Equivariant Gröbner bases

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[work unfinished] Equivariant Gröbner “basics” (with Chris Hillar and Robert Krone)
[to appear in ANT] Noetherianity for infinite-dimensional toric varieties (with Jan Draisma, Rob Eggermont, Robert Krone)
[ISSAC 2014] Equivariant lattice generators and Markov bases (with Thomas Kahle, Robert Krone)
[defended one month ago] Robert Krone’s thesis
Ideals finitely generated up to symmetry

Ideals in some $\infty$-dimensional rings can be represented finitely:

- $K[x_1, x_2, \ldots]$ with the action of $\mathcal{G}_\infty$ is Noetherian up to symmetry: every equivariant ideal is generated by orbits of finitely many elements.
- ... still true if $\mathcal{G}_\infty$ is replaced with the monoid

$$\text{Inc}(\mathbb{N}) = \{\text{increasing maps } \mathbb{N} \to \mathbb{N}\}.$$  

- E.g., $\langle x_1, x_2, \ldots \rangle = \langle x_{2015} \rangle_{\mathcal{G}_\infty} = \langle x_1 \rangle_{\text{Inc}(\mathbb{N})}$. or perhaps are represented finitely:

- $K[y_{i,j} \mid i, j \in \mathbb{N}]$ and $K[y_{\{i,j\}} \mid i, j \in \mathbb{N}]$ with the diagonal action of $\mathcal{G}_\infty$ are not equivariantly Noetherian.
Ideals in some $\infty$-dimensional rings can be represented finitely:

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Equivariant maps

General goal: study families with $\mathfrak{S}_n$ symmetry as $n \to \infty$.

Let $K[Y]$ and $K[X]$ be polynomial rings with $\mathfrak{S}_\infty$-actions.

- A map $\phi : K[Y] \to K[X]$ is a $\mathfrak{S}_\infty$-equivariant map if
  \[ \sigma \phi(f) = \phi(\sigma f) \quad \text{for all } \sigma \in \mathfrak{S}_\infty, f \in K[Y]. \]

- An ideal $I \subset K[Y]$ is a $\mathfrak{S}_\infty$-invariant ideal if
  \[ \sigma I \subseteq I \quad \text{for all } \sigma \in \mathfrak{S}_\infty. \]

- $\phi$ is $\mathfrak{S}_\infty$-equivariant $\Rightarrow$ ker $\phi$ is a $\mathfrak{S}_\infty$-invariant ideal.
Some known kernels of equivariant monomial maps

[de Loera-Sturmfels-Thomas, 1995] Let $\phi : y_{i,j} \mapsto x_i x_j$ for $i \neq j \in \mathbb{N}$,
$$
\ker \phi = \langle y_{1,2}y_{3,4} - y_{1,4}y_{3,2} \rangle_{S_\infty}.
$$

[Aoki-Takemura, 2005] Let $\phi : y_{i,j} \mapsto x_i x_j$ for $i \neq j \in \mathbb{N}$,
$$
\ker \phi = \langle y_{12}y_{34} - y_{14}y_{32}, \ y_{12}y_{23}y_{31} - y_{21}y_{32}y_{13} \rangle_{S_\infty}.
$$

A possible proof, 2015: eliminate (using EGB) in the ring $K[x_i, y_{i,j} | i, j \in \mathbb{N}]$. 

Kernel of $\phi : y_{ij} \mapsto x_i^2 x_j$

Using EGB for elimination get a generating set:

$$
\begin{align*}
&y_1, 3y_0, 2 - y_1, 2y_0, 3 \\
y_2, 0y_1, 0 - y_1, 2y_0, 2 \\
y_2, 1y_0, 1 - y_1, 2y_0, 2 \\
y_2, 3y_0, 1 - y_2, 1y_0, 3 \\
y_2, 3y_1, 0 - y_2, 0y_1, 3 \\
y_3, 1y_2, 0 - y_3, 0y_2, 1 \\
y_3, 2y_0, 1 - y_3, 1y_0, 2 \\
y_3, 2y_1, 0 - y_3, 0y_1, 2 \\
y_1, 2y_0^2, 1 - y_1, 0y_0, 2 \\
y_2, 0y_0^2, 1 - y_1, 0y_0^2, 2 \\
y_2, 2y_0^2, 1 - y_2, 0y_0, 1 \\
y_2, 1y_0, 0 - y_2, 0y_0, 1 \\
y_2, 1y_2, 0 - y_2, 0y_1, 2 \\
y_2, 1y_0, 2 - y_2, 0y_1, 2 \\
y_2, 1y_1, 0 - y_2, 0y_1, 2 \\
y_2, 1y_0, 3 - y_2, 0y_1, 3 \\
y_2, 1y_2, 0 - y_2, 0y_1, 3 \\
y_3, 0y_1, 2y_0, 2 - y_2, 0y_1, 3y_0, 3 \\
y_3, 0y_1, 2 - y_2, 0y_1, 3 \\
y_3, 0y_2, 1 - y_2, 2y_0, 1, 3 \\
y_3, 1y_0, 2 - y_2, 1y_0, 3 \\
y_3, 2y_1, 0 - y_3, 1y_2, 0y_0, 2y_1, 2 \\
y_3, 2y_1, 0 - y_3, 2y_0, 1, 4y_0, 2 \\
y_3, 2y_1, 0y_0, 2 - y_3, 1y_2, 3y_0, 2 \\
y_3, 2y_1, 0 - y_3, 1y_2, 4y_0, 2 \\
y_3, 2y_1, 0 - y_3, 1y_2, 4y_1, 4 \\
y_4, 0y_2, 3y_1, 3 - y_3, 0y_2, 4y_1, 4 \\
y_4, 1y_2, 3y_0, 0 - y_3, 1y_2, 4y_0, 4 \\
y_4, 2y_1, 3y_0, 3 - y_3, 2y_1, 4y_0, 4 \\
y_4, 2y_0, 1y_1, 3 - y_4, 0y_2, 3y_1, 2 \\
y_4, 2y_0, 1y_1, 3 - y_4, 1y_2, 3y_0, 2 \\
y_5, 1y_4, 2y_3, 5y_0, 3 - y_5, 0y_4, 3y_3, 2y_3, 1 \\
y_2, 1y_1, 2y_0, 3y_0, 2 - y_2, 0y_1, 3y_0, 1 \\
y_3, 1y_2, 3y_1, 3y_0, 4 - y_3, 0y_2, 1y_1, 4 \\
y_3, 1y_2, 3y_0, 4 - y_3, 0y_2, 4y_2, 1 \\
y_3, 2y_2, 3y_1, 3y_0, 4 - y_3, 0y_2, 4y_1, 1 \\
y_4, 1y_2, 3y_1, 4y_0, 4 - y_4, 0y_3, 1y_1, 2 \\
y_4, 1y_3, 2y_1, 4y_0, 4 - y_4, 0y_3, 1y_1, 2 \\
y_4, 1y_3, 4y_2, 4y_0, 5 - y_4, 0y_3, 1y_2, 5 \\
y_2, 1y_2, 3y_0, 2 - y_2, 0y_1, 3y_0, 1 \\
y_2, 1y_0, 2y_0, 4y_0, 3 - y_2, 0y_1, 4y_1, 3y_0, 1 \\
y_3, 2y_2, 3y_1, 4y_0, 4 - y_3, 0y_2, 4y_1, 2 \\
y_3, 2y_2, 3y_1, 4y_0, 5 - y_3, 0y_2, 5y_2, 4y_1, 2 \\
y_4, 1y_2, 3y_1, 4y_0, 5 - y_4, 0y_2, 1y_1, 5y_1, 3 \\
y_4, 1y_3, 2y_1, 4y_0, 5 - y_4, 0y_3, 1y_1, 5y_1, 2 \\
y_4, 3y_2, 3y_3, 2y_3, 1 - y_4, 2y_4, 1y_2, 3y_3, 4y_0, 3 \\
y_5, 1y_4, 2y_3, 5y_0, 3 - y_5, 0y_4, 3y_3, 2y_3, 1
\end{align*}
$$

51 generators, width 6, degree 5.
(Computation in 4ti2 by Draisma, EquivariantGB in M2 by Krone)
Questions

- Does a finite up-to-symmetry generating set exist?
- Can we compute one?
- Can we find small bases?
  - Size: number of generators.
  - Degree: maximum degree of the generators.
  - Width: largest index value used by the generators.

Can we answer these for the one-monomial map

$$\phi : K[y_{ij} \mid i \neq j \in \mathbb{N}] \rightarrow K[x_1, x_2, \ldots]$$

$$y_{ij} \mapsto x_i^a x_j^b$$

where $a > b$ are coprime?
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where \( a > b \) are coprime?
Up-to-symmetry generating set [Kahle-Krone-L.]

A $\mathcal{S}_\infty$-generating set for $\phi$ is

(i) $y_{12}y_{34} - y_{14}y_{32}$;

(ii) $y_{12}y_{23}y_{31} - y_{21}y_{32}y_{13}$;

(iii) for each $0 \leq n \leq a - b$,

$$y_{12}^{b+n} \prod_{j \geq 3} y_{1j}^{c_{1j}} y_{2j}^{c_{2j}} - y_{21}^{b+n} \prod_{j \geq 3} y_{1j}^{c_{1j}} y_{1j}^{c_{2j}} \begin{pmatrix} y_{12}y_{32} - y_{21}y_{31}, & n = 0; \\ y_{12}^2y_{23} - y_{21}^2y_{13}, & n = 1; \end{pmatrix}$$

where $\sum_{j \geq 3} c_{1j} = a - b - n$ and $\sum_{j \geq 3} c_{2j} = n$;

(iv) for each $1 \leq n \leq b$,

$$y_{12}^{b-n} y_{13} y_{32}^{a-b+n} - y_{21}^{b-n} y_{23} y_{31}^{a-b+n} \cdot \begin{pmatrix} y_{13}^2 y_{32} - y_{23}^2 y_{31}. \end{pmatrix}$$

Size = $O((a - b)^{(a-b)})$,

degree = $\max(a + b, 2a - b)$,

width = $\max(4, a - b + 2)$. (5) (3) (4)
Theorem (Draisma, Eggermont, Krone, L.)

If \( Y \) has a finite number of \( \mathfrak{S}_\infty \) orbits, any \( \mathfrak{S}_\infty \)-equivariant toric map

\[
\phi : K[Y] \to K \\
\begin{bmatrix}
  x_{11}, x_{12}, & \cdots \\
  \vdots \\
  x_{k1}, x_{k2}, & \cdots
\end{bmatrix}
\]

has a finite \( \mathfrak{S}_\infty \)-generating set.

- There exists an algorithm to construct a generating set above... (not implemented!)
- Implies [de Loera-Sturmfels-Thomas], [Aoki-Takemura], Independent Set Theorem [Hillar-Sullivant].
Π-divisibility and equivariant GB

Let $M$ be a monoid with an action of $\Pi = \{ \rho_i : i \in \mathbb{N} \} \subset \text{Inc}(\mathbb{N})$ where

$$\rho_i(j) = \begin{cases} j, & j < i, \\ j + 1, & j \geq i \end{cases}$$

- For $a, b \in M$, we say that $a$ $\Pi$-divides $b$ if $\alpha a | b$ for some $\alpha \in \Pi$. E.g.,

$$x_1 x_2^2 |_{\Pi} x_2 x_4^3, \text{ since } \rho_1 \rho_2 x_1 x_2^2 | x_2 x_4^3.$$ 

Equip $M$ with a monomial order $<$ that respects $\Pi$ and refines $|_{\Pi}$. E.g.,

- for $M = [x_i : i \in \mathbb{N}]$, lex or grlex with $x_1 < x_2 < \cdots$;
- for $M = [x_i : i \in \mathbb{N}; y_{ij} : i < j]$, block order with $y < x$ and grlex on $x$ and $y$.

For a polynomial $f$ in $R = K M$ define: $\text{lm}(f)$, $\text{lc}(f)$, $\Pi$-reduction, ...

... and, for ideal $I \subset R$, the initial ideal $\text{in}(I)$ and an equivariant GB.
\section*{Π-divisibility and equivariant GB}

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What is necessary for an EGB computation attempt?

For \( a, b \in M \) we define the set of least common multiples

\[
lcm(a, b) = \min\{l \in M : a \text{ and } b \text{ }\Pi\text{-divide } l\}.
\]

Condition EGB34

For all \( f, g \in K[M] \), the set of triples in \( M \times \Pi f \times \Pi g \) defined by

\[
T_{f,g} = \{(l', f', g') \mid l' \in lcm(lm(f'), lm(g'))\},
\]

is a union of a finite number of \( \Pi \)-orbits:

\[
T_{f,g} = \bigcup_{i} \Pi(l_i, f_i, g_i), \quad i \in [r].
\]

E.g., \( f = x_1^2 x_2 + \ldots \) and \( g = x_1 x_2^2 + \ldots \) give

\[
T_{f,g} = \Pi\{(x_1^2 x_2^2, f, g), (x_1 x_2 x_3, f, \rho_1 g), \cdots , (x_1 x_2 x_3 x_4, \rho_1^2 f, g), (x_1^2 x_2 x_3 x_4^2, f, \rho_1^2 g) \}.
\]
\(\Pi\)-S-polynomials

- Denote a finite set of orbit generators that “cover” \(T_{f,g}\) by
  \[
  O_{f,g} = \{(l_i, f_i, g_i) \mid i \in [r]\}.
  \]

- If \(\Pi\)-divisibility order is wpo (well partial order), take
  \[
  O_{f,g} = \text{minimal orbit generators}.
  \]

- For monic \(f, g \in K[M]\) define the set of corresponding S-polynomials to be
  \[
  S_{f,g} = \{af' - bg' \mid (l', f', g') \in O_{f,g}; a, b \in M \text{ and } a \text{ lm}(f') = b \text{ lm}(g') = l'\}.
  \]
Buchberger’s algorithm

Buchberger’s criterion. Let $G \subset K[M]$ be a set such that $\forall f, g \in G$ all S-polynomials $S_{f,g}$ $\Pi$-reduce to 0 by $G$. Then $G$ is a $\Pi$-Gröbner basis of $\langle G \rangle_\Pi$.

Input: $F$ is a finite subset of $K[M]$  
Output: $G$ is an equivariant Gröbner basis of $\langle F \rangle_\Pi$.

\begin{verbatim}
1: $G \leftarrow F$
2: $S \leftarrow \bigcup_{f,g \in G} S_{f,g}$
3: while $S \neq \emptyset$ do
4:   pick $f \in S$
5:   $S \leftarrow S \setminus \{f\}$
6:   $h \leftarrow$ the $\Pi$-reduction of $f$ with respect to $G$
7:   if $h \neq 0$ then
8:     $G \leftarrow G \cup \{g\}$
9:     $S \leftarrow S \cup \left( \bigcup_{g \in G} S_{g,h} \right)$
10: end if
11: end while
\end{verbatim}
Recall: \( \infty \)-dimensional toric ideals

**Key idea** of the proof of Noetherianity: factor the monomial map \( \phi : K[Y] \to K[X] \).

**Example:** factor \( \phi(y_{ij}) = x_i^2 x_j \) as \( \phi = \psi \circ \pi \):
- let \( Z = \{ z_{lm} : l \in [2], m \in \mathbb{N} \} \)
- \( \pi : K[Y] \to K[Z] \) is given by \( \pi(y_{ij}) = z_{1i}z_{2j} \)
- \[
\pi(y_{14}^2y_{24}) = (z_{11}z_{24})(z_{12}z_{24})^2 = z_1 \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & \cdots \end{bmatrix} \]
- \( \psi : K[Z] \to K[X] \) is given by \( \psi(z_{1i}) = x_i^2 \) and \( \psi(z_{2i}) = x_i \).
- **matching monoid:** the exponents of monomials of \( \text{im}(\pi) \subseteq K[Z] \) = matrices with
  - row sums all equal to the same number \( d \)
  - columns summing up to \( \leq d \)
Recall: $\infty$-dimensional toric ideals

**Effective** proof of $\text{Sym}(\mathbb{N})$-Noetherianity of $\ker \phi$:
- find an $\text{Inc}(\mathbb{N})$-EGB of $\ker \psi$ and
- show a finite $\text{Sym}(\mathbb{N})$-generating set for $\ker \pi$ ([Yamaguchi-Ogawa-Takemura, 2014]).

What this does **not** show:
- $\ker \phi$ is $\text{Inc}(\mathbb{N})$-finitely generated;
- equivariant Buchberger for $\text{Inc}(\mathbb{N})$-elimination on the graph of $\phi$ terminates.

**Proposition (Krone, 2015)**

*There is an monomial order on $K[Y]$ such that there is a finite $\text{Inc}(\mathbb{N})$-EGB of $\ker \phi$.***
Strong GB [Gao-Volny-Wang]

Let $I = \langle F \rangle \subset R = K[x_1, \ldots, x_n]$, where $|F| = r \in \mathbb{N}$.

- A subset $G$ of

$$P = \{(s, f) \in R^r \times R \mid f = s \cdot F = \sum_{i=1}^{r} s_i F_i\}.$$

is called a strong Gröbner basis if every non-zero pair is top-reducible by some pair in $G$.

- $(s_f, f)$ is top-reducible by $(s_g, g)$ if $\text{lcm } g | \text{lcm } f$ and for some $a$, $\text{lcm } f = a \text{lcm } g$, we have $a \text{lcm } s_g \leq \text{lcm } s_f$. If the reduction,

$$(f', s_{f'}) := (f - ag, s_f - as_g),$$

has $\text{lcm } s_{f'} = \text{lcm } s_f$, then it is regular top-reducible.

- If $G$ is a strong GB, then
  - $\{f : (s, f) \in G\}$ is a GB
  - $\{s : (s, 0) \in G\}$ is a GB for the module of syzygies
J-pairs

Take two pairs $p_f = (s_f, f)$ and $p = (s_g, g)$.

- **J-pairs** $J_{p_f, p_g}$: for $a$ and $b$, $a \text{lm } f = b \text{lm } g \in \text{lcm}(\text{lm } f, \text{lm } g)$, form a J-pair by taking the “larger side” of the corresponding S-polynomial. E.g.,

  $$p_f = (e_1 + \ldots, x_1^2x_2 + \ldots)$$
  $$p_g = (x_2e_1 + \ldots, x_1^2x_2 + \ldots)$$

- Since $x_2 \text{lm } s_f < x_1 \text{ lm } s_g$,

  $$J_{p_g, p_f} = x_1p_g = \{(x_1x_2e_1 + \ldots, x_1^2x_2^2 + \ldots)\}$$
StrongBuchberger

Input: $F$
Output: $G$, strong GB.

1: $G \leftarrow \emptyset$, $S \leftarrow \emptyset$
2: $J \leftarrow \{(e_i, F_i) : i \in r = |F|\} \subset R^r \times R$
3: while $J \neq \emptyset$ do
4: pick $p_f = (s_f, f) \in J$; $J \leftarrow J \setminus \{p_f\}$
5: $(s_h, h) \leftarrow \text{regular top-reduction of } (s_f, f) \text{ with respect to } G$
6: if $h \neq 0$ then
7: $G \leftarrow G \cup \{g\}$
8: append to $J$ all $J$-pairs $\bigcup_{(p_g) \in G} J_{p_g, p_f}$ not covered by $G \cup S$
9: else
10: $S \leftarrow S \cup \{(s_h, 0)\}$
11: end if
12: end while

A pair $(s_f, f)$ is covered by $(s_g, g)$ if $\text{lm } g | \text{lm } f$ and for some $a$ such that $\text{lm } f = a \text{lm } g$ we have $a \text{lm } s_g < \text{lm } s_f$. 
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A pair $(s_f, f)$ is covered by $(s_g, g)$ if $\text{lm } g | \text{lm } f$ and for some $a$ such that $\text{lm } f = a \text{lm } g$ we have $a \text{lm } s_g < \text{lm } s_f$. For a pair $(s, f)$ store only $\text{SIGNATURE } \text{lm } s$ and a "pointer" to $f$.
Adaptation to the equivariant case?

Proof of termination relies on Noetherianity of module $R^r$, but...

- Let $R = K[x_i, i \in \mathbb{N}]$ with a $\Pi$-compatible order, $\Pi = \text{Inc}(\mathbb{N})$.
- Need to work with pairs

$$P = \{(s, f) \in (R \rtimes \Pi)^r \times R \mid f = s \cdot F\}.$$ 

Recall: $\Pi = \{\rho_i : i \in \mathbb{N}\} \subset \text{Inc}(\mathbb{N})$, so

$$R \rtimes \Pi = K[X] \rtimes \Pi = K([X] \rtimes \Pi)$$

where the semidirect product $[X] \rtimes \Pi$ is a non-Noetherian noncommutative monoid.

$(R \rtimes \Pi)^r$ is not $\Pi$-Noetherian! A strong EGB is infinite!

How to make the equivariant StrongBuchberger terminate?

- “Make” the syzygy module Noetherian (some preliminary results with Draisma and Krone).
- Do not insist on being strong.
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- Do not insist on being strong.
Macaulay2 package EvidariantGB

i1 : needsPackage "EvidariantGB"

i2 : R = buildERing({symbol x, symbol y}, {1,2}, QQ, 2, MonomialOrder=>Lex, Degrees=>{1,2});

i3 : gens R

o3 = {x , x , y , y , y , y }
     1 0 1,1 1,0 0,1 0,0

i4 : F = {y_(1,0) - x_0*x_1};

i5 : egbSignature(F, PrincipalSyzygies=>false)

... 
... 
  processing pair: (x_3*x_1*y_(2,0)-x_2*x_1*y_(3,0), x_3*x_1*(0, 2)*[0])
-- 9th syzygy is: (0, x_3*x_1*(0, 2)*[0])
  covered pair in JP: (x_3*x_1*y_(2,0)-x_3*x_0*y_(2,1), x_3*x_1*(0, 2)*[0])
  processing pair: (-x_2*x_1*y_(3,0)+y_(3,0)*y_(2,1), y_(3,0)*(1)*[0])
-- 6th basis element is: (-y_(3,1)*y_(2,0)+y_(3,0)*y_(2,1), y_(3,0)*(1)*[0])
  new J-pairs: 26
  new NOT covered J-pairs: 21
  new J-pairs: 94
  new NOT covered J-pairs: 64
  new J-pairs: 94
  new NOT covered J-pairs: 53
  new J-pairs: 94
  new NOT covered J-pairs: 49
  new J-pairs: 251
  new NOT covered J-pairs: 135
  new J-pairs: 250
  new NOT covered J-pairs: 116
...
...

o5 = {- x x + y , x y - x y , x y - x y , - x y + y y , - y y
       1 0 1,0 1,2,0 2,1 2,1,0 1,2,0 0 2,1 2,0 1,0 3,2 1,0
       + y y , - y y + y y }
       3,1 2,0 3,1 2,0 3,0 2,1
Future

- Improve EGB algorithms further.
- Complexity?
- More “Noetherian up to symmetry” commutative algebra.
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