)\, d\mathrm{vol}_f < \infty \\ \text{② (Bakry-Emery est.)} \\ \quad {}^{\forall} \varphi \in H^{1,2}(\cancel{*}^{M^n}, d_{g_{M^n}}, \cancel{m}^{\mathrm{vol}_f}) \\ \qquad (C^\infty(M^n) \cap L^2(\cancel{*}^{M^n}, \mathrm{vol}_f)) \end{cases}$

$\cancel{s}$ ①

$\forall t>0$

$$|\nabla (h_t\varphi)|^2 \le e^{-2Kt}\, h_t|\nabla\varphi|^2 \qquad m\text{-a.e}$$

($\forall x\in M^n$)

because

$\Leftarrow$ these are C^∞.

(c.f. Ambrosio – Gigli – Savaré, Ann. Prob.
Bakry – Émery, Diffusions hypercontractives.)

"Pf. of ② in ⇒)"

Let $\varphi\in C^\infty(M^n)\cap L^2(M^n,\mathrm{vol}_f)$

Recall Bochner's formula;

$$-\frac{1}{2}\Delta^{\mathrm{vol}_f}|\nabla\varphi|^2 = |\mathrm{Hess}_\varphi|^2 - \langle\nabla\Delta^{\mathrm{vol}_f}\varphi,\nabla\varphi\rangle + \underset{2K|\nabla\varphi|^2}{\underline{\mathrm{Ric}^f_{M^n}(\nabla\varphi,\nabla\varphi)}}$$

Applying this for $h_t\varphi$ shows

$$\Phi'(s) \geq 2k\,\Phi(s), \quad \cdots (*)$$

where $\Phi(s) := h_s |\nabla_\# h_{t-s} \varphi|^2 \quad (s \in [0,t])$.

Integrating $(*)$ over $[0,t]$ yields ② //

Def 2.2 ($\mathrm{Ric}_{(X,d,m)} \geq K$)

Let $K \in \mathbb{R}$.

so called $BE(K,\infty)$-condition

$\mathrm{Ric}_{(X,d,m)} \geq K$ (or equivalently $RCD(K,\infty)$-cond.) in our setting.

$\overset{\text{def}}{\Longleftrightarrow}$

⓪ $H^{1,2}(X,d,m)$ is a Hilbert sp.

① (Volume estimate)

$\exists c > 0 \quad \exists x_0 \in X$

$$\int_X \exp(-c\, d(x_0,x)^2)\, dm < \infty$$

② (Bakry-Emery estimate)

$$\begin{cases} {}^{\forall}\varphi \in H^{1,2}(X,d,m) \quad {}^{\forall}t>0 \\ |\nabla h_t \varphi|^2 \le e^{-2kt} h_t |\nabla\varphi|^2 \quad m\text{-a.e.} \end{cases}$$

$\Rightarrow$ If φ is L^∞, then $h_t\varphi \in LIP(X,d)$.

<u>Thm 2.3</u> (L^1-Bakry-Émery estimate)

~~Under the same setting as in~~

If $\mathrm{Ric}_{(X,d,m)} \ge K$, then

${}^{\forall} f \in H^{1,2}(X,d,m)$.

$$|\nabla h_t f| \le e^{-kt} h_t |\nabla f| \quad m\text{-a.e.}$$

See. Savaré, Self-improvement of the Bakry-Émery condition and Wasserstein contraction of the heat flow in $RCD(K,\infty)$ metric measure spaces.

for the proof.

Rem 2.9

This was new even for C^2-mfd.

(Note L^1-Bakry-Émery est. $\Rightarrow$ $L^{p'}$-B-E est $\forall p \le \forall p'$)

§3. Convergence of m. m. sps and of heat flows

Def. 3.1 (pointed measured Gromov convergence)

(1) $\begin{cases} \cdot\ (X_i, d_i, m_i) \\ \cdot\ (X, d, m) \\ \cdot\ x_i \in X_i, \quad x \in X \end{cases}$: m. m. sps

$$(X_i, d_i, x_i, m_i) \xrightarrow{pmG} (X, d, x, m)$$

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists\, Z$: cple sep. met sp

usually $Z = \bigsqcup_i X_i \sqcup X$

$\exists\, \varphi_i : X_i \hookrightarrow Z$

$\exists\, \varphi : X \hookrightarrow Z$: isom. emb. s.

s.t. $\begin{cases} (1.1) \quad \varphi_i(x_i) \longrightarrow \varphi(x) \text{ in } Z \\ \end{cases}$

(1.2)

$(\varphi_i)_\# m_i \to (\varphi)_\# m$ in $(C_{bs}(Z))^*$

i.e.

$$\int_Z f \, d(\varphi_i)_\# m_i \longrightarrow \int_Z f \, d(\varphi)_\# m$$

recall

$(\varphi_i)_\# m_i(A) := m_i(\varphi_i^{-1}(A))$.

$\forall f \in \underline{C_{bd}(Z)}$

The set of all cont. fct on Z with bdd supp.

(2) $$(X_i, d_i, m_i) \xrightarrow{mG} (X, d, m)$$

$\overset{\text{def}}{\iff}$ $\exists x_i \in X_i$ $\exists x \in X$ s.t.

$$(X_i, d_i, x_i, m_i) \xrightarrow{pmG} (X, d, x, m)$$

Rem 3.12

(1) $\exists$ (p)mGH : another (famous) notion of conv. of m.m.-sps.

(2) (p)mGH $\Rightarrow$ (p)mG.

However in general the converse is not true.

$\mathrm{Ric} \geq n-1$

$$\left(\text{c.f} \quad \boxed{\left([0,\pi],\ d_{[0,\pi]}\ \frac{\sin^{n-1} t \, dt}{\int_0^\pi \sin^{n-1} t \, dt}\right)} \xrightarrow{mG} \left(\{\tfrac{\pi}{2}\},\ \delta_{\frac{\pi}{2}}\right) \right)$$

(but is not mGH.)

(3) However

mGH ⊕ "$\underline{\text{nice geom. assump}}$"

e.g. doubling

$\Longrightarrow$ mGH.

$\lrcorner$

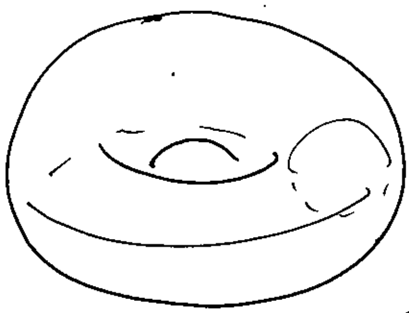
Open prob.

> Find a nice geometric assumption for seq (X_i, d_i, m_i), which implies cptness w.r.t mG-conv.

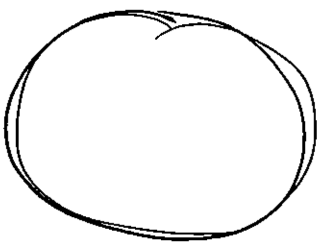
Example 3.3

- $g_n \to g$ in $L^1(\mathbb{R}^n, \mathcal{L}^n)$ $g_n \geq 0$ n-a.e
- $X_n := \operatorname{supp} g_n$ $X = \operatorname{supp} g$.

$\Longrightarrow (X_n, d_{\mathbb{R}^n}, g_n \, d\mathcal{L}^n) \xrightarrow{mG} (X, d_{\mathbb{R}^n}, g \, d\mathcal{L}^n)$



$\xrightarrow[(\varepsilon_i \downarrow 0)]{mGH}$



$$\left(\mathbb{S}^1(1) \times \mathbb{S}^1(\varepsilon_i), \frac{\overset{\text{2-dim. Haus. meas}}{H^2}}{\boxed{H^2(\mathbb{S}^1(1) \times \mathbb{S}^1(\varepsilon_i))}} \right) \qquad \left(\mathbb{S}^1(1), \frac{H^1}{\boxed{2\pi}} \right)$$

$\mathbb{S}^1(\varepsilon_i) = \{x \in \mathbb{R}^2 \mid |x| = \varepsilon_i\}$

$H^2(\mathbb{S}^1(1) \times \mathbb{S}^1(\varepsilon_i)) = 4\pi^2 \varepsilon_i$

$2\pi = H^1(\mathbb{S}^1(1))$

$\boxed{\text{Fix}}$

$$(X_i, d_i, m_i) \xrightarrow{mG} (X, d, m)$$

~~and~~ Z, φ_i, φ and so on.

Def. 3.4 (L^2-conv. w.r.t. mG-conv.)

Let $f_i \in L^2(X_i, m_i)$ and let $f \in L^2(X, m)$.

this can be seen as $f_i \in L^2(Z, (\varphi_i)_\# m_i)$

① $f_i \xrightarrow{L^2\text{-weak}} f$

$\overset{\text{def}}{\Leftrightarrow}$
(1.1) $\sup_i \|f_i\|_{L^2} < \infty$

(1.2) $f_i (\varphi_i)_\# m_i \longrightarrow f (\varphi)_\# m$ in $(C_{bs}(Z))^*$

i.e.

$$\int_Z f_i g \, d(\varphi_i)_\# m_i \longrightarrow \int_Z f g \, d\varphi_\# m.$$

$\forall g \in C_{bs}(Z)$.

② $f_i \xrightarrow{L^2\text{-str}} f$

$\overset{\text{def}}{\Leftrightarrow}$
(2.1) It is L^2-weakly conv.

(2.2) $\limsup_{i\to\infty} \|f_i\|_{L^2} \le \|f\|_{L^2}$.

Rem 3.5

① Similarly we can define L^p-conv. $^\forall p \in [1,\infty)$

② Well-known results on a fixed sp. are also satisfied in this setting.

(e.g. $^\forall L^p$-bdd seq has a L^p weak conv. seq. and so on).

Similarly,

Def. 3.6 ($H^{1,2}$-conv.)

Let $f_i \in H^{1,2}(X_i, d_i, m_i)$ and let $f \in H^{1,2}(X, d, m)$.

① $f_i \xrightarrow{H^{1,2}\text{-weak}} f$

$\overset{\text{def}}{\Longleftrightarrow}$ (1.1) $\sup_i \|f_i\|_{H^{1,2}} < \infty$

(1.2) $f_i \xrightarrow{L^2\text{-weak}} f$.

~~trivial $\Rightarrow f_i \xrightarrow{L^2\text{-weak}} f$~~

② $f_i \xrightarrow{H^{1,2}\text{-str}} f$.

$\overset{\text{def}}{\Leftrightarrow}$ $\begin{cases} \text{(2\_1)} & I_t \text{ is } H^{1,2}\text{-weakly conv.} \\ \text{(2\_2)} & \limsup_{i\to\infty} \|f_i\|_{H^{1,2}} \le \|f\|_{H^{1,2}} \end{cases}$

Thm 3.17 (Conv of heat flow and stability of lower bounds on Ricci)

Assume $\exists K \in \mathbb{R}$ s.t

$$\mathrm{Ric}(X_i, d_i, m_i) \ge K \quad \forall i$$

Then

① $\mathrm{Ric}(X, d, m) \ge K$.

② If $f_i \xrightarrow{L^2\text{-str}} f$, then

$$f_i \in L^2(X_i, m_i), \quad f \in L^2(X, m)$$

$$\begin{cases} h_t f_i \xrightarrow{H^{1,2}\text{-str}} h_t f \\ \Delta h_t f_i \xrightarrow{L^2\text{-weak}} \Delta h_t f \end{cases} \quad \forall t > 0$$

See Gigli-Mondino-Savaré Proc. LMS.

Ambrosio-H. New stability ...

for proofs.

§4 Continuous geometric/analytic quantities w.r.t mG-conv.

$\boxed{\text{Fundamental problem. on mGH/mG-conv. theory}}$

Find geometric/analytic quantities which are cont. w.r.t mGH/mG-conv.

Example 4.1 (trivial)

Diameters are cont. w.r.t. mGH-conv.

(not mG, e.g. (2) of Rem 3.2) ┘

Thm 4.2 (Spectral conv.)

$$\begin{cases} ⓐ \ (X_i, d_i, m_i) \xrightarrow{mG} (X, d, m) \\ ⓑ \ {}^{\exists}K \in \mathbb{R} \text{ s.t } Ric_{(X_i, d_i, m_i)} \geq K \ {}^{\forall} i \end{cases}$$

(Recall: then. $Ric_{(X,d,m)} \geq K$)

(☆$_{L^2}$) $\forall$ $H^{1,2}$-bdd seq has an L^2-str. conv. subseq. (w.r.t. both a fixed X_i and $(X_i, d_i, m_i) \to (X, d, m)$)

$\Rightarrow$ The spectrum of Δ^{m_i} is discrete and unbounded.

Then

"analytic nontrivial quantity"

$$\underbrace{\lambda_\ell(X_i, d_i, m_i)}_{\ell\text{th eig. value of } \Delta^{m_i}} \to \lambda_\ell(X, d, m)$$

$\forall \ell$

See K. Fukaya. Invent.

Cheeger-Colding JDG.

Gigli-Mondino-Savaré P.LMS. for the proof.

Rem 4.3

(☆$_{L^2}$) is satisfied under "mild geometric assumption."

"$\dim(X_i, d_i, m_i) \leq N < \infty$ with $\operatorname{diam}(X_i, d_i) \leq d$ and so on.."

Question

$\exists$ other quantity (geometric) ?

Answer (Ambrosio-H.)

Yes, Cheeger constant (isoperimetric)

<u>Recall</u>

- (M^n, g_{M^n}) : cplt Riem. mfd with $\mathrm{vol}\, M^n < \infty$

~~(1) The Cheeger (isoperimetric) constant $Ch(M^n)$ is defined by~~

$$\boxed{Ch(M^n)} \overset{\text{def}}{=} \inf_{\substack{\Omega \subset M^n : \text{open} \\ \partial\Omega : C^\infty \\ \mathrm{vol}\,\Omega \le \frac{1}{2}\mathrm{vol}\, M^n}} \frac{\mathrm{vol}_{n-1}\partial\Omega}{\mathrm{vol}\,\Omega}$$

Def 4.4 ($Ch(X, d, m)$)

Let (X, d, m) be a m.m. sp.

with $\underline{m(X) < \infty}$

without loss of gen. $m(X) = 1$.

$$Ch(X, d, m) \overset{\text{def}}{:=} \inf_{\substack{A \subset X \\ m(A) \leq \frac{1}{2} m(X)}} \frac{m_{-1}(A)}{m(A)},$$

where

ε-nbd of A.

$$m_{-1}(A) \overset{\text{def}}{:=} \liminf_{\varepsilon \downarrow 0} \frac{m(B_\varepsilon(A)) - m(A)}{\varepsilon}$$

Rem 4.5

$$Ch(M^n, d_{g_{M^n}}, vol) = Ch(M^n)$$

Thm 4.6 (Continuity of Cheeger constants)

ⓐ $(X_i, d_i, m_i) \xrightarrow{mG} (X, d, m)$ with

$m_i(X_i) = m(X) = 1$

(b) $\exists K \in \mathbb{R}$ s.t. $\mathrm{Ric}_{(X_i, d_i, m_i)} \geq K \quad \forall i$

($\star_{L^1}$) $\forall$ BV-bdd seq has a L^1-str. conv. sub. seq.

Then

$$Ch(X_i, d_i, m_i) \to Ch(X, d, m)$$

$\lrcorner$

Rem 4.7

($\star_{L^1}$) is also satisfied under mild geometric assumption

$\lrcorner$

Sketch of the proof of Thm 4.6

" $-\mathrm{div}\left(\frac{\nabla f}{|\nabla f|}\right) = \lambda\, \mathrm{sgn} f$ "

Step 1 (Ch is the 1st pos. eig of 1-Lap.)

it is know that this is equal to the 1st pos eig value of $\boxed{1}$ in C^2 setting.

Prove.

$$Ch(X, d, m) = \boxed{\inf_{\substack{f \in LIP(X,d) \cap L^1(X,m) \\ \neq \text{const}}} \frac{\int_X |\nabla f| \, dm}{\inf_{c \in \mathbb{R}} \int_X |f - c| \, dm}}$$

$(\ast)$

(c.f see E. Milman Invent. for the proof).

Step 2 Find a minimizer $f_X = f_{(X,d,m)}$ of (*) as BV-fct.

so called "total var. of f_X"

$$\Rightarrow \quad Ch(X,d,m) = \frac{\|Df_X\|}{\int_X |f_X|\,dm}$$

Step 3 Applying L^1-Bakry-Émery est. to f_{X_i}.

yields

$$\|D(h_t f_{X_i})\| \le e^{kt} \|Df_{X_i}\|$$

Step 4 letting $i \to \infty$ and then $t \downarrow 0$

(with some technical arg.)

shows

$$\liminf_{i\to\infty} \|Df_{X_i}\| \ge \|Df_X\|$$

which implies, with (A_{L^1}),

$$\liminf_{i\to\infty} Ch(X_i, d_i, m_i) \ge Ch(X, d, m)$$

Final step

$^\forall f \in LIP(X, d)$

Take $f_i \in L^2(X_i, m_i) \cap L^\infty(X_i, m_i)$ s.t

$$f_i \xrightarrow{L^2\text{-str}} f.$$

Then since $h_t f_i \xrightarrow{H^{1,2}\text{-str}} h_t f$ $\quad ^\forall t > 0$,

$$\limsup_{i\to\infty} Ch(X_i, d_i, m_i)$$

$$\leq \limsup_{i\to\infty} \frac{\int_{X_i} |\nabla h_t f_i| \, dm_i}{\inf_{c\in\mathbb{R}} \int_{X_i} |h_t f_i - c| \, dm_i}$$

$$= \frac{\int_X |\nabla h_t f| \, dm}{\inf_{c\in\mathbb{R}} \int_X |h_t f - c| \, dm}.$$

letting $t \downarrow 0$ and taking the inf. over f. show

$$\limsup_{i\to\infty} Ch(X_i, d_i, m_i) \leq Ch(X, d, m). \quad //$$

See L. Ambrosio - H.

New stability ..

for the proof.