Bouquet algebra of toric ideals

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The 8th Mathematical Society of Japan Seasonal Institute
Current Trends on Gröbner Bases
The 50th Anniversary of Gröbner Bases
Hotel Nikko Osaka, Osaka, Japan

1-10 July 2015
... What is this talk about?

$A := [a_1 \ldots a_n] \in \mathbb{Z}^{m \times n}$, \quad $K$ arbitrary field.

Toric ideal $I_A$ is:

kernel of homomorphism $K[x_1, \ldots, x_n] \rightarrow K[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$, \quad $x_i \mapsto t^{a_i}$.

$$I_A = (x^u - x^{-u} \mid u \in \ker \mathbb{Z}(A)).$$

**Application of toric ideals in statistics:** Shinzo Abe and Japanese corporate network

- Framework for studying combinatorial signatures of toric ideals
- Basic object: Bouquet ideal of $A$ - encodes properties via a structural decomposition of an associated matroid
- Characterize equality of bases
  - recover known results e.g., 2nd Lawrence configurations;
- Recover structural results of toric ideals of hypergraphs and graphs
- Provide some new constructions
  - infinite families of robust and of generic toric ideals;
  - hypergraphs that encode bases of any general toric ideal.
Key definitions

\[ A := [a_1 \ldots a_n] \in \mathbb{Z}^{m \times n} \quad \text{Gale}(A) := [b_1, \ldots, b_n]. \]

\[
A = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix}, \quad \text{Gale}(A) = \begin{pmatrix}
1 & 1 \\
-1 & 0 \\
0 & -1 \\
0 & -1 \\
-1 & 0 \\
1 & 1 \\
0 & 0
\end{pmatrix}
\]

The bouquet graph \( G_A \):

- \( V(G_A) = \{a_1, \ldots, a_n\} \),
- \( \{a_i, a_j\} \in E(G_A) \) if and only if either \( b_i = b_j = 0 \) or \( \dim_{Q} \langle b_i, b_j \rangle = 1 \).

A bouquet of \( A \) is a connected component of \( G_A \).

In this example, there are 4 bouquets.

\[ G_A \] has 3 edges and 1 isolated vertex.]
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\[ A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad Gale(A) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \\ -1 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \]

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In this example, there are 4 bouquets.

\[ G_A \text{ has 3 edges and 1 isolated vertex.} \]
Key definitions, continued

(for A to $A_B$)

Let $A = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix}$

$Gale(A) = \begin{pmatrix}
1 & 1 \\
-1 & 0 \\
0 & -1 \\
0 & -1 \\
-1 & 0 \\
1 & 1 \\
0 & 0
\end{pmatrix}$.

- **Mixed bouquet:** contains an edge $\{a_i, a_j\}$ with $b_i = \lambda b_j \neq 0$ for some $\lambda < 0$;
- **Non-mixed bouquet:** $b_i = \lambda_{ij} b_j \neq 0$ with $\lambda_{ij} > 0$ for all edges $\{a_i, a_j\}$;
- **Free bouquet:** $b_i = b_j = 0$ for all edges.

The bouquet matrix:

$A_B = [a_{B_1}, \ldots, a_{B_s}]$

where $B_1, \ldots, B_s$ are all of the bouquets,

$a_{B_i} = \sum_{i=1}^{n} (c_B)_i a_i$.

$[c_B_i$ is a bouquet-index encoding vector.]
Key bijection - General behavior

Theorem

Bijection between the elements of $\ker_{\mathbb{Z}}(A)$ and the elements of $\ker_{\mathbb{Z}}(A_B)$:

$$u = (u_1, \ldots, u_s) \in \ker_{\mathbb{Z}}(A_B) \implies B(u) := c_{B_1}u_1 + \cdots + c_{B_s}u_s \in \ker_{\mathbb{Z}}(A).$$

Converse: any $v \in \ker_{\mathbb{Z}}(A)$ is of the form of $B(u)$ for some $u \in \ker_{\mathbb{Z}}(A_B)$.

$$A_B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

This bijection preserves:

- Graver basis,
- Circuits.

$A$ unimodular $\implies$ $A_B$ uni.
Key bijection - General behavior

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$$A_B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$  

$$\ker_\mathbb{Z}(A_B) = \{ (\alpha + \beta, -\alpha, -\beta, 0) | \alpha, \beta \in \mathbb{Z} \}$$

$$B((\alpha + \beta, -\alpha, -\beta, 0)) = (\alpha + \beta)c_{B_1} - \alpha c_{B_2} - \beta c_{B_3}$$

$$= (\alpha + \beta, -\alpha, -\beta, -\beta, -\alpha, \alpha + \beta, 0) \in \ker_\mathbb{Z}(A).$$

This bijection preserves:

- Graver basis,
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**Key bijection - General behavior**

**Theorem**

Bijection between the elements of \( \ker \mathbb{Z}(A) \) and the elements of \( \ker \mathbb{Z}(A_B) \):

\[
\mathbf{u} = (u_1, \ldots, u_s) \in \ker \mathbb{Z}(A_B) \iff B(\mathbf{u}) := c_{B_1} u_1 + \cdots + c_{B_s} u_s \in \ker \mathbb{Z}(A).
\]

**Converse:** any \( \mathbf{v} \in \ker \mathbb{Z}(A) \) is of the form of \( B(\mathbf{u}) \) for some \( \mathbf{u} \in \ker \mathbb{Z}(A_B) \).

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A_B = \begin{pmatrix}
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\[
\ker \mathbb{Z}(A_B) = \{ (\alpha + \beta, -\alpha, -\beta, 0) \mid \alpha, \beta \in \mathbb{Z} \}
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\[
B((\alpha + \beta, -\alpha, -\beta, 0)) = (\alpha + \beta) c_{B_1} - \alpha c_{B_2} - \beta c_{B_3} = (\alpha + \beta, -\alpha, -\beta, -\beta, -\alpha, \alpha + \beta, 0) \in \ker \mathbb{Z}(A).
\]

This bijection preserves:

- Graver basis,
- Circuits.

\( A \) unimodular \( \iff \) \( A_B \) uni.

\[
A_B : \quad \{(1, -1, 0, 0), (0, 1, -1, 0), (1, 0, -1, 0)\}
\]

\[
B((0, 1, -1, 0)) = c_{B_2} - c_{B_3} = (0, 1, -1, -1, 1, 0, 0).
\]
Natural inverse procedure
(from $A_B$ to $A$)

What structure can matrices $A_B$ have?

**Theorem**

There is a natural inverse procedure:
Given any set of vectors $a_1, \ldots, a_s$, one can (appropriately) construct a toric ideal $I_A$ such that the given vectors are precisely the encoding vectors of subbouquets.

(We call these matrices generalized Lawrence matrices.)

**Corollary**

For any $A$ there exists a generalized Lawrence matrix $A'$ such that $I_A = I_{A'}$. 
Bouquets: combinatorial essence of toric ideals

In the remainder of the talk, we ask (and answer in various ways):

**Question:**
What does the bouquet structure of $A$ say about the toric ideal $I_A$?
Non-mixed bouquets encode stability

Stable toric ideals:
Defined by $A$ all of whose bouquets are non-mixed.

$A_B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

[$\iff$ coordinates of bouquet-index-encoding vector are all positive.]

Theorem

For stable $A$, correspondence $\ker \mathbb{Z}(A) \leftrightarrow \ker \mathbb{Z}(A_B)$ is preserved on:
- Graver basis and circuits (in general);
- Minimal generating sets (Markov bases);
- Indispensable binomials;
- Reduced Gröbner bases [explicit weight vectors correspondence.]
- Universal Gröbner basis.

→ There are infinitely many stable toric ideals. ←
[Enough: constructed using inverse procedure, any $a_i$’s, restricted $c_i$’s.]
Applications of stability

\( I_A \subset S \) positively graded, stable, \( I_{AB} \subset R \); \( F_\bullet \) the minimal graded free resolution of \( R/I_{AB} \). \( \implies F_\bullet \otimes_R S \) is minimal graded free resolution of \( I_A \).
Applications of stability

\[ I_A \subset S \text{ positively graded, stable, } I_{AB} \subset R; \ F_\bullet \text{ the minimal graded free resolution of } R/I_{AB}. \implies F_\bullet \otimes_R S \text{ is minimal graded free resolution of } I_A. \]

* Generic toric ideals: \( \text{MinGen}(I_A) \) have full support. [Miller-Sturmfels’05]

\( I_A \text{ generic } \iff I_{AB} \text{ generic.} \)

[Infinitely many examples; e.g. use [Ojeda’08] as starting point.]
Applications of stability

\( I_A \subset S \) positively graded, stable, \( I_{AB} \subset R \); \( F_* \) the minimal graded free resolution of \( R/I_{AB} \). \( \implies F_* \otimes_R S \) is minimal graded free resolution of \( I_A \).

* Generic toric ideals: \( \text{MinGen}(I_A) \) have full support. [Miller-Sturmfels’05] \n\( I_A \) generic \( \iff \) \( I_{AB} \) generic.

[Infinitely many examples; e.g. use [Ojeda’08] as starting point.]

* Robust toric ideals: \( \text{MinGen}(I_A) = \text{Graver}(I_A) \) [Boocher et al ’13, ’15]. \n\( I_A \) robust \( \iff \) \( I_{AB} \) robust.

[New classes of examples \( \neq \) toric ideals of graphs, toric ideals generated by quadrics [Boocher-Robeva’15, BoocherEtAl’13].]
Mixed bouquets encode equality of bases

- \( \text{Circuits}(I_A) \subseteq UGB(I_A) \subseteq \text{Graver}(I_A) \); [well-known]; = for:
  - 2nd Lawrence configurations [Sturmfels’95]
  - unimodular matrices folklore? St., etc?
  - high-dimensional low-degree rational normal scrolls [BHP’12]
  - 2-regular hypergraphs [Gross-P.’13]

**Theorem**

Characterization of \( A \) for which bases equal; captured by mixed bouquets.

[Equality if and only if \( I_A \) is \( S \)-Lawrence for \( S = \{ i \colon a_i \in \text{mixed bouquet} \} \).]

**Corollary**

If all non-free bouquets of \( A \) are mixed, then

\[ \text{MinGen}(I_A) = \text{Reduced Gröbner}(I_A) = UGB(I_A) = \text{Graver}(I_A). \]

An application: infinite families of examples where equality holds.
Complexity questions

What can we say when $A$ is a 0/1 matrix?

...and: is that really a restriction?

Why would 0/1 matrices be special?

$I_A = \text{toric ideal of a hypergraph}; \text{encodes algebraic relations among edges.}$

... Abundant literature for graphs.

... Some recent work on hypergraphs.
Interlude - monomial walks

Theorem (Ohsugi-Hibi ’99-’00, Villarreal ’95-’01)

*The toric ideal* $I_G$ *is generated by binomials arising from (primitive) even closed walks on G.*

Theorem (Ohsugi-Hibi ’99-’00, Villarreal ’01, Reyes-Tatakis-Thoma ’12)

Primitive even closed walks on G are of 3 types (1st row below).

![Graphs](image-url)

hypergraph equivalents

[P-Stasi’14]
Interlude: detecting primitive monomial walks [PS’14]

- **[Theorems]** In ‘sparse’ (←informal!) cases, counting sunflower petals (with multiplicities) detects balanced sets on $H$, and thus primitivity.

- For arbitrary binomials, there is no efficient way to detect primitivity.

**Theorem [P-Stasi’14]**

For squarefree binomials in $I_H$, detecting primitivity is dual to the discrepancy problem in hypergraphs.

**Problem:** finding good “sunflower decompositions” ???
Bouquets with bases

Definition (on an example)
Consider this hypergraph $\mathcal{H}$:
- $A \in \mathbb{Z}^{23} \times \mathbb{Z}^{20}$
- 3 bouquets with bases
- bases consist of $v_i$’s (no $x$).

Theorem
A bouquet with basis of the hypergraph $\mathcal{H}$ is either a free subbouquet or a non-free mixed subbouquet of its incidence matrix.

Consequence:
If the edge set of $\mathcal{H}$ can be partitioned into bouquets with bases, then the toric ideal is easier to describe.

[And it is also $\emptyset$-Lawrence, so equality of bases holds too.]
Bouquets with bases

Example: If the edge set of $\mathcal{H}$ can be partitioned into bouquets with bases, then the toric ideal is easier to describe.

Columns of bouquet matrix $A_B$: $(3, 0 \ldots 0), (4, 0 \ldots 0), (5, 0 \ldots 0) \in \mathbb{Z}^{23}$.

$l_\mathcal{H}$ is encoded by monomial curve $[3, 4, 5]$. Graver basis has 7 elements.

$$x_1x_2x_3x_{10}^2x_{11}^2x_{12}^2x_{13}x_{14}^2x_{15}x_{16} - x_4x_5x_6^2x_7^2x_8^2x_9^2x_{17}x_{18}x_{19}x_{20} \in \ker A$$

$$\iff y_1y_3 - y_2^2 \in \ker[3, 4, 5].$$
Bouquets with bases: Sunflowers + an Application

Interesting examples of bouquets with bases: “sparse bouquets”; built on sunflowers [Jukna’01].

- Extend results from [PS’14] [non-uniform]
- Identify Graver elements of any hypergraph containing sunflowers
- [for experts only:] Monomial walks ↔ vectors $a_B = 0$ [it’s really, a vector of color-imbalances].

Upshot: matched-petal sunflowers ↔ monomial curves (determined by sunflower cores).

[Tatakis-Thoma’11]: $UGB \neq Graver$ for graph $K_n$ iff $n \geq 9$.

Application

$UGB \neq Graver$ for $K^d_n$ if $n \geq (d + 1)^2$ vertices.
Complexity of hypergraphs captures general toric ideals

- Well-known: Graver(toric ideal of graph) has rather special form, and completely determined by support
- Hypergraphs: none of this holds
- **Question**: how complicated are various bases for toric ideals of 0/1 matrices?

[Bouquet ideals $\Rightarrow$ 0/1 toric ideals are as complicated as general ones:]

**Theorem**

Given any integer matrix $A$, there exists a hypergraph $\mathcal{H}$ such that:

- There is a one-to-one correspondence $\ker_{\mathbb{Z}}(A) \leftrightarrow \ker_{\mathbb{Z}}(\mathcal{H})$,
- For every $u \in Gr(A)$ we have $\deg(x^u^+ - x^u^-) \leq \deg(x^{B(u^+)} - x^{B(u^-)})$,
- Equality of bases holds for $I_{\mathcal{H}}$.

Note: these hypergraphs have edges of size at most 3.
Hypergraphs encode all finely graded toric ideals

Theorem

Let $I_D$ be an arbitrary positively graded nonzero toric ideal. Then there exists a hypergraph $H$ such that there is a one-to-one correspondence between the Graver basis, minimal generating sets, reduced Gröbner bases, circuits, indispensable binomials of $I_D$ and $I_H$.

[This $I_H$ is stable.]

Application [and open problem]

Problems about arbitrary positively graded toric ideals can be reduced to problems about $I_H$. For example:

- If $I_G$ is robust, then $\text{Graver}(I_G) = \text{MinGen}(I_G)$ [Boocher et al].
- They ask if this property holds for all robust toric ideals.
- It is enough to prove this for toric ideals of hypergraphs.
A problem to think about?

On Tuesday, Akihiro Shikama presented joint work with Takayuki Hibi, Kenta Nishiyama and Hidefumi Ohsugi. They find an infinite class of graphs whose toric ideals are generated by quadrics, but have no quadratic Gröbner basis.

Question:
What is the bouquet structure of these ideals?
A problem to think about?

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Question:
What is the bouquet structure of these ideals?

Thank you!
References

... will be added before slides are posted online.
Example: application of toric ideals in statistics

Fact (algebraic statistics):

Toric ideals ↔ log-linear discrete models. MCMC requires computing $I_A$.

- We study linear ERGMs: log-linear random graph (network) models.
- Bases of $I_A$ provide a justified way for testing model fit.
- Matrices $A$ are too large to compute Markov/Graver bases directly.


We implemented and tested the $p_1$ model approach based on hypergraphs.

What does this mean for Shinzo Abe’s Bid to Shake Up Corporate Japan?

(↑ This text is a link to NYT article from 2014.)

[... Your conclusion? Cannot reject model; share ownership connections seem mutually independent.]
Example: The inverse procedure (from $A_B$ to $A$)

**Theorem:** $A = [a_1 \ldots a_s] \subset \mathbb{Z}^m$ arbitrary; $c_1, \ldots, c_s$ any (primitive) vectors with full support, $c_i \in \mathbb{Z}^{m_i}$. Define $p = m + \sum_{i=1}^{s} (m_i - 1)$ and $q = \sum_{i=1}^{s} m_i$. There exists a matrix $A \in \mathbb{Z}^{p \times q}$ with $s$ subbouquets, $B_1, \ldots, B_s$, encoded by: $a_{B_i} = (a_i, 0, \ldots, 0) \in \mathbb{Z}^p$ and $c_{B_i} = (0, \ldots, c_i, \ldots, 0) \in \mathbb{Z}^q$, where the support of $c_{B_i}$ is precisely in the $i$th block of $\mathbb{Z}^q$ of size $m_i$.

For each $i = 1, \ldots, s$, let $c_i = (c_{i1}, \ldots, c_{im_i}) \in \mathbb{Z}^{m_i}$ and define

$$C_i = \begin{pmatrix} -c_{i2} & c_{i1} \\ -c_{i3} & c_{i1} \\ \vdots \\ -c_{imi} & c_{i1} \end{pmatrix} \in \mathbb{Z}^{(m_i - 1) \times m_i}.$$ 

Define $A_i = [\lambda_{i1} a_i, \ldots, \lambda_{imi} a_i] \in \mathbb{Z}^{m \times m_i}$, where $1 = \lambda_{i1} c_{i1} + \cdots + \lambda_{imi} c_{imi}$

$$A = \begin{pmatrix} A_1 & 0 & A_2 & \cdots & A_s \\ C_1 & 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & C_s \end{pmatrix} \in \mathbb{Z}^{p \times q},$$

where $p = m + (m_1 - 1) + \cdots + (m_s - 1)$ and $q = m_1 + \cdots + m_s$. 
**Example: Complexity of $\mathcal{H}$ captures complexity of all $A$**

\[
A = \begin{pmatrix}
-1 & -1 & 2 & 2 \\
-2 & 2 & -1 & 0
\end{pmatrix},
\]

Every non-zero entry of the matrix $A$ is used to construct a sunflower, which will be the building blocks of the desired hypergraph. The sunflowers corresponding to matrix entries $-1$ (L), 2, and $-2$ (R):

Next we construct for each column $a_j$ a connected matched-petal partitioned-core sunflower $\mathcal{H}_j$.

Here is $\mathcal{H}_1$:

The other 3 bouquets with basis corresponding to $a_2, a_3, a_4$:
Ex: Hypergraphs encode all positively graded toric ideals

\[ D = (d_{ij}) = \begin{pmatrix} 1 & 3 & 2 & 0 & 1 \\ 3 & 2 & 1 & 3 & 2 \\ 3 & 0 & 2 & 2 & 1 \end{pmatrix} \in \mathbb{N}^{3 \times 5}. \]

\[ A = M_H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]