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Weighted Completion of Arithmetic Fundamental Groups

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1. Preface

This is joint work with Makoto Matsumoto. We have a general program to understand the Zariski closure of the image of certain representations of the absolute Galois group on various completions of fundamental groups of algebraic varieties. In our two talks, we aim to sketch a proof of one part of the Deligne-Ihara Conjecture on the action of the absolute Galois group on the fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$.

2. Preliminaries

The algebraic fundamental group of a scheme (geometrically connected and finite type) over a field k will be denoted by

$$\pi_1(X,x)^{\text{arith}}$$

Fix a separable closure K of k. Set

$$\pi_1(X,x)^{\text{geom}} = \pi_1(X \otimes_k K, x)^{\text{arith}}.$$

There is an exact sequence

$$1 \to \pi_1(X,x)^{\text{geom}} \to \pi_1(X,x)^{\text{arith}} \to G_k \to 1,$$

where $G_k := \text{Gal}(K/k)$. We therefore have a representation

$$\phi : G_k \to \text{Out}\, \pi_1(X,x)^{\text{geom}}.$$

This representation is independent of $x \in X$. If k is a sub-field of $\mathbb{C}$, then

$$\pi_1(X,x)^{\text{geom}} = \pi_1(X(\mathbb{C}),x)\hat{}$$

the profinite completion of the topological fundamental group of the complex points of X.

This homomorphism generalizes to many functorially defined completions of $\pi_1(X(\mathbb{C}),x)$.

Pro-ℓ Completion: Fix a prime number ℓ. The pro-ℓ completion

$$\Gamma^{(\ell)}$$

of a group Γ is, by definition, the inverse limit of all finite quotients of Γ of ℓ-power order. We view $\Gamma^{(\ell)}$ as a topological group. When k is a number field, we set

$$\pi_1(X,x)^{\mathrm{geom}(\ell)} = \pi_1(X(\mathbb{C}),x)^{(\ell)}$$

It is also the topological pro-ℓ completion of $\pi_1(X,x)^{\mathrm{geom}}$. Consequently, the representation ϕ induces

$$\phi_\ell : G_k \to \operatorname{Out}\pi_1(X(\mathbb{C}),x)^{(\ell)}$$

Filtrations of $\mathrm{Out}\,\pi_1(X)^{\mathrm{geom}}$

Denote the lower central series of

$$\pi^{(\ell)} := \pi_1(X)^{\mathrm{geom}(\ell)}$$

by

$$\pi_1(X)^{\mathrm{geom}(\ell)} = L^1 \supseteq L^2 \supseteq L^3 \supseteq \cdots$$

This is a filtration by characteristic subgroups and can thus be used to induce filtrations

$$L^0 \supseteq L^1 \supseteq L^2 \supseteq \cdots$$

on $\mathrm{Aut}\,\pi_1(X)^{\mathrm{geom}(\ell)}$ and $\mathrm{Out}\,\pi_1(X)^{\mathrm{geom}(\ell)}$. Define

$$L^n \,\mathrm{Aut}\,\pi^{(\ell)} = \{\phi \in \mathrm{Aut}\,\pi^{(\ell)} : \phi \text{ is trivial mod } L^{n+1}\pi^{(\ell)}\}.$$

One then defines

$$L^n \,\mathrm{Out}\,\pi_1(X)^{\mathrm{geom}(\ell)} = \mathrm{im}\{L^n \,\mathrm{Aut}\,\pi^{(\ell)} \to \mathrm{Out}\,\pi^{(\ell)}\}.$$

4. The Deligne-Ihara Conjecture

Here we take $X = \mathbb{P}^1 - \{0, 1, \infty\}$ and $k = \mathbb{Q}$. We have the representation

$$\phi_\ell : G_\mathbb{Q} \to \mathrm{Out}\,\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})^{\mathrm{geom}(\ell)}$$

and the filtration

$$L^0 \supseteq L^1 \supseteq L^2 \supseteq \cdots$$

of $\mathrm{Out}\,\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})^{\mathrm{geom}(\ell)}$. We can pull back this filtration along ϕ to obtain the *Ihara filtration*

$$G_\mathbb{Q} = I^0 \supseteq I^1 \supseteq I^2 \supseteq \cdots$$

of $G_\mathbb{Q}$. Specifically,

$$I^n G_\mathbb{Q} := \phi_\ell^{-1} L^n\,\mathrm{Out}\,\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})^{\mathrm{geom}(\ell)}.$$

This filtration depends on ℓ.

We have $\mathrm{Gr}_W^0 G_{\mathbb{Q}} \cong \mathbb{Z}_\ell^\times$ and the exact sequence

$$1 \to I^1 G_{\mathbb{Q}} \to G_{\mathbb{Q}} \to \mathrm{Gr}_I^0 G_{\mathbb{Q}} \to 1.$$

Conjecture (Deligne-Ihara): The associated graded Lie algebra

$$(\mathrm{Gr}_I^{>0} G_{\mathbb{Q}}) \otimes \mathbb{Q}_\ell := \Big(\bigoplus_{n>0} \mathrm{Gr}_I^n G_{\mathbb{Q}} \Big) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is freely generated by elements s_3, s_5, s_7, ... where

$$s_{2n+1} \in \mathrm{Gr}_I^{2n+1} G_{\mathbb{Q}}.$$

The fact that it is a Lie algebra is easily established. Modulo the commutator subalgebra of $(\mathrm{Gr}_I^{>0} G_{\mathbb{Q}}) \otimes \mathbb{Q}_\ell$, the element s_{2n+1} is to be dual to the image of a generator of $K_{4n+1}(\mathbb{Z})/\mathrm{tors}$ under the regulator

$$ch_{2n+1} : K_{4n+1}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \xrightarrow{\approx} H^1(G_{\mathbb{Q}}, \mathbb{Q}_\ell(2n+1)).$$

The reasons for this should become clearer during the lecture.

Some General Comments about our Approach:

- The conjecture is expressed in terms of graded objects.

- We will generally work with filtered objects, and only pass to the associated graded module at the last step — cf. homological algebra where often it is best to work with complexes and pass to homology at the last step.

- The property of morphisms of filtered modules that will be crucial in the argument is *strictness*.

5. Strictness

A linear mapping $f : (V_1, W_\bullet) \to (V_2, W_\bullet)$ between two filtered vector spaces is said to be *strict* with respect to the filtrations $W_\bullet$ if it is filtration preserving and if

$$\operatorname{im} f \cap W_m V_2 = f(W_m V_1)$$

for all $m \in \mathbb{Z}$.

Natural examples of categories of filtered vector spaces where the morphisms are strict include:

- the category of (variations of) mixed Hodge structures (Hodge or weight filtrations);
- the category of ℓ-adic local systems of geometric origin (weight filtration);
- the (conjectural) category of mixed motives (weight filtration).

A consequence:

A filtration on a vector space V induces one on every subspace (by intersection) and on every quotient (by projection). In particular, the kernel and image of a filtration preserving mapping $f : (V_1, W_\bullet) \to (V_2, W_\bullet)$ have natural filtrations. If f is strict with respect to $W_\bullet$, then there are natural isomorphisms

$$\ker \mathrm{Gr}^W_\bullet f \cong \mathrm{Gr}^W_\bullet \ker f$$

and

$$\mathrm{im}\, \mathrm{Gr}^W_\bullet f \cong \mathrm{Gr}^W_\bullet \mathrm{im}\, f.$$

This consequence of strictness is key in our argument.

Goal: To replace certain $\pi_1(X, x)^{\text{arith}}$ by a proalgebraic group, each of whose modules has a natural weight filtration such that equivariant maps between modules are strictly compatible with it.

Example: Let

$$V = V' = \mathbb{C}^n \text{ where } n \geq 2.$$

Define

$$W_j V = W_{j-1} V' = \{(z_1, \ldots, z_j, 0, \ldots, 0)\} \cong \mathbb{C}^j.$$

The filtrations $W_\bullet$ of V and V' are increasing. The identity $\mathbb{C}^n \to \mathbb{C}^n$ induces a filtration preserving mapping

$$f : (V, W_\bullet) \to (V', W_\bullet).$$

Note that $f : V \to V'$ is an isomorphism and that

$$\mathrm{Gr}_j^W V \to \mathrm{Gr}_j^W V'$$

is trivial for all j. That is:

$$\mathrm{Gr}_\bullet^W f = 0$$

so that

$$\mathrm{Gr}_\bullet^W V = \ker \mathrm{Gr}_\bullet^W f \neq \mathrm{Gr}_\bullet^W \ker f = 0$$

and

$$0 = \mathrm{im}\, \mathrm{Gr}_\bullet^W f \neq \mathrm{Gr}_\bullet^W \mathrm{im}\, f = \mathrm{Gr}_\bullet^W V.$$

6. ℓ-adic Relative Completion

This is our first attempt at finding such a proalgebraic group. It is an obvious variant of the relative unipotent (or Malcev) completion of a discrete group suggested by Deligne.

Setup:

- Γ is a profinite group;

- S is a reductive algebraic group defined over $\mathbb{Q}_\ell$;

- $\rho : \Gamma \to S(\mathbb{Q}_\ell)$ is a continuous, Zariski dense homomorphism.

Examples:

1. S is the trivial group, Γ any profinite group.

2. $S = \mathbb{G}_m$, $\Gamma = \mathbb{Z}_\ell^\times$, and $\rho : \mathbb{Z}_\ell^\times \to \mathbb{Q}_\ell^\times$ is the inclusion.

3. $S = \mathbb{G}_m$,

   $$\Gamma = G_\ell = \pi_1(\mathrm{Spec}\,\mathbb{Z}[1/\ell]) := \mathrm{Gal}(K^{\mathrm{nr}}/\mathbb{Q}),$$

   where K^{nr} is the maximal algebraic extension of $\mathbb{Q}$ unramified outside ℓ, and ρ is the composite

   $$G_\ell \xrightarrow{\chi_\ell} \mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Q}_\ell^\times,$$

   where χ_ℓ is the cyclotomic character.

4. $S = Sp_{g/\mathbb{Q}_\ell}$, Γ is the profinite completion of any mapping class group associated to genus g curves, and ρ is the natural representation.

The definition: The ℓ-adic completion of Γ relative to ρ consists of:

- a proalgebraic $\mathbb{Q}_\ell$-group $\mathcal{G}$ which is an extension of S by a prounipotent group $\mathcal{U}$,
- a continuous homomorphism $\tilde{\rho} : \Gamma \to \mathcal{G}(\mathbb{Q}_\ell)$ which lifts ρ:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{U}(\mathbb{Q}_\ell) & \longrightarrow & \mathcal{G}(\mathbb{Q}_\ell) & \longrightarrow & S(\mathbb{Q}_\ell) & \longrightarrow & 1 \\ & & & & \uparrow{\scriptstyle \tilde{\rho}} & & \uparrow{\scriptstyle \rho} & & \\ & & & & \Gamma & = & \Gamma & & \end{array}$$

It is required to satisfy the following universal mapping property:
If G is a proalgebraic $\mathbb{Q}_\ell$-group which is an extension of S by a prounipotent group, and if $\phi : \Gamma \to G(\mathbb{Q}_\ell)$ is a continuous homomorphism which lifts ρ, then there is a unique homomorphism of proalgebraic groups $\Phi : \mathcal{G} \to G$ that commutes with the projections to S and such that $\phi = \Phi\tilde{\rho}$.

Remarks:

We view $\mathcal{G}$ as a topological group — the neighbourhoods of the identity are the kernels of the homomorphisms to its finite dimensional quotients. Give $\mathcal{U}$ the induced topology.

Denote the Lie algebras of S, $\mathcal{G}$ and $\mathcal{U}$ by $\mathfrak{s}$, $\mathfrak{g}$ and $\mathfrak{u}$. Then we have an exact sequence

$$0 \to \mathfrak{u} \to \mathfrak{g} \to \mathfrak{s} \to 0.$$

The Lie algebras $\mathfrak{g}$ and $\mathfrak{u}$ are topological Lie algebras — they inherit their topology from those of $\mathcal{G}$ and $\mathcal{U}$.

We can therefore define the continuous cohomology groups

$$H^\bullet_{\mathrm{cts}}(\mathfrak{g}) \text{ and } H^\bullet_{\mathrm{cts}}(\mathfrak{u}).$$

6. Computation of Relative Completions

Let $\{V_\alpha\}$ be a set of representatives of the isomorphism classes of irreducible representations of S.

Theorem 1 *If each $H^j_{\mathrm{cts}}(\Gamma, V_\alpha)$ is finite dimensional when $j = 1, 2$, then*

$$H^1_{\mathrm{cts}}(\mathfrak{u}) = \bigoplus_\alpha H^1_{\mathrm{cts}}(\Gamma, V_\alpha) \otimes V_\alpha^*$$

and there is a natural injection

$$H^2_{\mathrm{cts}}(\mathfrak{u}) \hookrightarrow \bigoplus_\alpha H^2_{\mathrm{cts}}(\Gamma, V_\alpha) \otimes V_\alpha^*$$

To some extent, this reduces the computation of relative completions to cohomology computations. Although this is usually not possible, it is often easier than understanding the actual group Γ.

Recall: The exponential and logarithm mappings give a bijective correspondence between a prounipotent group $\mathcal{U}$ and its Lie algebra $\mathfrak{u}$. So there is no loss of information if we work with the Lie algebra $\mathfrak{u}$ of the prounipotent radical $\mathcal{U}$.

Example 1: If we take S to be the trivial group, $\mathcal{G} = \mathcal{U}$ is just the standard unipotent completion. One can show that if Γ is a finitely generated group, and if U is its unipotent completion over $\mathbb{Q}$, then $U \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is the unipotent completion of its profinite and pro-ℓ completions.

For example, suppose that $F = \langle x_1, \ldots, x_n \rangle$ is a free group on n generators. The Lie algebra of its unipotent completion is the free $\mathbb{Q}$-Lie algebra on n-generators.

The completion of its pro-ℓ (or profinite) completion Γ, has Lie algebra the completion of a free $\mathbb{Q}_\ell$-Lie algebra with n generators.

Example 2: Here $S = \mathbb{G}_m$, $\Gamma = \mathbb{Z}_\ell^\times$ and ρ is the inclusion $\mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Q}_\ell^\times$. Denote the nth power of the standard representation of $\mathbb{G}_m$ by T^n. We have

$$H^1_{\mathrm{cts}}(\mathbb{Z}_\ell^\times, T^n) = \begin{cases} 0 & n \neq 0; \\ \mathbb{Q}_\ell & n = 0 \end{cases}$$

and $H^2_{\mathrm{cts}}(\mathbb{Z}_\ell^\times, T^n) = 0$ for all n. This can be seen using the spectral sequence associated to the extension

$$0 \to \mathbb{Z}_\ell \to \mathbb{Z}_\ell^\times \to (\mathbb{Z}/\ell\mathbb{Z})^\times \to 1$$

when $\ell \neq 2$, and its analogue when $\ell = 2$.

It follows that $\mathcal{U} = \mathbb{G}_a$ and that the relative completion of $\mathbb{Z}_\ell^\times$ with respect to ρ is

$$\mathbb{G}_m \times \mathbb{G}_a.$$

The homomorphism $\tilde{\rho}$ is ρ on the first factor, and

$$\mathbb{Z}_\ell^\times \xrightarrow{\text{projn}} \mathbb{Z}_\ell \hookrightarrow \mathbb{Q}_\ell$$

on the second.

Example 3: Take $S = \mathbb{G}_m$,

$$\Gamma = G_\ell = \pi_1(\mathsf{Spec}\,\mathbb{Z}[1/\ell]) = \mathsf{Gal}(K^{\mathsf{nr}}/\mathbb{Q})$$

and ρ to be the composite

$$G_\ell \xrightarrow{\chi_\ell} \mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Q}_\ell^\times,$$

where χ_ℓ is the cyclotomic character.

We skip this example for the time being as we do not know

$$H^j_{\mathsf{cts}}(G_\ell, \mathbb{Q}_\ell(n))$$

for all n when $j = 1, 2$. Note that the groups that are most interesting are those when $n \geq j$, which are known. Later we shall define the *weighted completion*, which is more relevant to our problem and is computable for this example.

Example 4: Here $S = Sp_{g/\mathbb{Q}_\ell}$, Γ is the profinite completion of the mapping class group

$$\Gamma_g = \pi_0 \,\mathrm{Diff}^+ F$$

where F is a compact oriented surface of genus g.

Here the relative completion $\mathcal{G}_g$ is completely known when $g \geq 6$. It is the $\mathbb{Q}_\ell$-points of the corresponding relative completion of Γ_g, which is a proalgebraic $\mathbb{Q}$-group:

$$\mathcal{G}_g \cong Sp_g \ltimes \mathcal{U}_g$$

where $\mathfrak{u}_g$ is the completion of the graded Lie algebra

$$\mathbb{L}(V)/(T)$$

where

$$V = \Lambda^3 H_1(F)/H_1(F)$$

and

$$T \subseteq \mathbb{L}_2(V) = \Lambda^2(V)$$

is the orthogonal complement of the part with highest weight $2\lambda_2$.

7. Weighted Completion

There is a modified version of relative completion which, in retrospect, is more natural.

The setup:

- Γ is a profinite group;
- S is a reductive algebraic group defined over $\mathbb{Q}_\ell$;
- $w : \mathbb{G}_m \to S$ is a central cocharacter of S;
- $\rho : \Gamma \to S(\mathbb{Q}_\ell)$ is a continuous, Zariski dense homomorphism.

Notation: Denote by T^n the 1-dimensional representation of $\mathbb{G}_m$ on which it acts via the nth power of the standard character. Denote the T^n isotypical component of a $\mathbb{G}_m$-module V by V_n.

Negatively weighted extensions: If

$$1 \to U \to G \to S \to 1$$

is an extension of $\mathbb{Q}_\ell$-groups, where U is unipotent, then $H_1(U)$ is an S-module. It can be regarded as a $\mathbb{G}_m$-module via w, so we can write

$$H_1(U) = \bigoplus_{n \in \mathbb{Z}} H_1(U)_n.$$

We say that this extension is *negatively weighted* if $H_1(U)_n = 0$ whenever $n \geq 0$.

The definition: The *weighted completion of* Γ *with respect to* ρ *and* w consists of:

- a pro-algebraic $\mathbb{Q}_\ell$-group $\mathcal{G}$ which is an extension of S by a prounipotent group $\mathcal{U}$;

- a continuous homomorphism $\tilde{\rho} : \Gamma \to \mathcal{G}(\mathbb{Q}_\ell)$ which lifts ρ:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{U}(\mathbb{Q}_\ell) & \longrightarrow & \mathcal{G}(\mathbb{Q}_\ell) & \longrightarrow & S(\mathbb{Q}_\ell) & \longrightarrow & 1 \\ & & & & \tilde{\rho}\Big\uparrow & & \Big\uparrow\rho & & \\ & & & & \Gamma & = & \Gamma & & \end{array}$$

It is required to satisfy the following universal mapping property:

Given $\phi : \Gamma \to G(\mathbb{Q}_\ell)$, where G is a negatively weighted extension of S such that

$$\begin{array}{ccc} \Gamma & \xrightarrow{\phi} & G(\mathbb{Q}_\ell) \\ \rho \downarrow & & \downarrow \\ S(\mathbb{Q}_\ell) & = & S(\mathbb{Q}_\ell) \end{array}$$

commutes, then we have a unique homomorphism $\Phi : \mathcal{G} \to G$ of $\mathbb{Q}_\ell$-groups such that:

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\tilde{\rho}} & \mathcal{G}(\mathbb{Q}_\ell) & \longrightarrow & S(\mathbb{Q}_\ell) \\ \| & & \Phi \downarrow & & \| \\ \Gamma & \xrightarrow[\phi]{} & G(\mathbb{Q}_\ell) & \longrightarrow & S(\mathbb{Q}_\ell) \end{array}$$

commutes.

The weight filtration:

If G is a negatively weighted extension of S, then every G-module V has a natural weight filtration

$$\cdots \subseteq W_{n-1}V \subseteq W_nV \subseteq W_{n+1}V \subseteq \cdots$$

It is constructed as follows:

Choose any lifting $\tilde{w} : \mathbb{G}_m \to G$ of the central cocharacter $w : \mathbb{G}_m \to S$. Now any G-module V can be regarded as a $\mathbb{G}_m$-module via $\tilde{w}$, and can thus be decomposed as a $\mathbb{G}_m$-module. Define

$$W_mV = \bigoplus_{n \leq m} V_n.$$

Naturality: For a fixed lift $\tilde{w}$, the weight filtration is clearly preserved by every G-module mapping $f : V \to V'$.

It is well defined:

1. First, fix a lift $\tilde{w}$, and use it to put a weight filtration on every G-module.

2. Using the adjoint action, put a weight filtration on $\mathfrak{g}$ and $\mathfrak{u}$. The facts that w is central and the extension is negatively weighted imply that

$$\mathfrak{g} = W_0\mathfrak{g} \text{ and that } \mathfrak{u} = W_{-1}\mathfrak{g}.$$

3. It is not hard to see that $W_\bullet$ behaves well with respect to tensor products. This can be used to show that the infinitesimal action

$$\mathfrak{g} \otimes V \to V$$

preserves the weight filtration.

4. Since $\mathfrak{u} = W_{-1}\mathfrak{g}$, we have $\mathfrak{u} \cdot W_n V \subseteq W_{n-1} V$, which implies that $\mathfrak{g}$ preserves $W_\bullet$ and that it acts trivially on $\mathrm{Gr}^W_\bullet V$.

5. Basic facts about the Levi decomposition imply that any two lifts $\tilde{w}$ are conjugate by an element u of U. This implies that u carries the weight filtration of $\tilde{w}$ onto that of $u\tilde{w}u^{-1}$. Since U preserves the weight filtration, $W_\bullet$ is independent of $\tilde{w}$.

Strictness: By fixing a lift $\tilde{w}$ of w, it is clear that every G-equivariant mapping $f : V \to V'$ is strict with respect to the weight filtration and that the functor

$$\mathrm{Gr}^W_\bullet$$

is exact on the category of G-modules. It is also clear that it behaves well with respect to Hom and $\otimes$.

8. Computation of Weighted Completions

Let $\{V_\alpha\}$ be a set of representatives of the isomorphism classes of irreducible representations of S. Schur's Lemma implies that for each α, there is an integer $n(\alpha)$ such $\mathbb{G}_m$ acts on V_α via w by the $n(\alpha)$th power of the standard character.

Theorem 2 *If each $H^j_{\mathrm{cts}}(\Gamma, V_\alpha)$ is finite dimensional when $j = 1,2$ and $n(\alpha) < 0$, then*

$$H^1_{\mathrm{cts}}(\mathfrak{u}) = \bigoplus_{\{\alpha : n(\alpha) \leq -1\}} H^1_{\mathrm{cts}}(\Gamma, V_\alpha) \otimes V_\alpha^*$$

and there is a natural injection

$$H^2_{\mathrm{cts}}(\mathfrak{u}) \hookrightarrow \bigoplus_{\{\alpha : n(\alpha) \leq -2\}} H^2_{\mathrm{cts}}(\Gamma, V_\alpha) \otimes V_\alpha^*$$

We shall need the stronger (and true) statement: If $H^1_{\mathrm{cts}}(\Gamma, V_\alpha) = 0$ when $-d < n(\alpha) < 0$, then

$$H^2_{\mathrm{cts}}(\mathfrak{u}) \subseteq \bigoplus_{\{\alpha : n(\alpha) \leq -2d\}} H^2_{\mathrm{cts}}(\Gamma, V_\alpha) \otimes V_\alpha^*$$

Example 1: Take $S = \mathbb{G}_m$, $w : \mathbb{G}_m \to \mathbb{G}_m$ to be the (-2)th power of the standard character, $\Gamma = \mathbb{Z}_\ell^\times$ and ρ to be the inclusion $\mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Q}_\ell^\times$.

As before, we have $H^1_{\text{cts}}(\mathbb{Z}_\ell^\times, T^n) = 0$ except when $n = 0$, and $H^2_{\text{cts}}(\mathbb{Z}_\ell^\times, T^n) = 0$ for all n. It follows from this that $\mathcal{U}$ is trivial and that the weighted completion of $\mathbb{Z}_\ell^\times$ with respect to ρ and w is just $\mathbb{G}_m$.

Why take $w = (\ )^{-2}$?

We want representation theoretic weights to equal Galois theoretic weights. The fact that

$$\chi_\ell(F_p) = p$$

then forces us to define $w = (\ )^{-2}$.

Example 2: This example is key. Here we take $\ell \neq 2$, $S = \mathbb{G}_m$, $w : \mathbb{G}_m \to \mathbb{G}_m$ to be the (-2)th power of the standard character,

$$\Gamma = G_\ell = \pi_1(\operatorname{Spec} \mathbb{Z}[1/\ell])$$

and ρ to be the composite

$$G_\ell \overset{\chi_\ell}{\to} \mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Q}_\ell^\times.$$

where χ_ℓ is the cyclotomic character. We shall denote the weighted completion of G_ℓ with respect to χ_ℓ and w by $\mathcal{A}_\ell$, and its unipotent radical by $\mathcal{K}_\ell$:

$$1 \to \mathcal{K}_\ell \to \mathcal{A}_\ell \to \mathbb{G}_m \to 1.$$

Their Lie algebras will be denoted by $\mathfrak{a}_\ell$ and $\mathfrak{k}_\ell$:

$$0 \to \mathfrak{k}_\ell \to \mathfrak{a}_\ell \to \mathbb{Q}_\ell \to 0.$$

Computation of $\mathfrak{k}_\ell$: The pullback of T^{-n} along w is $\mathbb{Q}_\ell(n)$. Thanks to Soulé, we know that when $n \geq 1$ and $\ell \neq 2$,

$$H^1_{\text{cts}}(G_\ell, \mathbb{Q}_\ell(n)) = \begin{cases} \mathbb{Q}_\ell & n = 2m+1, m \geq 0; \\ 0 & \text{otherwise,} \end{cases}$$

and that $H^2_{\text{cts}}(G_\ell, \mathbb{Q}_\ell(n))$ vanishes when $n \geq 2$.

This implies that

$$\begin{aligned} H^1_{\text{cts}}(\mathfrak{k}_\ell) &= \bigoplus_{n>0} H^1_{\text{cts}}(G_\ell, \mathbb{Q}_\ell(n)) \otimes \mathbb{Q}_\ell(-n) \\ &= \bigoplus_{m\geq 0} \mathbb{Q}_\ell(-2m-1). \end{aligned}$$

Taking duals, this implies that

$$\begin{aligned} H_1(\mathfrak{k}_\ell) &= \prod_{n>0} H^1_{\text{cts}}(G_\ell, \mathbb{Q}_\ell(n))^* \otimes \mathbb{Q}_\ell(n) \\ &= \prod_{m\geq 0} \mathbb{Q}_\ell(2m+1). \end{aligned}$$

The vanishing of $H^2_{\text{cts}}(G_\ell, \mathbb{Q}_\ell(n))$ for all $n \geq 2$ implies that $H^2_{\text{cts}}(\mathfrak{k}_\ell) = 0$, and that $\mathfrak{k}_\ell$ is free.

Theorem 3 *If $\ell \neq 2$, the Lie algebra $\mathfrak{k}_\ell$ is a free pro-nilpotent Lie algebra over $\mathbb{Q}_\ell$. It is (un-naturally) isomorphic to the completion of its associated graded*

$$\mathrm{Gr}^W_\bullet \mathfrak{k}_\ell := \bigoplus_{n>0} \mathrm{Gr}^W_{-2n} \mathfrak{k}_\ell,$$

which is a free Lie algebra:

$$\mathrm{Gr}^W_\bullet \mathfrak{k}_\ell = \mathbb{L}\Big(\mathrm{Gr}^W_\bullet H_1(\mathfrak{k}_\ell)\Big) = \mathbb{L}\Big(\bigoplus_{m\geq 0} \mathbb{Q}_\ell(2m+1)\Big).$$

9. Philosophy

Belief: (due to Deligne and/or Beilinson?) To each scheme X, there should be a proalgebraic $\mathbb{Q}$-group $\mathcal{T}_X$ which is an extension of $\mathbb{G}_m$ by a prounipotent group. Denote its Lie algebra by $\mathfrak{t}_X$. The one-dimensional $\mathcal{T}_X$-module corresponding to the nth power of the standard representation of $\mathbb{G}_m$ will be denoted by $\mathbb{Q}(n)$. The Lie algebra $\mathfrak{t}_X$ should have the property that

$$H^m_{\text{cts}}(\mathfrak{t}_X, \mathbb{Q}(n)) \cong K_{2n-m}(X)^{(n)} := \left\{ \begin{matrix} \text{the weight } n \text{ part of} \\ K_{2n-m}(X) \otimes \mathbb{Q} \text{ under} \\ \text{the Adams operations} \end{matrix} \right\}$$

Heuristic: The category of representations of $\mathcal{T}_X$ should be equivalent to the category $\mathcal{T}(X)$ of mixed Tate motives over X, and so

$$H^m(\mathfrak{t}_X, \mathbb{Q}(n)) \cong \operatorname{Ext}^m_{\mathcal{T}(X)}(\mathbb{Q}_X, \mathbb{Q}_X(n))$$

should hold. Motivic philosophy says the RHS is the weight n part of $K_{2n-m}(X)$.

Is $\mathfrak{a}_\ell = \mathfrak{t}_{\mathrm{Spec}\,\mathbb{Z}[1/\ell]} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ **?**

An easy spectral sequence argument shows that

$$H^m_{\mathrm{cts}}(\mathfrak{a}_\ell, \mathbb{Q}_\ell(n)) = \begin{cases} \mathbb{Q}_\ell & m = n = 0; \\ \mathbb{Q}_\ell & m = 1, n \geq 1 \text{ odd}; \\ 0 & \text{otherwise.} \end{cases}$$

Computations of Borel show that

$$K_m(\mathbb{Z}[1/\ell]) = \begin{cases} \mathbb{Q} & m = 0; \\ \mathbb{Q} & m \equiv 1 \mod 4; \\ 0 & \text{otherwise} \end{cases}$$

By results of Soulé, Beilinson and Gillet, we have

$$K_{4n+1}(\mathbb{Z}[1/\ell]) \otimes \mathbb{Q}_\ell = K_{4n+1}(\mathbb{Z}[1/\ell])^{(2n+1)}_{\mathbb{Q}_\ell} \xrightarrow[ch_{2n+1}]{\approx} H^1_{\mathrm{cts}}(G_\ell, \mathbb{Q}_\ell(2n+1)).$$

Thus the regulator induces an isomorphism

$$H^m_{\mathrm{cts}}(\mathfrak{a}_\ell, \mathbb{Q}_\ell(n)) \cong K_{2n-m}(X)^{(n)} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$$

for all m and n, and is thus a candidate for the $\mathbb{Q}_\ell$-form of the motivic Lie algebra of $\mathrm{Spec}\,\mathbb{Z}[1/\ell]$.