Strong Koszulness of the toric ring associated to a cut ideal

Kazuki Shibata (Rikkyo University)*

Let $G$ be a finite simple graph on the vertex set $[n] = \{1, \ldots, n\}$ with the edge set $E(G)$. For two subsets $A$ and $B$ of $[n]$ such that $A \cap B = \emptyset$ and $A \cup B = [n]$, the $(0, 1)$-vector $\delta_{A|B}(G) \in \mathbb{Z}^{E(G)}$ is defined as

$$\delta_{A|B}(G)_{ij} = \begin{cases} 1 & \text{if } |A \cap \{i, j\}| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $ij$ is an edge of $G$. Let

$$X_G = \left\{ \left( \delta_{A_1|B_1}(G) \right), \ldots, \left( \delta_{A_N|B_N}(G) \right) \right\} \subset \mathbb{Z}^{E(G)}_{N = 2^{n-1}}.$$

Let $K$ be a field and

$$K[q] = K[q_{A_1|B_1}, \ldots, q_{A_N|B_N}],$$
$$K[s, T] = K[s, t_{ij} | ij \in E(G)]$$

be two polynomial rings over $K$. Then the ring homomorphism $\pi_G$ is defined as follows:

$$\pi_G : K[q] \to K[s, T], \quad q_{A_l|B_l} \mapsto s \cdot \prod_{ij \in E(G), |A_l \cap \{i, j\}| = 1} t_{ij}$$

for $1 \leq l \leq N$. The cut ideal $I_G$ of $G$ is the kernel of $\pi_G$ and the toric ring $R_G$ of $X_G$ is the image of $\pi_G$ [2]. In [2], Sturmfels and Sullivant introduced a cut ideal and posed the problem of relating properties of cut ideals to the class of graphs.

On the other hand, the notion of strongly Koszul algebras was introduced by Herzog, Hibi and Restuccia [1]. A strongly Koszul algebra is a stronger notion of Koszulness.

In this talk, we introduce a sufficient condition for cut ideals to have quadratic Gröbner bases and characterization of the class of graphs such that $R_G$ is strongly Koszul.

References


*The author is a Research Fellow of Japan Society for the promotion of Science. This work is supported by Grant-in-Aid for JSPS Fellows 26-4365.

e-mail: k-shibata@rikkyo.ac.jp