Geometric Analysis in Geometry and Topology 2017

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Foundations and developments of Poincaré-Einstein metrics

Yoshihiko Matsumoto

Disclaimer: Not much "developments" is discussed.

References

- O. Biquard, Métrique d'Einstein asymptotiquement symétrique, Astérisque, 2000
  (English translation by S. Wilson, 2006)
- J. Lee, Fredholm Operators and Einstein Metrics on Conformally Compact Manifolds, Memoirs of the AMS, 2006
- O. Biquard, ed., AdS/CFT Correspondence: Einstein Metrics and Their Conformal Boundaries, IRMA Lect. in Math. and Theo. Phys., 2005

GAGT 2017 Lecture # 1

Definition, Examples

Poincaré model of $\mathbb{H}^{n+1}$:

$$B^{n+1} = \{ x \in \mathbb{R}^{n+1} \mid |x| < 1 \}$$

$$g_{\mathbb{H}^{n+1}} = \frac{4\left((dx^1)^2 + \cdots + (dx^{n+1})^2 \right)}{(1-|x|^2)^2} \qquad \text{sect. curv.} \equiv -1$$

$(B^{n+1}, g_{\mathbb{H}^{n+1}})$ is conformally compact :

$(1-|x|^2)^2 g_{\mathbb{H}^{n+1}}$ extends to $\overline{B^{n+1}} \subset \mathbb{R}^{n+1}$

Let $\overline{X}$ be a cpt. mfd-with-bdry. $X := \operatorname{int}(\overline{X})$.

<u>Def. 1.1</u>

$\rho \in C^\infty(\overline{X})$ is a (positive) <u>bdry. dfn. fn.</u>

$\overset{\text{def}}{\Longleftrightarrow}$ $\rho > 0$ in X,

$\rho = 0$, $d\rho$ nowhere vanishing on $\partial\overline{X}$ $(=: \partial X)$.

<u>Ex.</u> $\overline{X} = \overline{B^{n+1}}$, $\rho(x) = 1 - |x|^2$. $d\rho = -2 \sum_{i=1}^{n+1} x^i dx^i$

<u>Lem. 1.2</u>

$\rho, \hat{\rho} \in C^\infty(\overline{X})$ bdry. dfn. fn.'s

$\Rightarrow$ $\exists\, \varphi \in C^\infty(\overline{X})$, $\varphi > 0$ s.t. $\hat{\rho} = \varphi \rho$.

<u>Def. 1.3</u>

A Riemannian metric g on X $(= \operatorname{int} \overline{X})$ is <u>C^∞ conformally compact</u> when

$\overline{g} = \rho^2 g$ extends to a C^∞ Riem. met. on $\overline{X}$.

Let $h := \overline{g}|_{T\partial X}$. $[h]$ is the <u>conf. infinity</u> of g.

Lem.-Def. 1.4

g on X, C^∞ conf. cpt.

Sect. curv. of $g \longrightarrow -1$ at ∂X uniformly

$\Longleftrightarrow \quad |d\rho|_{\bar g} = 1$ on ∂X. (ρ bdf, $\bar g = \rho^2 g$).

g is called <u>asymptotically hyperbolic</u> (<u>AH</u>) in this case.

<u>Ex.</u> $\bar X = \overline{B^{n+1}}$, $g = g_{\mathbb{H}^{n+1}}$. $\rho(x) = 1 - |x|^2$, $\bar g = 4 g_{Euc}$.

$$|d\rho|^2_{\bar g} = \left| -2 \sum_{i=1}^{n+1} x^i dx^i \right|^2_{\bar g} = \left| \sum x^i dx^i \right|^2_{g_{Euc}} = \sum (x^i)^2 \underset{\substack{\uparrow \\ \text{on } \partial B^{n+1}}}{=} 1.$$

<u>Rem.</u> g C^∞ conf. cpt., AH, Einstein $\Rightarrow$ $\mathrm{Ric}(g) = -ng$.

Convex-cocompact quotients of $\mathbb{H}^{n+1}$

Γ discrete, torsion-free subgp of

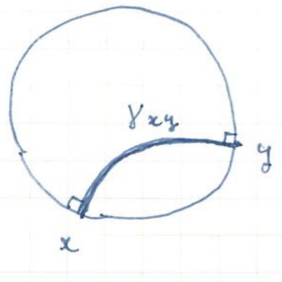
$\mathrm{Isom}(\mathbb{H}^{n+1}) = PO(n+1,1) = O(n+1,1)/\{\pm I\}$

$$\left(O(n+1,1) \curvearrowright \left\{ (\xi^1, \dots, \xi^{n+1}, \xi^0) \in \mathbb{R}^{n+2} \;\middle|\; \sum_{i=1}^{n+1} |\xi^i|^2 - |\xi^0|^2 < 0 \right\} \atop \qquad\qquad\qquad\qquad\qquad \downarrow \atop \qquad\qquad\qquad\qquad\qquad B^{n+1} \right)$$

$\rightsquigarrow$ $X = \Gamma \backslash \mathbb{H}^{n+1}$ a hyperbolic manifold.

$\Lambda(\Gamma) := \overline{\Gamma \cdot o} \cap \partial B^{n+1}$ ($o \in B^{n+1}$ arbitrary) the <u>limit set</u>

$CH(\Lambda(\Gamma)) :=$ (convex hull of $\bigcup_{\substack{x,y \in \Lambda(\Gamma) \\ x \neq y}} \gamma_{xy}$)



$\Gamma \backslash CH(\Lambda(\Gamma))$ the <u>convex core</u>

<u>Def. 1.5</u>

Γ is <u>convex-cocompact</u> $\overset{\text{def}}{\Longleftrightarrow}$ $\Gamma \backslash CH(\Lambda(\Gamma))$ compact.

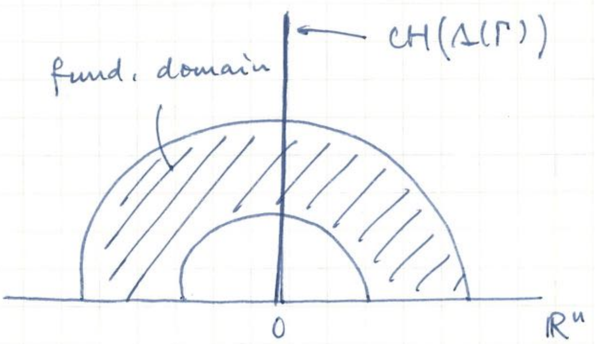
<u>Fact</u>   If Γ convex-cocompact, then $X = \Gamma \backslash \mathbb{H}^{n+1}$ C^∞ conf. cpt.
with $\overline{X} = \Gamma \backslash (\mathbb{H}^{n+1} \cup \Omega(\Gamma))$, $\Omega(\Gamma) = \partial B^{n+1} \backslash \Lambda(\Gamma)$.

<u>Ex.</u>

$\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$ upper-half sp. model

$\gamma \in \mathrm{Isom}(\mathbb{H}^{n+1})$: $x \longmapsto 2x$ (in $\mathbb{R}^n \times \mathbb{R}_+$)

$\Gamma = \langle \gamma \rangle$

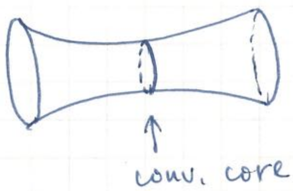


$\Lambda(\Gamma) = \{0, \infty\}$

$CH(\Lambda(\Gamma)) = \{0\} \times \mathbb{R}_+$

convex core $\approx S^1$

$n = 1$



$\partial X = S^1 \sqcup S^1$

$n \geqq 2$

$\partial X = S^1 \times S^{n-1}$
as a C^∞ manifold
(not conformally)

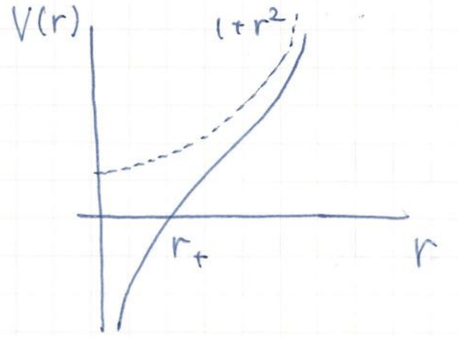
AdS Schwarzschild metric

Hawking - Page (1983)

$$g = V(r)^{-1} dr^2 + V(r)\, dt^2 + r^2 g_{S^2}$$

$$V(r) = 1 + r^2 - \frac{m}{r}, \qquad m > 0 \text{ const. ("mass")}$$

On which space?



g defines a Riem. met. on $(r_+, \infty) \times \mathbb{R}/2\pi\beta\mathbb{Z} \times S^2$

2 ends : $\underbrace{r = r_+}_{\text{fake end}}, \infty$

Let $r = r_+ + s^2$ and $\theta = t/\beta$. When $r \downarrow r_+$,

$$V(r) \sim V'(r_+) \cdot s^2 = \left(2r_+ + \frac{m}{r_+^2}\right) s^2 = \frac{1 + 3r_+^2}{r_+} \cdot s^2.$$

$$\therefore \quad g \sim \left(\frac{1+3r_+^2}{r_+} \cdot s^2\right)^{-1} \cdot (2s\, ds)^2 + \frac{1+3r_+^2}{r_+} s^2 (\beta\, d\theta)^2$$

$$+ (r_+ + s^2)^2 g_{S^2}$$

$$= 4 \cdot \frac{r_+}{1+3r_+^2} ds^2 + \beta^2 \cdot \frac{1+3r_+^2}{r_+} s^2 d\theta^2 + (r_+ + s^2)^2 g_{S^2}$$

Regard (s, θ) as the polar coord. of $\mathbb{R}^2$.

$s = 0$ can be filled smoothly when

$$4 \cdot \frac{r_+}{1+3r_+^2} = \beta^2 \cdot \frac{1+3r_+^2}{r_+},$$

i.e., $\beta = \dfrac{2r_+}{1+3r_+^2}$.

g is a Riem. metric on $X = \mathbb{R}^2 \times S^2$.

Conformal compactification?

$$g = V(r)^{-1} dr^2 + V(r) \cdot \beta^2 d\theta^2 + r^2 g_{S^2}$$

Let $\rho = 1/r$. Then

$$g = \frac{1}{1 + \frac{1}{\rho^2} - m\rho} \cdot \left(\frac{d\rho}{\rho^2}\right)^2 + \left(1 + \frac{1}{\rho^2} - m\rho\right) \cdot \beta^2 d\theta^2 + \frac{1}{\rho^2} g_{S^2}$$

$$= \frac{1}{\rho^2}\left(\frac{1}{\rho^2 + 1 - m\rho^3} d\rho^2 + \beta^2 (\rho^2 + 1 - m\rho^3) d\theta^2 + g_{S^2} \right).$$

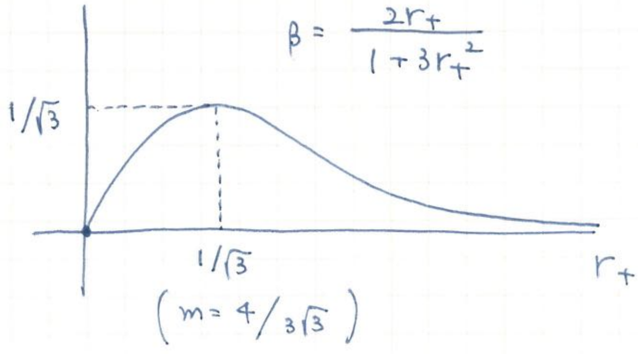
$\bar{X} = X \cup \{\rho = 0\}$ $\quad (\rho, \theta, \underset{S^2}{\underset{\cap}{x}})$ smooth coords away from $r = r_+$

Conformal infinity $\gamma = [\rho^2 g |_{T\partial X}] = [\beta^2 d\theta^2 + g_{S^2}]$

$\therefore \quad \partial X = S^1(\beta) \times S^2(1)$ conformally.

Non-uniqueness of AHE filling (Anderson 2003)

r_+ is monotonically increasing in m.



$0 < \beta < \frac{1}{\sqrt{3}}$

$\Rightarrow \exists\, m_1, m_2, \ m_1 \neq m_2,$

with conf. inf. $S^1(\beta) \times S^2(1)$.

g_{m_1}, g_{m_2} not isometric. In fact,

$$\sup_{\mathbb{R}^2 \times S^2} |\mathrm{Riem}(g_m)|^2 = 6 + 3 \cdot \left(1 + \frac{1}{r_+^2}\right)^2.$$

Non-existence phenomenon

Thm. 1.6 (Gursky - Han 2017)

$\exists \infty$ conformal classes on S^7

which are not realized as conf. infinity of AHE on B^8.

GAGT 2017 Lecture #2

The Graham-Lee Theorem (I)

Thm. 2.1 (Graham-Lee 1991)

Let $n \geqq 3$, $k \geqq 2$, $0 < \alpha < 1$.

h a $C^{k,\alpha}$ Riem. met. on S^n, $C^{k,\alpha}$-close to the std. met. h_{std}

$\Longrightarrow$ $\exists g$ an Einstein AH metric of class $C^{k,\alpha}$ on B^{n+1}

w/ conf. inf. $[h]$. $\uparrow$ To be defined in Def. 2.3

Rem. If n even, for generic h g is not C^∞ conf. cpt.
(Fefferman-Graham 1985)

Function spaces

E "tensor bundle" over $\mathbb{H}^{n+1}$
(e.g., $\Sigma^r = \mathrm{Sym}^r T^*\mathbb{H}^{n+1}$, $\Lambda^r = \Lambda^r T^*\mathbb{H}^{n+1}$)

$H^s(E)$ $(s \in \mathbb{Z})$

$s \geqq 0$ $\quad \| u \|_{H^s}^2 = \sum_{j=0}^{s} \| \nabla^j u \|$

$s < 0$ $\quad H^s(E) = (H^{|s|}(E^*))^*$, $\quad \| \cdot \|_{H^s}$ = operator norm

Rem. $C_c^\infty(E) \subset H^s(E)$ dense, $s \geqq 0$

$C^{k,\alpha}(E)$ $\quad (k \in \mathbb{Z}_{\geq 0},\ 0 \leq \alpha < 1)$

$\alpha = 0$ $\qquad \| u \|_{C^k} = \sum_{j=0}^{k} \sup_{x \in \mathbb{H}^{n+1}} | \nabla^j u(x) |_{g_{\mathbb{H}^{n+1}}}$

$0 < \alpha < 1$ $\qquad [u]_\alpha = \sup_{\substack{x, y \in \mathbb{H}^{n+1} \\ x \neq y}} \dfrac{| \Pi_{y \to x} u(y) - u(x) |_{g_{\mathbb{H}^{n+1}}}}{d_{\mathbb{H}^{n+1}}(x, y)^\alpha}$

($\Pi_{y \to x}$ parallel transl. along geodesic)

$$\| u \|_{C^{k,\alpha}} = \| u \|_{C^k} + [\nabla^k u]_\alpha$$

Def. 2.2

(1) Fix a bdf $\rho \in C^\infty(\overline{B^{n+1}})$. For $\delta \in \mathbb{R}$,

$H^s_\delta(E) := \rho^\delta H^s(E)$, $\qquad \| u \|_{H^s_\delta} := \| \rho^{-\delta} u \|_{H^s}$,

$C^{k,\alpha}_\delta(E) := \rho^\delta C^{k,\alpha}(E)$, $\qquad \| u \|_{C^{k,\alpha}_\delta} := \| \rho^{-\delta} u \|_{C^{k,\alpha}}$.

(2) $C^\infty(E) := \bigcap_{k=0}^{\infty} C^k(E)$,

$C^\infty_\delta(E) := \bigcap_{k=0}^{\infty} C^k_\delta(E) = \rho^\delta C^\infty(E)$.

Def. 2.3

Let $k \geq 2$, $0 \leq \alpha < 1$. $\quad g$ a $C^{k,\alpha}$ Riem. met. on B^{n+1}.

(1) g is <u>$C^{k,\alpha}$ conf. cpt.</u> if $\bar{g} = \rho^2 g$ extends to a $C^{k,\alpha}$ Riem. met. on $\overline{B^{n+1}}$ ($\rho \in C^\infty(\overline{B^{n+1}})$ bdf).

g is <u>AH</u> $\overset{\text{def}}{\iff} K_g \to -1$ at ∂X

$\iff |d\rho|_{\bar{g}} \equiv 1$ on ∂X.

(2) g is an AH metric of class $C^{k,\alpha}$ if

$$g = g_0 + u, \quad g_0 \ \ C^{k,\alpha} \text{ conf. cpt., AH,}$$
$$u \in C^{k,\alpha}_\delta(\Sigma^2) \quad \text{for some } \delta > 0.$$

(" AH metric of class C^∞ " is similarly defined.)

Let $\bar{g}_0 = \rho^2 g_0$ and $h = \bar{g}_0|_{T\partial X}$. $[h]$ is the conf. infinity of g.

Ex. $u = \frac{1}{\rho^2} \cdot \rho^k \log \rho \cdot \bar{v}$, $\quad \bar{v} \in C^\infty(\overline{B^{n+1}}, \Sigma^2)$

$\rho^2 u$ is only $C^{k-1,\alpha}$ up to $\overline{B^{n+1}}$, but $u \in C^\infty_\delta$, $\forall \delta < 1$.

Rem. Typical choices of ρ:

(1) $\rho(x) = 1 - |x|^2$.

(2) $\rho(x) = e^{-r(x)}$, $\quad r(x) := d_{\mathbb{H}^{n+1}}(0, x) = \frac{1-|x|}{1+|x|}$

(regularized around 0).

Lem. 2.4

g an AH metric of class C^∞.

(1) $R(g) \in C^\infty$.

(2) ∇ maps $H^s_\delta(E)$ into $H^{s-1}_\delta(\Lambda^1 \otimes E)$,

$C^{k,\alpha}_\delta(E)$ into $C^{k-1,\alpha}_\delta(\Lambda^1 \otimes E)$.

[Proof] Note that $\nabla \rho \in C^\infty_1(\Lambda^1)$. //

Cor. 2.5

Any "geometric" diff. op. $\Gamma(E) \to \Gamma(E)$ of order m gives rise to bdd operators

$$H^{s+m}_\delta(E) \longrightarrow H^s_\delta(E),$$
$$C^{k+m,\alpha}_\delta(E) \longrightarrow C^{k,\alpha}_\delta(E).$$

Basic properties

Prop. 2.6 (Sobolev embedding)

(1) $s > k + \frac{n+1}{2}$ $\quad H^s_\delta \hookrightarrow C^k_\delta$ cont.

(2) $0 < \alpha < 1$

$s \geq k + \alpha + \frac{n+1}{2}$ $\quad H^s_\delta \hookrightarrow C^{k,\alpha}_\delta$ cont.

Prop. 2.7 $\quad C^0_{n/2+\varepsilon} \hookrightarrow L^2$ cont., $\quad \varepsilon > 0$.

[Proof]

$$g_{\mathbb{H}^{n+1}} = \frac{4 g_{Euc}}{\rho^2}, \quad \therefore \; dV_{g_{\mathbb{H}^{n+1}}} = \frac{2^{n+1} dV_{Euc}}{\rho^{n+1}}.$$

$$\int_0^{\rho_0} (\rho^{n/2+\varepsilon})^2 \cdot \frac{1}{\rho^{n+1}} \, d\rho = \int_0^{\rho_0} \rho^{-1+2\varepsilon} \, d\rho < \infty. \quad //$$

Prop. 2.8

(1) (Rellich compactness)

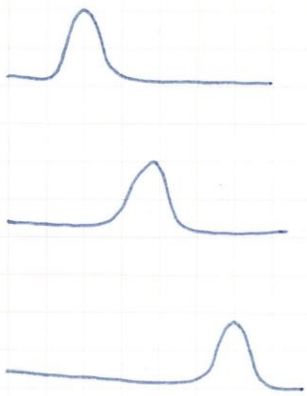
$$s_1 > s_2, \quad \delta_1 > \delta_2 \quad \Rightarrow \quad H^{s_1}_{\delta_1} \hookrightarrow H^{s_2}_{\delta_2} \quad \text{compact.}$$

(2) (Ascoli-Arzelà compactness)

$$k_1 + \alpha_1 > k_2 + \alpha_2, \quad \delta_1 > \delta_2 \qquad (\alpha_1, \alpha_2 \in [0,1))$$

$$\Rightarrow \quad C^{k_1, \alpha_1}_{\delta_1} \hookrightarrow C^{k_2, \alpha_2}_{\delta_2} \quad \text{compact.}$$

Rem. Compactness fails when $\delta_1 = \delta_2$.



Framework of the proof of Thm. 2.1

h given on S^n. Want to solve $Ric(g) = -ng$.

Lem. 2.9 $\quad g_1 := \frac{1}{\rho^2} \bar{g}_1, \quad \bar{g}_1 \in C^{k,\alpha}(\overline{B^{n+1}}, \Sigma^2)$

any extention of h, $|d\rho|_{\bar{g}_1} \equiv 1$ on ∂B^{n+1}.

This is a 1^{ST} approximate sol'n, i.e.,

$$Ric(g_1) + ng_1 \in C^{k-2,\alpha}_1(\Sigma^2).$$

Consider

$$(*) \quad \begin{cases} \mathrm{Ric}(g) = -ng \\ \mathcal{B}_{g_1}(g) := \delta_{g_1} g + \frac{1}{2} d\, \mathrm{tr}_{g_1} g = 0 \end{cases}$$

gauge-fixing cond.

<u>Lem. 2.10</u>

If $\mathrm{Ric}(g_1) < 0$, $u \in C^{k,\alpha}_1(\Sigma^2)$ small,

$(*) \iff g = g_1 + u$ satisfies

$$\mathrm{Ric}(g) + ng + \delta_g^* \mathcal{B}_{g_1}(u) = 0.$$

[Proof] $\underline{\Rightarrow}$ Trivial (note: $\mathcal{B}_{g_1}(g) = \mathcal{B}_{g_1}(u)$).

$\underline{\Leftarrow}$ Since $\mathcal{B}_g \mathrm{Ric}(g) = 0$ (Bianchi identity), we get

$$\underbrace{\mathcal{B}_g \delta_g^*}_{= \frac{1}{2}\left(\nabla_g^* \nabla_g - \mathrm{Ric}(g)\right)} \mathcal{B}_{g_1}(u) = 0.$$

Cor. 2.5 implies $\mathcal{B}_{g_1}(u) \in C_1^{k-2,\alpha}$. Integration-by-parts gives no bdry term, hence $\mathcal{B}_{g_1}(u) = 0$ ($\because \mathrm{Ric}(g) < 0$). //

Let $\mathcal{E}_{g_1}(u) := \mathrm{Ric}(g_1 + u) + n(g_1 + u) + \delta^*_{g_1 + u} \mathcal{B}_{g_1}(u)$.

$$\begin{array}{ccc} C^{k,\alpha}(\mathrm{Sym}^2 T^* S^n) \times C_1^{k,\alpha}(\Sigma^2) & \longrightarrow & C_1^{k-2,\alpha}(\Sigma^2) \\ \Psi & & \Psi \\ (h, \quad u) & \longmapsto & \mathcal{E}_{g_1}(u) \\ \text{near } (h_{std}, 0) & & \end{array}$$

Taken smoothly in h so that $\mathrm{Ric}(g_1) < 0$

Thm. 2.1 follows if $\mathcal{E}'_{g_{\mathbb{H}^{n+1}}}(0) : C_1^{k,\alpha}(\Sigma^2) \to C_1^{k-2,\alpha}(\Sigma^2)$ invertible.

Linearization.

$$L := \mathcal{E}'_{g_{\mathbb{H}^{n+1}}}(0)$$
$$= \frac{1}{2}\left(\nabla^*\nabla - 2\mathring{R}\right) \quad : \quad C^{k+2,\alpha}_\delta \longrightarrow C^{k,\alpha}_\delta$$

$$(\mathring{R}u)_{ij} = - R_i{}^k{}_j{}^l u_{kl}$$
$$= -\left(g_{ij}g^{kl} - \delta_i{}^l \delta_j{}^k\right) u_{kl}$$
$$= u_{ij} - (\mathrm{tr}_{g_{\mathbb{H}^{n+1}}} u)(g_{\mathbb{H}^{n+1}})_{ij}.$$

Claim 2.11

$L : C^{k+2,\alpha}_\delta \longrightarrow C^{k,\alpha}_\delta$

is invertible for $\forall k \geq 0$, $0 < \forall \alpha < 1$, $0 < \forall \delta < n$.

● Green kernel of L

Lem. 2.12 (Coerciveness)

$\exists C > 0$ s.t.

$$\| u \|_{L^2} \leq C \| Lu \|_{L^2}, \qquad u \in C^\infty_c(\Sigma^2).$$

[Proof] Regard u as a Λ^1-valued 1-form.

Use a Weizenböck formula involving cov. ext. diff. //

Cor. 2.13 $\quad L : H^{s+2} \longrightarrow H^s$ invertible.

In the proof of Cor. 2.13, we need:

Prop. 2.14 (Elliptic estimates)

E a tensor bundle.

$P : \Gamma(E) \to \Gamma(E)$ elliptic geometric linear diff. op. order m

(1) $\|u\|_{H^{s+m}_\delta} \leqq C\left(\|Pu\|_{H^s_\delta} + \|u\|_{H^s_\delta}\right)$ $\quad \forall s \in \mathbb{Z}$

(2) $\|u\|_{C^{k+m,\alpha}_\delta} \leqq C\left(\|Pu\|_{C^{k,\alpha}_\delta} + \|u\|_{C^{k,\alpha}_\delta}\right)$ $\quad \forall k \geqq 0,\ 0 < \forall \alpha < 1$

$L^{-1} : C^\infty_c \longrightarrow C^\infty \subset \mathcal{D}'$ (distributions) well-def'd by Cor. 2.13.

Def. 2.15

$G \in \mathcal{D}'(\mathbb{H}^{n+1} \times \mathbb{H}^{n+1}, \mathrm{Hom}(\pi_2^* \Sigma^2_1, \pi_1^* \Sigma^2))$

is the Schwarz kernel of $L^{-1} : C^\infty_c \longrightarrow \mathcal{D}'$.

Prop. 2.16

(1) $G|_{(\mathbb{H}^{n+1} \times \mathbb{H}^{n+1}) \setminus (\mathrm{diag})}$ is C^∞.

(2) Let $G(x) := G(x, 0) \in C^\infty(\mathbb{H}^{n+1} \setminus \{0\}, \mathrm{Hom}(\Sigma^2_0, \Sigma^2))$.

Then

(2-1) $G(x)$ is L^2 at infinity (i.e., $G(x) \in L^2(\mathbb{H}^{n+1} \setminus U)$)

nbhd of 0

(2-2) $G(x) \in L^1_{loc}$ around 0 and

$$(L^{-1}u)(x) = \int_{\mathbb{H}^{n+1}} G(x, y)\, u(y)\, dy, \quad u \in C^\infty_c.$$

GAGT 2017 Lecture #3

The Graham-Lee Theorem (2)

Thm. 2.1

$n \geqq 3$, $k \geqq 2$, $0 < \alpha < 1$

h a $C^{k,\alpha}$ Riem. met. on S^n, close to h_{std}

$\Rightarrow$ $\exists g$ an Einstein AH metric of class $C^{k,\alpha}$ on B^{n+1} with conf. inf. $[h]$

Claim 2.11

$$L = \frac{1}{2}(\nabla^* \nabla - 2\mathring{R}) : \Gamma(\Sigma^2) \to \Gamma(\Sigma^2),$$

the linearized gauged Einstein op. associated to $g_{\mathbb{H}^{n+1}}$, is invertible as $C^{k+2,\alpha}_\delta \to C^{k,\alpha}_\delta$, $\forall k \geqq 0$, $0 < \forall \alpha < 1$, when $0 < \delta < n$.

General Setting 3.1

E a tensor bundle over $\mathbb{H}^{n+1}$

P an elliptic, formally self-adjoint geometric linear diff. op. $\Gamma(E) \to \Gamma(E)$

P satisfies the coerciveness estimate

$$\|u\|_{L^2} \leqq C \|Pu\|_{L^2}, \qquad u \in C_c^\infty(E).$$

$P^{-1} : C_c^\infty \to \mathcal{D}'$ well-def'd, continuous

$G \in \mathcal{D}'(\mathbb{H}^{n+1} \times \mathbb{H}^{n+1}, \mathrm{Hom}(\pi_2^* E, \pi_1^* E))$ the Schwarz kernel

the Green kernel of P.

Prop. 2.16'

(1) $G|_{(\mathbb{H}^{n+1}\times\mathbb{H}^{n+1})\setminus(\mathrm{diag})}$ is C^∞, $G(y,x) = G(x,y)^*$.

(2) Let $V := E_0$.

$G(x) := G(x,0) \in C^\infty(\mathbb{H}^{n+1}\setminus\{0\}, \mathrm{Hom}(V, E))$.

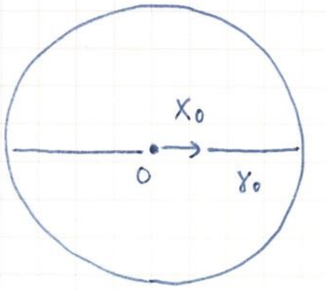
(2-1) $G(x)$ is L^2 at infinity.

(2-2) $G(x) \in L^1_{loc}$ around 0 and

$$(P^{-1}u)(x) = \int_{\mathbb{H}^{n+1}} G(x,y)\, u(y)\, dy, \qquad u \in C_c^\infty.$$

In particular, $PG = \delta_0 \cdot \mathrm{id}_V$.

Reduction to a 1-variable function



$$\mathbb{H}^{n+1} = G/K$$

$$G = \mathrm{Isom}(\mathbb{H}^{n+1}) = PO(n+1,1)$$

$$= O(n+1,1)/\{\pm I\}$$

$K = O(n+1)$ isotropic subgp at 0

We fix a unit vector $X_0 \in T_0\mathbb{H}^{n+1}$, giving a geodesic γ_0.

$$H = O(n) = \{k \in K \mid k_* X_0 = X_0\}.$$

Since P is G-invariant,

$$G(g\cdot x, g\cdot y) = g_* \circ G(x,y) \circ (g^{-1})_*, \qquad g \in G.$$

In particular,

$$G(k\cdot x) = k_* \circ G(x) \circ (k^{-1})_*, \qquad k \in K.$$

Thus it suffices to investigate $G(\gamma_0(r))$ for $r>0$.

<u>Def. 3.2</u>

$$G(r) := \Pi_{\gamma(r)\to 0} \circ G(\gamma_0(r)) : \quad V \to V.$$

This gives a C^∞ function $(0,\infty) \to \mathrm{End}(V)$.

<u>Rem.</u> Actually, $G = G(r)$ takes values in $\mathrm{End}_H(V)$.

Geometric differential operators in polar coordinates

$\mathbb{H}^{n+1}$ is a symmetric sp.

$$\rightsquigarrow \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \quad K\text{-invariant}, \qquad \mathfrak{m} \cong T_0\mathbb{H}^{n+1}$$

$$E = G \times_K V = (G\times V)/\sim \qquad (g, v) \sim (gk, k^{-1}\cdot v),\ k \in K$$

$$\Gamma(E) = C^\infty(G, V)^K$$
$$= \{ f \in C^\infty(G, V) \mid f(gk) = k^{-1}\cdot f(g),\ k \in K \}$$

<u>Lem. 3.3</u>

Any G-inv. lin. diff. op. $\Gamma(E) \to \Gamma(E)$ can be expressed as

$$\sum_i A_i X_1 \cdots X_{m_i}, \qquad A_i \in \mathrm{End}(V),$$
$$X_j \in \mathfrak{m} \left(\begin{array}{l} \text{interpreted as left-inv} \\ \text{v.f. on } G \end{array} \right)$$

For $r \in (0,\infty)$, $E|_{S_r}$ K-homog. bundle over $S_r = K/H$.

If we identify $E_{\gamma(r)}$ and V by $\Pi_{\gamma(r)\to o}$,

$$\Gamma(E|_{S_r}) \cong C^\infty(K, V)^H.$$

Thus

$$\Gamma(E|_{\mathbb{H}^{n+1}\setminus\{o\}}) \cong C^\infty((0,\infty)\times K, V)^H.$$

Let $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{q}$ H-invariant decomp ($\leftarrow$ S_r symm. sp.)

Prop. 3.4

Any G-inv. diff. op. $P : \Gamma(E) \to \Gamma(E)$ of order m is expressed as

$$\sum_{i=0}^{m} \underbrace{A_i}_{\in \operatorname{End}(V)} \partial_r^i + \sum_j O(e^{-r}) \cdot \underbrace{Y_1 \cdots Y_{n_j}}_{\substack{\text{elements of } \mathfrak{q} \\ \text{(interpreted as left-inv} \\ \text{v.f. on } K)}} \partial_r^{l_j} \ (l_j < m).$$

Def. 3.5

$$I_P(s) := \sum_{i=0}^{m} (-1)^i A_i s^i \in \operatorname{End}(V)[s]$$

(in fact, $\in \operatorname{End}(V)^H[s]$)

the indicial polynomial of P.

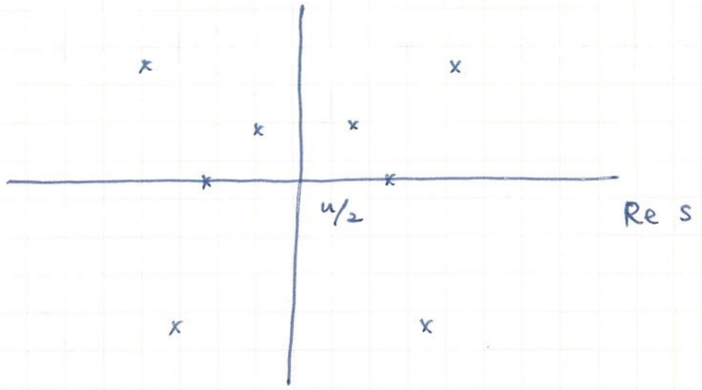
$$\Sigma_P := \{ s \in \mathbb{C} \mid I_P(s) \notin GL(V_{\mathbb{C}}) \}$$

$\ni s$ indicial root

Prop. 3.6

(1) $I_{P^*}(s) = I_P(n-\bar{s})^*$.

(2) If $P^* = P$, then Σ_P is symmetric about $\mathrm{Re}\, s = \frac{n}{2}$.



Def. 3.5 (cont'd) $R_P := \inf_{s \in \Sigma_P} \left| \mathrm{Re}\, s - \frac{n}{2} \right|$ indicial radius

Ex. $P = L = \frac{1}{2}(\nabla^*\nabla - 2\mathring{R})$

If we take $X_1, \dots, X_n \in \mathfrak{m}_0 = \langle X_0 \rangle^\perp \subset \mathfrak{m}$,

$$(Lu)_{00} = -\frac{1}{2}(\partial_r^2 + n\partial_r - 2n)\, u_{00} + \left(\begin{array}{c}\text{terms containing} \\ q\text{-derivatives}\end{array}\right)$$

$$(Lu)_{0i} = -\frac{1}{2}(\partial_r^2 + n\partial_r - 2)\, u_{0i} + \cdots$$

$$(Lu)_{ij} = -\frac{1}{2}(\partial_r^2 + n\partial_r)\, u_{ij} + (g^{kl}u_{kl})\, g_{ij} + \cdots$$

$\therefore$ W.r.t. $\Sigma^2 \mathfrak{m}^* = \eta^2 \oplus \Sigma(\eta \otimes \mathfrak{m}_0) \oplus \langle g|_{\mathfrak{m}_0} \rangle \oplus (\mathrm{tf}\, \Sigma^2 \mathfrak{m}_0^*)$,

$$I_L(s) = -\frac{1}{2}\begin{pmatrix} s^2 - ns - 2n & & & \\ & s^2 - ns - 2 & & \\ & & s^2 - ns - 2n & \\ & & & s^2 - ns \end{pmatrix}.$$

$$\therefore \quad \Sigma_L = \left\{ 0, n, \frac{n \pm \sqrt{n^2+8}}{2}, \frac{n \pm \sqrt{n^2+8n}}{2} \right\}$$

$$R_L = n/2.$$

Estimate of $G(r)$, invertibility of P

Since $P(G(x)v) = 0 \in C^\infty(\mathbb{H}^{n+1} \setminus \{0\}, E)$, $v \in V$,

$$I_P(-\partial_r)(G(r)v) = \sum_{l=0}^{m-1} O(e^{-r}) \cdot \partial_r^l (G(r)v).$$

One concludes:

<u>Prop. 3.7</u>

$G = G(r) \in C^\infty((0,\infty), \mathrm{End}(V))$ satisfies

$$|G(r)| = O\left(e^{-(n/2 + R_P) r}\right) \quad \text{as } r \to \infty.$$

<u>Thm. 3.8</u>

$P : \Gamma(E) \to \Gamma(E)$ satisfying Gen. Setting 3.1 with $R_P > 0$.

Then

$$P : H_\delta^{s+m} \to H_\delta^s, \qquad s \in \mathbb{R}$$

$$P : C_{n/2+\delta}^{k+m,\alpha} \to C_{n/2+\delta}^{k,\alpha}, \qquad k \geq 0,\ 0 < \alpha < 1$$

are invertible for $-R_P < \delta < R_P$.

Claim 2.11 follows, and so does Thm. 2.1.

GAGT 2017 Lecture #4

Fredholm theorem for geometric operators on AH spaces

$\overline{X}$ a compact C^∞ mfd-with-bdry

Thm. 4.1

g an AH metric of class C^∞ on X.

$P : \Gamma(E) \to \Gamma(E)$ satisfying Gen. Setting 3.1
(coerciveness is assumed on $\mathbb{H}^{n+1}$) with $R_P > 0$.

(1) $s \in \mathbb{Z}$, $-R_P < \delta < R_P$

$\exists Q_1 : H^s_\delta(X, E) \to H^{s+m}_\delta(X, E)$ s.t.

$Q_1 P u = u + K_1 u$, $u \in H^{s+m}_\delta$,

$K_1 : H^{s+m}_\delta \to H^{s+m+1}_{\delta_1}$ bdd, $\delta < \forall \delta_1 < \min(\delta+1, R_P)$,

$\exists Q_2 : H^{s+m}_\delta(X, E) \to H^s_\delta(X, E)$ s.t.

$P Q_2 u = u + K_2 u$, $u \in H^{s+m}_\delta$,

$K_2 : H^{s+m}_\delta \to H^{s+m+1}_{\delta_1}$ bdd, ______ .

(2) The same for Hölder spaces.

Rem.

(1) AH metric
$H^s_\delta(X, E)$, $C^{k,\alpha}_\delta(X, E)$ } to be defined.

(2) "class C^∞" is assumed for simplicity.

Cor. 4.2 For $-R_P < \delta < R_P$,

(1) $P : H^{s+m}_\delta \to H^s_\delta$

$P : C^{k+m,\alpha}_{n/2+\delta} \to C^{k,\alpha}_{n/2+\delta}$ } Fredholm.

(dim ker $P < \infty$, dim coker $P < \infty$)

(2) $\ker(P : H^{s+m}_\delta \to H^s_\delta)$ is independent of s, δ.

$\operatorname{coker}(P : H^{s+m}_\delta \to H^s_\delta)$ is ———————

$$\left(H^s_\delta / P(H^{s+m}_\delta) \cong H^{s'}_{\delta'} / P(H^{s'+m}_{\delta'}) \quad \text{naturally} \right).$$

(2') The same for Hölder spaces.
Moreover, $(\text{co})\ker_{\text{Sobolev}} = (\text{co})\ker_{\text{Hölder}}$.

(3) If

$$\ker_{L^2} P = \{ u \in L^2 \mid Pu = 0 \text{ as distribution} \}$$
$$= \ker(P : H^m \to L^2)$$

vanishes, then

$P : H^{s+m}_\delta \to H^s_\delta$

$P : C^{k+m,\alpha}_{n/2+\delta} \to C^{k,\alpha}_{n/2+\delta}$ } invertible.

Cor. 4.3

Graham-Lee Thm extends to
AH-Einstein manifolds (X, g) of class C^∞ with $\ker_{L^2} L = 0$.

Rem. $K_g < 0 \Rightarrow \ker_{L^2} L = 0$.

Lee's Möbius coordinates

Prop. 4.4

(X, g) AH C^∞ conf. cpt. mfd, Then

$\exists \{ \Phi_\lambda : \underset{\substack{\text{open} \\ \cap \\ X}}{U_\lambda} \xrightarrow{\text{diffeo}} B_2 \}$ $\quad (B_2 := B(0,2) \subset \mathbb{H}^{n+1})$ s.t.

(i) $\{U_\lambda\}$ covers X.

Moreover, $\{U'_\lambda\}$ already covers X, where $U'_\lambda := \Phi_\lambda^{-1}(B_1)$.

(ii) $\| (\Phi_\lambda)_* g - g_{\mathbb{H}^{n+1}} \|_{C^l(B_2)} < C_l$ for $\forall l$,

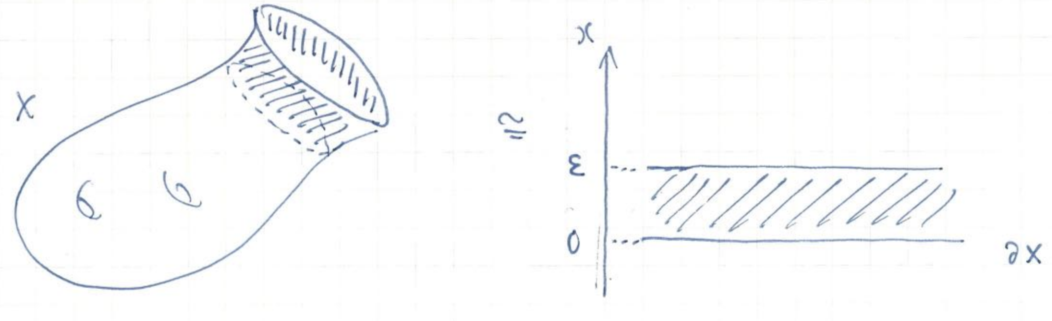
$\sup_{B_2} |((\Phi_\lambda)_* g)^{-1}|_{g_{\mathbb{H}^{n+1}}} < C$.

(iii) $\{U_\lambda\}$ is uniformly locally finite.

Lem. 4.5

(X, g) AH conf. cpt. $h \in$ (conf. infinity).

Then one can identify as

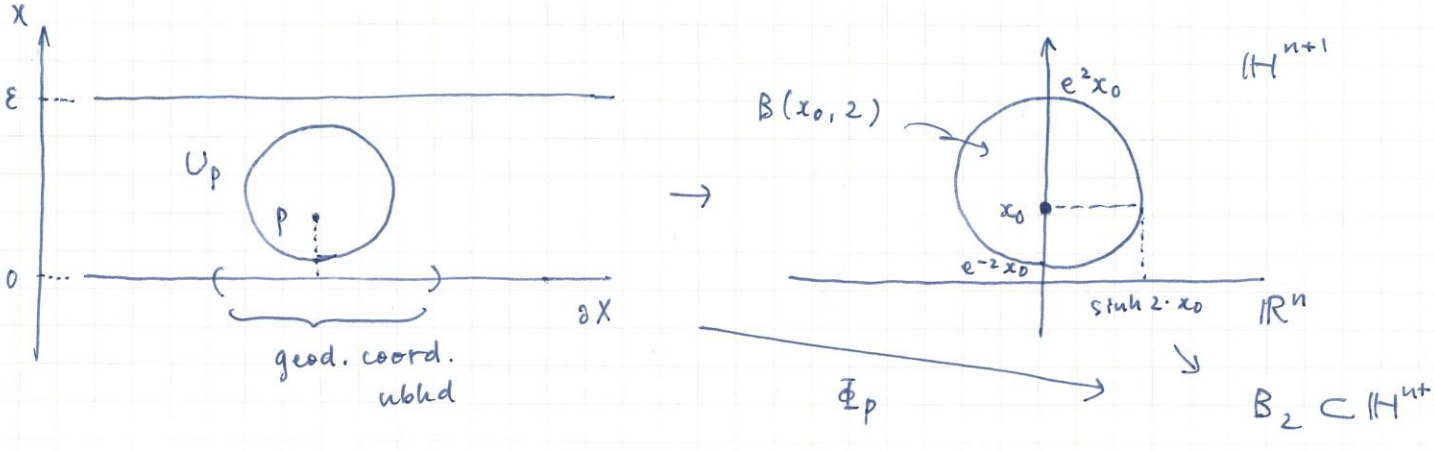


so that $g = \dfrac{dx^2 + h_x}{x^2}$ in this region,

where h_x a Riem. met. on each $\{x = \text{const.}\}$, $h_0 = h$.

[Proof of Prop. 4.4]

Introduce the identification in Lem. 4.3.



Let $r_{\partial X} = \left(\text{inj. rad. of } (\partial X, h)\right)$.

For $p \in X$ with $x < \min\left(\frac{\varepsilon}{e^2}, \frac{r_h}{\sinh 2}\right)$, we take

$\Phi_p : U_p \to B_2$ as illustrated above. Then (ii) is satisfied.

$\exists r_0 < \exists r_1$ s.t.

$$B_g(p, r_0) \subset U_p' \subset U_p \subset B_g(p, r_1).$$

If $\{p_\lambda\}$ is a maximal set of points for which $B_g(p_\lambda, r_0/2)$ are disjoint, then by the maximality $\{U_{p_\lambda}'\}$ covers X. If $p \in X$ and $p \in U_{p_\lambda}$, then

$$B_g(p_\lambda, r_0/2) \subset B_g(p, r_1 + r_0/2).$$

Since $B_g(p_\lambda, r_0/2)$ are disjoint, the # of such λ can be uniformly bounded by the volume comparison theorem. //

Function spaces

$H^s(X,E)$, $s \geq 0$

$$\|u\|_{H^s}^2 := \sum_{j=0}^{s} \|\nabla^j u\|_{L^2}^2$$

Equivalent norm is given by

$$\|u\|_{H^s}^2 := \sum_{\lambda} \|\Phi_* u\|_{H^s(B_2, g_{\mathbb{H}^{n+1}})}^2 .$$

$H^s(X,E)$, $s<0$ $\qquad H^s(X,E) := (H^{|s|}(X, E^*))^*.$

$C^{k,\alpha}(X,E)$

$$\|u\|_{C^{k,\alpha}} := \sum_{\lambda} \|\Phi_* u\|_{C^{k,\alpha}(B_2, g_{\mathbb{H}^{n+1}})} .$$

$$H^s_\delta := \rho^\delta H^s, \qquad C^{k,\alpha}_\delta := \rho^\delta C^{k,\alpha}$$

Props. 2.6, 2.7, 2.8 remain true.

Def. 4.6

$\overline{X}$ compact C^∞ mfd-with-bdry.

g a $C^{k,\alpha}$ $[C^\infty]$ Riem. met. on X

(X,g) is an AH space of class $C^{k,\alpha}$ $[C^\infty]$

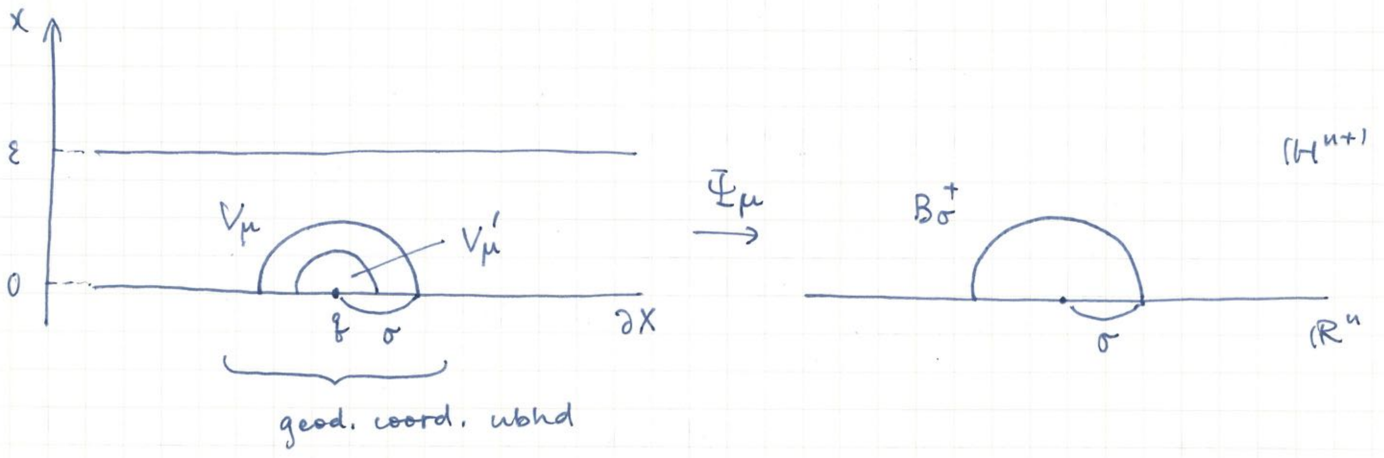
$\overset{\text{def}}{\Longleftrightarrow}$ $g = g_0 + u$, g_0 $C^{k,\alpha}$ $[C^\infty]$ conf. cpt., AH,

$u \in C^{k,\alpha}_\kappa(X, \Sigma^2)$ $[C^\infty_\kappa(X,\Sigma^2)]$

for some $\kappa > 0$.

Construction of a parametrix (Proof of Thm. 4.1)

Boundary Möbius coordinates.



(i) $\{V_\mu\}$ covers ∂X.

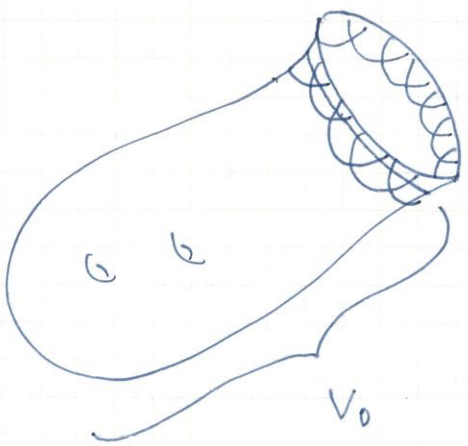
Moreover, $\{V'_\mu\}$ already covers ∂X.

(ii) $\|(\Phi_\mu)_* g - g_{\mathbb{H}^{n+1}}\|_{C^\ell(B^+_\sigma)} \leqq C_\ell\, \sigma^\kappa$,

$$\sup_{B^+_\sigma} \left|((\Phi_\mu)_* g)^{-1}\right|_{g_{\mathbb{H}^{n+1}}} \leqq C \sigma^\kappa.$$

(iii) $\{V_\mu\}$ uniformly locally finite,

bound N being independent of σ.



$$X = \left(\bigcup_\mu V_\mu\right) \cup V_0.$$

Partition of unity

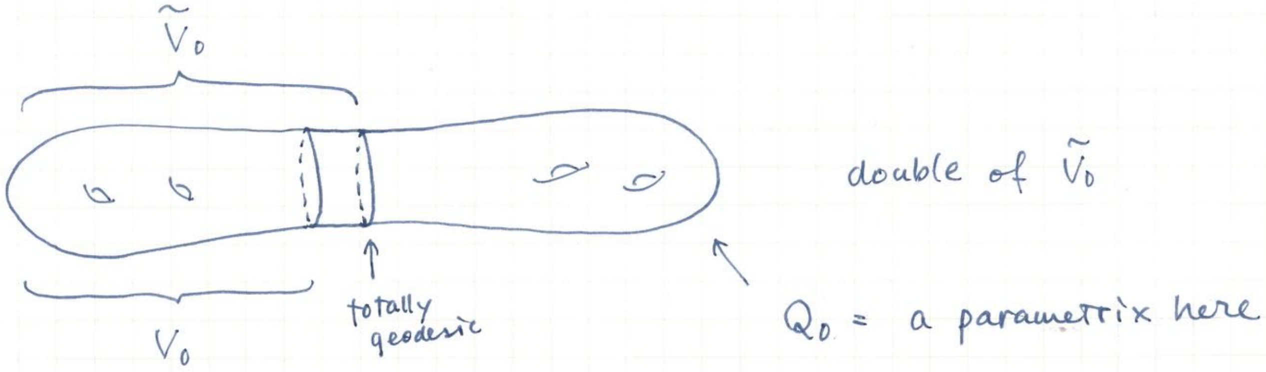
$\{\chi_\mu^2\}$, χ_0^2

so that $\chi_\mu \in C^\infty(\bar{X})$.

$\rightsquigarrow \nabla^j \chi_\mu \in C_1^\infty(\bar{X})$, $\forall j \geqq 1$.

Let $Q_1' : H^s_\delta \to H^{s+m}_\delta$ be defined by

$$Q_1' u = \sum_\mu x_\mu P^{-1}_{\mathbb{H}^{n+1}} (x_\mu u) + x_0 Q_0 (x_0 u)$$



$$Q_1' P u = \sum_\mu x_\mu P^{-1}_{\mathbb{H}^{n+1}} P (x_\mu u) - \sum_\mu x_\mu P^{-1}_{\mathbb{H}^{n+1}} ([P, x_\mu] u)$$

$$+ x_0 Q_0 P (x_0 u) - x_0 Q_0 ([P, x_0] u)$$

$$= u + \boxed{\sum_\mu x_\mu P^{-1}_{\mathbb{H}^{n+1}} (P - P_{\mathbb{H}^{n+1}}) (x_\mu u)} =: Su$$

$$+ x_0 K_{0,1} (x_0 u) \qquad (K_{0,1} : H^{s+m} \to H^{s+m+1} \text{ on the double})$$

$$- \sum x_\mu P^{-1}_{\mathbb{H}^{n+1}} ([P, x_\mu] u) \qquad H^{s+m}_\delta \to H^{s+m+1}_{\delta_1}$$

$$- x_0 Q_0 ([P, x_0] u) \qquad H^{s+m}_\delta \to H^{s+m+1}_{\delta_1}$$

$S : H^{s+m}_\delta \to H^{s+m}_\delta$ with norm $N C \sigma^k$.

$\rightsquigarrow$ If $\sigma \ll 1$, $I + S$ invertible.

$Q_1 = (I + S)^{-1} Q_1'$ has the desired property.

The rest can be shown similarly.