Galois Actions and Geometry

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Fields of Definition and Families of Covers

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Part 1: Fields of Definition? models of covers?
$\rightarrow$ Descent Theory
$\rightarrow$ Weil, Grothendieck et al.
$\leftarrow$ Regular Inverse Galois Problem

Typical question: If action of G_K leaves invariant the set of models of a given K_s-cover f — K is called the *field of moduli* —, is there a model of f over K?

Part 2: Moduli Spaces of covers
Existence of Families of Covers?

Typical question: Given a moduli space $\mathcal{H}$ of covers, is there a family above $\mathcal{H}$?
[Action of $\pi_1(\mathcal{H})$ leaves invariant the set of local families above $\mathcal{H}$ (defined above an étale cover).]

Part 3: Covers, Families and Gerbes

Fields of definition of covers

Category: covers $f : X \to B$ *a priori* defined over the separable closure K_s of a given field K;
B is a given algebraic variety defined over K.

Main example:

covers $f : X \to \mathbb{P}^1$ defined over $\overline{\mathbb{Q}}$

Two situations:

- G-covers: Galois covers f given an isomorphism $\mathrm{Aut}(f) \simeq G$
- mere covers: not necessarily Galois covers, given without automorphisms

k-model of f $(K \subset k \subset K_s)$:

k-cover $f_k : X_k \to B$ such that $f_k \otimes_k K_s \underset{K_s}{\simeq} f$

Some problems

Problem 1: If a K_s-cover $f : X \to B$ is isomorphic to each of its conjugates $f^\tau : X^\tau \to B$ ($\tau \in \mathrm{G}_K$), is it defined over K?   [No in general]

Problem 2: If a $\overline{\mathbb{Q}}$-cover $f : X \to B$ is defined over each $\mathbb{Q}_p$, is it defined over $\mathbb{Q}$?

Problem 3: An intersection of fields of definition need not be a field of definition.

Field of moduli

The *field of moduli* K_{m} of a K_{s}-cover $f : X \to B$ is the fixed field in K_{s} of the subgroup

$$M(f) = \{\tau \in \mathrm{G}_K \mid f^\tau \underset{K_{\mathrm{s}}}{\simeq} f\}$$

That is, K_{m} is the smallest subfield of K_{s} such that $\mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{m}})$ maps f to an isomorphic copy of itself.

Basic observation: The field of moduli K_{m} is contained in each field of definition k of f. Thus it is the smallest field of definition if it *is* a field of definition.

Problem 1: Is the field of moduli a field of definition?

Weil's descent criterion

Theorem (Weil) — For each $\tau \in \mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{m}})$, we have $f \underset{K_{\mathrm{s}}}{\simeq} f^\tau$, *i.e.*, there exists an isomorphism $\chi_\tau : X \to X^\tau$ such that $f^\tau \circ \chi_\tau = f$.

Then K_{m} is a field of definition of f iff there exists such isomorphisms χ_τs such that

$$\chi_{uv} = \chi_v^u \chi_u \qquad \text{(cocycle condition)}$$

→ <u>first results</u>:

for mere covers of $\mathbb{P}^1$*:* K_{m} is a field of definition if

- $\mathrm{Aut}(f) = \{1\}$ (Fried)
- $f : X \to \mathbb{P}^1$ is Galois (Coombes-Harbater)

for G*-covers of* $\mathbb{P}^1$*:* K_{m} is a field of definition if

- $Z(G) = \{1\}$ (Belyi et al.)
- G is abelian (—)

→ cohomological problem → $H^2(K_{\mathrm{m}}, \mathrm{Aut}(f))$ ...

Cohomological approach

k-cover	$\longleftrightarrow$	representation
$f : X \to B$		$\phi : \pi_1(B_k^*) \twoheadrightarrow G \subset S_d$
étale above B^*		

for G-covers: $\mathrm{Nor}_{S_d}(G) \leftrightarrow G$; $\mathrm{Cen}_{S_d}(G) \leftrightarrow Z(G)$

Theorem (D. - Douai) — *Assume λ has some lifting Λ. Then there is, explicitly attached to Λ, a 2-cocycle*

$$\Omega_\Lambda \in H^2(K_{\rm m}, Z(G))$$

such that $K_{\rm m}$ is a field of definition of f iff

$$\Omega_\Lambda \in \delta^1 \left(H^1(K_{\rm m}, {\rm Cen}_{S_d}(G)/Z(G))\right)$$

where

$$\delta^1 : H^1(K_{\rm m}, {\rm Cen}_{S_d}(G)/Z(G)) \to H^2(K_{\rm m}, Z(G))$$

is the connecting morphism associated with sequence $1 \to Z(G) \to {\rm Cen}_{S_d}(G) \to {\rm Cen}_{S_d}(G)/Z(G) \to 1$

<u>for G-covers</u>: the *iff* condition is that

$$\Omega_\Lambda \textit{ vanishes in } H^2(K_{\rm m}, Z(G))$$

Consequences

• No obstruction to field of moduli being a field of definition

- if $Z(G) = \{1\}$
- if $\mathrm{G}_{K_{\mathrm{m}}}$ projective, *e.g.* $\mathrm{cd}(K_{\mathrm{m}}) \leq 1$
- for Galois covers of $\mathbb{P}^1$

• The field of moduli of a cover is the intersection of its fields of definition.

• (For covers of $\mathbb{P}^1$). There exists a field of definition K_d with degree $[K_{\mathrm{d}} : K_{\mathrm{m}}]$ over the field of moduli K_{m} bounded as follows.

(for G-covers): $$[K_{\mathrm{d}} : K_{\mathrm{m}}] \leq \frac{|G|}{|Z(G)|}$$

(for mere covers): $$[K_{\mathrm{d}} : K_{\mathrm{m}}] \leq \frac{|\mathrm{Nor}_{S_d} G|}{|\mathrm{Cen}_{S_d} G|}$$

More recent progress

Data: $f : X \to \mathbb{P}^1$ $\overline{\mathbb{Q}}$-cover of field of moduli $\mathbb{Q}$

• $\mathbb{Q}$ need not be a field of definition.
(Coombes-Harbater / Couveignes-Granboulan)

• f is defined over $\mathbb{Q}_p$ for all but finitely many p.
[essentially because the map

$$H^2(\mathbb{Q}, Z(G)) \to \prod_p H^2(\mathbb{Q}_p, Z(G))$$

has values in the direct sum $\coprod_p H^2(\mathbb{Q}_p, Z(G))$]

• f is defined over $\mathbb{Q}_p$ if
  - p does not divide $|G|$, and
  - the branch points do not coalesce modulo p.

(D. - Harbater / Emsalem)

• (Local-global principle for G-covers $f : X \to B$):
f is defined over $\mathbb{Q}$ iff f is defined over each $\mathbb{Q}_p$.
(D. / Douai)

The local-global principle

$f : X \to \mathbb{P}^1$ G-cover over $\overline{\mathbb{Q}}$

defined over each $\mathbb{Q}_p$

• The field of moduli K_{m} is contained in each $\mathbb{Q}_p$, thus equals $\mathbb{Q}$.

• Obstruction to $\mathbb{Q}$ being a field of definition lies in $H^2(\mathbb{Q}, Z(G))$, more precisely in kernel of the map

$$H^2(\mathbb{Q}, Z(G)) \to \prod_p H^2(\mathbb{Q}_p, Z(G))$$

• This map is injective:

- restrict to the case $Z(G) = \mathbb{Z}/n$

- from the Tate-Poitou duality theorem, the above kernel is in duality with the kernel of the map

$$H^1(\mathbb{Q}, \mu_n) \to \prod_p H^1(\mathbb{Q}_p, \mu_n)$$

or, equiv.
$$\mathbb{Q}^*/(\mathbb{Q}^*)^n \to \prod_p \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^n$$

which is clearly injective.

Further questions

• How "often" is the field of moduli a field of definition? Hurwitz spaces make this more precise: how big is the subset of a given Hurwitz space of points/covers defined over their f.o.m.?

• Under assumptions of the local case result, does the cover f have a model over the field $\mathbb{Q}^{\mathrm{t}p}$ of algebraic totally p-adic numbers (and not just over $\mathbb{Q}_p$)?

• Where do mysterious cocycle calculations come from?

$\Rightarrow$ need for more structure: moduli spaces, gerbes ...

Families of covers

Notation: Given $r > 2$, G and $\mathbf{C} = (C_1, \ldots, C_r)$

• $\mathcal{H}_G(\mathbf{C})$: Hurwitz space of covers of $\mathbb{P}^1$ with r branch points, group G and inertia invariant $\mathbf{C}$

• $\mathcal{H}$: an irreducible component of $\mathcal{H}_G(\mathbf{C})$ defined over a field k

Main question: Does there exist a Hurwitz family above $\mathcal{H}$, *i.e.* a finite flat morphism

$$\mathcal{F} : \mathcal{T} \to \mathcal{H} \times \mathbb{P}^1$$

with $\mathcal{F} : \mathcal{T} \to \mathcal{H}$ smooth, projective and with irreducible generic fiber,

such that, for each $h = [f] \in \mathcal{H}$, the fiber cover

$$\mathcal{F} : \mathcal{T}_h \to \mathbb{P}^1$$

is isomorphic to the cover f?

Variants: Hurwitz families of G-covers, with ordered branch points, etc.

→ more structure in definition...

Observations:

• If $\mathcal{H}$ is a given Hurwitz space, assumed irreducible and defined over k, then, for $h = [f_h] \in \mathcal{H}$

$$\text{residue field } k(h) \text{ of } h \in \mathcal{H} = \text{f.o.m. of } f_h$$

<u>If</u> there is a family $\mathcal{F}$ above $\mathcal{H}$, f_h does have a model over its f.o.m. $k(h)$: namely the fiber cover $\mathcal{F}_h$.

• Families exist locally, for the complex topology (Fried), or for the étale topology [DeDoEm].

• If covers have no automorphism ($\mathrm{Cen}_{S_d}(G) = \{1\}$), local families can be glued to provide a global family above $\mathcal{H}$: $\mathcal{H}$ is a *fine* moduli space.

• cohomological pb. → $H^2(\pi_1(\mathcal{H}), \mathrm{Cen}_{S_d}(G))$ ...

Cohomological approach

k-family	$\longleftrightarrow$	representation
$\mathcal{F} : \mathcal{T} \to \mathcal{H} \times \mathbb{P}^1$		$\Phi : \pi_1((\mathcal{H} \times \mathbb{P}^1)^*_k) \twoheadrightarrow G \subset S_d$
étale above $(\mathcal{H} \times \mathbb{P}^1)^*$		

for G-covers: $\mathrm{Nor}_{S_d}(G) \leftrightarrow G$; $\mathrm{Cen}_{S_d}(G) \leftrightarrow Z(G)$

Theorem (D. - Douai - Emsalem) — *Assume λ has some lifting Λ. Then there is a 2-cocycle*

$$\Omega_\Lambda \in H^2(\pi_1(\mathcal{H}), Z(G))$$

such that there is a family above $\mathcal{H}$ iff

$$\Omega_\Lambda \in \delta^1 \left(H^1(\pi_1(\mathcal{H}), \mathrm{Cen}_{S_d}(G)/Z(G))\right)$$

with

$\delta^1 : H^1(\pi_1(\mathcal{H}), \mathrm{Cen}_{S_d}(G)/Z(G)) \to H^2(\pi_1(\mathcal{H}), Z(G))$

the connecting morphism associated with the sequence $1 \to Z(G) \to \mathrm{Cen}_{S_d}(G) \to \mathrm{Cen}_{S_d}(G)/Z(G) \to 1$

<u>for G-covers</u>: the *iff* condition is that

$$\Omega_\Lambda \text{ vanishes in } H^2(\pi_1(\mathcal{H}), Z(G))$$

Consequences

- <u>No obstruction</u> to existence of family

  - above Hurwitz space $\mathcal{H}$ if $Z(G) = \{1\}$

  - above any affine $\overline{k}$-curve $C \subset \mathcal{H}$

  - above $\mathcal{H}_{\overline{k}}$, in the situation of Galois covers with ordered branch points

- The function field $\overline{\mathbb{Q}}(\mathcal{H})$ of the Hurwitz space $\mathcal{H}$ is the intersection of all function fields of étale covers of $\mathcal{H}$ over which there exists a family.

- (For covers with ordered branch points). There exists a family above an étale cover $\widetilde{\mathcal{H}} \to \mathcal{H}_{\overline{\mathbb{Q}}}$ whose degree D can be bounded as follows.

(for G-covers): $$D \leq \frac{|G|}{|Z(G)|}$$

(for mere covers): $$D \leq \frac{|\mathrm{Nor}_{S_d} G|}{|\mathrm{Cen}_{S_d} G|}$$

Gerbes, Covers and Families

Gerbe: a fibered category $\mathcal{G}$ above an étale site $\mathcal{S}$, satisfying the following conditions

1. $\mathcal{G}$ is a prestack: given any open subset $U \subset \mathcal{S}$ and two objects/sections $x, y \in \mathcal{G}(U)$, $\mathrm{Hom}(x, y)$ is a sheaf. [local-global condition on morphisms]

2. $\mathcal{G}$ is a stack: ...
[local-global condition on objects locally given with compatible isomorphisms]

3. Locally, *i.e.*, up to shrinking open subset U, the fiber $\mathcal{G}(U)$ is non-empty.

4. Locally, any two objects in $\mathcal{G}(U)$ are isomorphic.

Patching data: $(U_i, x_i, \chi_{ij})_{i,j\in I}$ where:
- $(U_i)_{i\in I}$ is an open covering of S,
- for $i \in I$, x_i is an object of the category $\mathcal{G}(U_i)$,
- for $i, j \in I$, χ_{ij} isomorphism of category $\mathcal{G}(U_i \cap U_j)$ between restrictions to $U_i \cap U_j$ of x_i and x_j.

$\mathcal{G}$ is **neutral** if global sections exist, *i.e.* $\mathcal{G}(\mathcal{S}) \neq \emptyset$.

The gerbe $\mathcal{G}_f$ of models of a cover $f : X \to \mathbb{P}^1$

Etale site:	$\mathrm{Spec}(K_{\mathrm{m}})_{\mathrm{et}}$
Open subsets:	$U = \mathrm{Spec}(E)/\mathrm{Spec}(K_{\mathrm{m}})$ with E/K_{m} finite Galois extension
Fiber $\mathcal{G}_f(U)$:	Category of E-models of f
Condition 1:	...
Condition 2:	Weil's descent criterion (cocycle condition)
Condition 3:	f has a E-model over suitably large Galois extension E/K_{m}
Condition 4:	Two models become isomorphic over suitably large Galois extension E/K_{m}
Patching data:	K_{m} field of moduli of f
$\mathcal{G}_f$ neutral:	K_{m} field of definition of f

The Hurwitz gerbe $\mathcal{G}$

Etale site:	$\mathcal{H}_{\mathrm{et}}$
Open subsets:	étale Galois cover $U = \widetilde{\mathcal{H}} \to \mathcal{H}$
Fiber $\mathcal{G}(U)$:	Category of families above $\widetilde{\mathcal{H}}$
Condition 1:	...
Condition 2:	Grothendieck's faithfully flat descent theorem
Condition 3:	Hurwitz families exist locally above $\mathcal{H}$
Condition 4:	Hurwitz families above $\mathcal{H}$ are locally isomorphic
Patching data:	$\mathcal{H}$ moduli space
$\mathcal{G}_f$ neutral:	There exists a family above $\mathcal{H}$

Applications of theory of gerbes

• Natural approach to previous problems:
→ conceptual support to cocycle calculations

• **Theorem** (D. -Douai-Emsalem) — *With* $\mathcal{H}$ *as above, if* $|\mathrm{Cen}_{S_d}(G)|$ *is prime to* $|\mathrm{Tor}(H^3(\mathcal{H}(\mathbb{C}), \mathbb{Z}))|$, *then there exists a family above a Zariski open subset of* $\mathcal{H}_{\overline{k}}$.

• **Theorem** (D. -Douai-Emsalem / Wewers) — *Let* $\mathcal{H}$ *be as above and* $h = [f]$ *be a closed point of* $\mathcal{H}$. *The gerbe* $\mathcal{G}(f)$ *of models of* f *is the pull-back of the Hurwitz gerbe* $\mathcal{G}$ *along the map* $\mathrm{Spec}(k(h)) \to \mathcal{H}$.

In general, there exists a specialization morphism

$$\mathrm{Sp}_h : H^2(\mathcal{H}_{et}, \mathcal{L}) \to H^2(k(h)_{\mathrm{et}}, \mathcal{L}(\mathcal{G}(f))$$

What is the kernel of $\prod_h \mathrm{Sp}_h$?
If no family above $\mathcal{H}$, subset of points/covers $h \in \mathcal{H}(\overline{k})$ not defined over their f.o.m.?

• *From p-adic to totally p-adic models.*

Theorem (D. -Douai) *Let $f : X \to \mathbb{P}^1$ be a cover defined over $\overline{\mathbb{Q}}$ with field of moduli $\mathbb{Q}$. If p does not divide $|G|$ and branch points do not coalesce mod. p, then f has a model over $\mathbb{Q}^{\mathrm{t}p}$ (and not only over $\mathbb{Q}_p$).*

Proof. Consider the gerbe $\mathcal{G}(f)$ of models of f. It is a gerbe over $\mathrm{Spec}(\mathbb{Q})$. From the local case result, this gerbe has sections over $\mathrm{Spec}(\mathbb{Q}_p)$. From recent work of Moret-Bailly, the Pop/Moret-Bailly result on totally p-adic points on varieties extends to stacks and gerbes (under suitable conditions). Conclude that the gerbe $\mathcal{G}(f)$ has sections over $\mathrm{Spec}(\mathbb{Q}^{\mathrm{t}p})$, that is, that f has a model over $\mathbb{Q}^{\mathrm{t}p}$.

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