Some aspects of Galois-Teichmüller modular groups of small type

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1. Moduli spaces $M_{0,4}, M_{0,5}, M_{1,1}, M_{1,2}$

Let $M_{g,n}$ denote the moduli space (stack defined over $\mathbb{Q}$) of the smooth proper curves of genus g with n marked points $(2-2g-n<0)$.

In this talk, we will mainly be concerned with the following four types of moduli spaces.

$M_{0,4} = \mathbf{P}^1 - \{0, 1, \infty\}$

$M_{0,5} = \mathbf{P}^2 - \{\text{complete quadrangle}\}$

$M_{1,1} =$ the fine version of "j-line"

$M_{1,2} =$ the universal family of affine elliptic curves

The etale fundamental group $\pi_1(M_{g,n}/_{\mathbb{Q}})$ is called the Galois-Teichmüller modular group of type (g, n): There is a group extension

$$1 \to \pi_1(M_{g,n}/_{\overline{\mathbb{Q}}}) \to \pi_1(M_{g,n}/_{\mathbb{Q}}) \to G_{\mathbb{Q}} \to 1,$$

where,

$G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$\pi_1(M_{g,n}/_{\overline{\mathbb{Q}}})$ is naturally identified with the profinite completion of the corresponding mapping class group.

2. $G_{\mathbb{Q}} \hookrightarrow GT$ (Belyi, Drinfeld, Ihara)

We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

The standard (tangential) base point $\overrightarrow{01}$ is by definition the generic geometric point

$$\overrightarrow{01} : \operatorname{Spec} \overline{\mathbb{Q}}\{\{\mathrm{t}\}\} \to \mathbf{P}^1_{\mathrm{t}} - \{0, 1, \infty\},$$

where t is the fixed coordinate of $\mathbf{P}^1$, and

$$\overline{\mathbb{Q}}\{\{t\}\} = \bigcup_{n=1}^{\infty} \overline{\mathbb{Q}}((t^{1/n}))$$

is the Puiseux power series field (known as algebraically closed).

The $G_{\mathbb{Q}}$ acts on the coefficients of each elements of $\overline{\mathbb{Q}}\{\{t\}\}$, which induces the $G_{\mathbb{Q}}$-action on $\pi_1(\mathbf{P}^1_t - \{0, 1, \infty\}, \overrightarrow{01})$.

In general, given (tangential) base points on an algebraic variety V over $\mathbb{Q}$:

$$\vec{v} : \mathrm{Spec}\, \overline{\mathbb{Q}}\{\{\mathrm{t}\}\} \to \mathrm{V},$$
$$\vec{v}' : \mathrm{Spec}\, \overline{\mathbb{Q}}\{\{\mathrm{t}'\}\} \to \mathrm{V},$$

there arises a natural $G_{\mathbb{Q}}$-action on the set of paths $\pi_1(V, \vec{v}, \vec{v}')$.

This notion can be rigorously defined by Grothendieck interpretation of a "path" $\gamma : \vec{v} \to \vec{v}'$ as a natural equivalence of set-valued functors $\Phi_{\vec{v}} \overset{\sim}{\leftarrow} \Phi_{\vec{v}'}$ on the Galois category $Et(V)$: For $\sigma \in G_{\mathbb{Q}}$, $\sigma(\gamma)$ is the compatible collection of isomorphisms

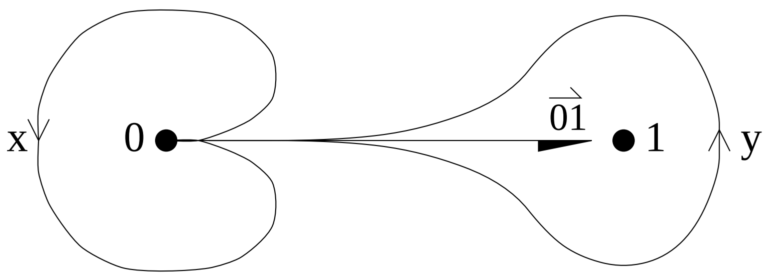
$$\left\{ \Phi_{\vec{v}}(Y) \overset{\sigma}{\leftarrow} \Phi_{\vec{v}}(Y) \overset{\gamma}{\leftarrow} \Phi_{\vec{v}'}(Y) \overset{\sigma^{-1}}{\leftarrow} \Phi_{\vec{v}'}(Y) \right\}_{Y \in Et(V)}.$$

Let x, y denote the standard loops running around the punctures $t = 0, 1$ so that $z = (xy)^{-1}$ runs around $t = \infty$.

The geometric fundamental group

$$\pi_1(\mathbf{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\}, \overrightarrow{01})$$

is a free profinite group generated by the x, y.



The $G_{\mathbb{Q}}$-action can be described in the following form:

$$\boxed{\begin{aligned}&\sigma(x) = x^{\lambda},\\ &\sigma(y) = f(x,y)^{-1} y^{\lambda} f(x,y)\\ &\sigma(z) = x^{\frac{\lambda-1}{2}} f(x,z)^{-1} z^{\lambda} f(x,z) x^{\frac{1-\lambda}{2}}\end{aligned}}$$

Here $\sigma \in G_{\mathbb{Q}}$, $\lambda = \lambda(\sigma) \in \hat{\mathbb{Z}}^*$ is the cyclotomic character, and $f(X,Y) = f_\sigma(X,Y)$ denotes an element (of the commutator subgroup) of the free profinite group generated by the symbols X, Y.

By freeness, $f(A,B)$ makes sense whenever substituting for (X,Y) any pair of elements (A,B) of any profinite group.

It is known by Drinfeld, Ihara that the pairs $(\lambda_\sigma, f_\sigma(X,Y))$ satisfies the following three equations :

$$\boxed{\begin{aligned} &\text{(I)}\ f(X,Y)f(Y,X)=1,\\ &\text{(II)}\ f(X,Y)X^{\frac{\lambda-1}{2}}f(Z,X)Z^{\frac{\lambda-1}{2}}\\ &\quad f(Y,Z)Y^{\frac{\lambda-1}{2}}=1,\\ &\text{(III)}\ f(x_{12},x_{23})f(x_{34},x_{45})f(x_{51},x_{12})\\ &\quad f(x_{23},x_{34})f(x_{45},x_{51})=1 \end{aligned}}$$

The last equation (III) holds in $\pi_1(M_{0,5}/\overline{\mathbb{Q}})$ for certain standard braid generators x_{ij} $(1 \le i,j \le 5)$ of $\pi_1(M_{0,5}/\overline{\mathbb{Q}})$. These equations reflect symmetry of the moduli spaces $M_{0,4} = \mathbf{P}^1 - \{0,1,\infty\}$ and $M_{0,5}$.

The Grothendieck-Teichmüller group $\widehat{GT}$ is the group formed by those pairs $(\lambda, f(X,Y))$ satisfying (I),(II),(III) and inducing automorphisms of the free profinite group $\hat{F}_2$ by $X \mapsto X^\lambda, Y \mapsto f(X,Y)^{-1}Y^\lambda f(X,Y)$. (Multiplication in $\widehat{GT}$ is the composition as automorphisms of $\hat{F}_2$.)

Theorem

(Ihara, Lochak-Schneps, Harbater-Schneps) $\widehat{GT}$ is the automorphism group ($\supset G_{\mathbb{Q}}$) of a certain tower of braid groups or Teichmüller modular groups $\pi_1(M_{0,n}/_{\overline{\mathbb{Q}}})$.

A new version of the Grothendieck-Techimüller group $\mathbb{I}\Gamma$ is introduced to be the group of the invertible pairs (λ, f) satisfying (I),(II),(III) and

(III$'$) $f(a_1 a_3, x_{23})$

$$= g(x_{45}, x_{51}) f(x_{12}, x_{23}) f(x_{34}, x_{45}),$$

(IV) $f(a_1, a_2^4) = a_2^{8\rho_2} f(a_1^2, a_2^2) a_1^{4\rho_2} (a_1 a_2)^{-6\rho_2}.$

Based on the fundamental topological work of Hatcher-Lochak-Schneps, we obtain:

Theorem(N.   -Schneps):

$\mathbb{I}\Gamma$ ($\supset G_{\mathbb{Q}}$) acts on a tower of all types of Teichmüller modular groups $\pi_1(M_{g,n}/_{\overline{\mathbb{Q}}})$ extending the natural $G_{\mathbb{Q}}$-action on them.

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——————- Observation ——————

(III′) $f(a_1a_3, x_{23})$

$$= g(x_{45}, x_{51})f(x_{12}, x_{23})f(x_{34}, x_{45}),$$

(IV) $f(a_1, a_2^4) = a_2^{8\rho_2} f(a_1^2, a_2^2) a_1^{4\rho_2}(a_1a_2)^{-6\rho_2}$.

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(III′) is an equation in $\pi_1(M_{0,5}/_{\overline{\mathbb{Q}}})$ (which implies (III)).

(IV) is an equation in the profinite braid group $\hat{B}_3$.

$a_1, a_2, a_3, \ldots$ denote (images of) the standard generators of braid groups ($a_ia_j = a_ja_i$ if $|i-j| \geq 2$; $a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1}$.)

- What are $g(X, Y), \rho_2$?

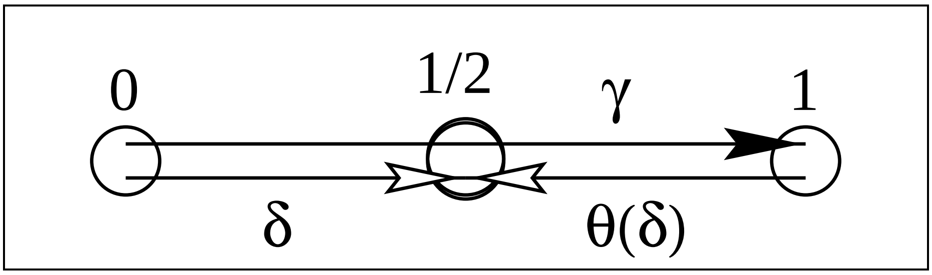
3. Auxiliary parameters g, ρ_2

The main parameter $f_\sigma(x,y)$ for $\sigma \in G_{\mathbb{Q}}$ has interpretation in $\pi_1(\mathbf{P}^1_{\mathbb{Q}} - \{0,1,\infty\})$ as follows. Let $\overrightarrow{01}$, $\overrightarrow{10}$ be tangential base points defined by $\overline{\mathbb{Q}}\{\{t\}\}$, $\overline{\mathbb{Q}}\{\{1-t\}\}$, and let γ be the shortest path from $\overrightarrow{01}$ to $\overrightarrow{10}$ along the real line. The Galois group $G_{\mathbb{Q}}$ acts on the paths $\pi_1(\overrightarrow{01}, \overrightarrow{10})$ and the composition $\gamma \cdot \sigma(\gamma)^{-1}$ gives a loop $\in \pi_1(\mathbf{P}^1_{\mathbb{Q}} - \{0,1,\infty\}, \overrightarrow{01})$. We have then:

$$\gamma \cdot \sigma(\gamma)^{-1} = f_\sigma(x,y) \quad (\sigma \in G_{\mathbb{Q}}).$$

Now we divide the path γ into the halfs:

$$\gamma = \delta \cdot \theta(\delta)^{-1},$$



where δ is the path from $\overrightarrow{01}$ to $\frac{1}{2}$ and

$$\theta \in \mathrm{Aut}(\mathbf{P}^1) : \theta(t) = 1 - t.$$

Lochak-Schneps introduced $g_\sigma(X, Y)$ by

$$\delta \cdot \sigma(\delta)^{-1} = g_\sigma(x, y) \quad (\sigma \in G_{\mathbb{Q}}).$$

The parameter $g = g_\sigma$ is characterized uniquely by the property

$$g(Y, X)^{-1} g(X, Y) = f(X, Y).$$

This characterization can be used to extend the parameter g from $G_{\mathbb{Q}}$ to $\widehat{GT}$.

Lochak-Schneps also introduced another useful 1-cocycle: i.e., instead of $t = \frac{1}{2}$, one can consider $t = \pm\exp(2\pi i/6)$ for additional base points, which are fixed points of the automorphism

$$\omega \in \mathrm{Aut}(\mathbf{P}^1_t) \quad (t \mapsto \frac{1}{1-t}).$$

Then, there arises another parameter $h(X, Y)$ on $\widehat{GT}$ characterized by

$$\begin{aligned} f(X, Y) &= y^{\frac{1-\lambda}{2}} h(Y, Z)^{-1} h(X, Y), \\ &\qquad \text{if } \lambda \equiv 1 \mod 6, \\ &= y^{\frac{1-\lambda}{2}} h(Y, Z)^{-1} y^{-1} h(X, Y), \\ &\qquad \text{if } \lambda \equiv -1 \mod 6. \end{aligned}$$

The "profinite-word" $g(X, Y)$ is in general not lying in the commutator subgroup of the free profinite group of generators X, Y. The abelianization is of the form

$$g(X, Y) \equiv (XY)^{\rho_2} \text{ mod commutators}$$

for some scalar $\rho_2 \in \hat{\mathbb{Z}}$.

It is known that ρ_2 extends the Kummer 1-cocycle for the roots of 2 on $G_{\mathbb{Q}}$, namely,

$$\frac{\sigma(\sqrt[n]{2})}{\sqrt[n]{2}} = \zeta_n^{\rho_2(\sigma)} \quad (\sigma \in G_{\mathbb{Q}}).$$

More systematic method to define Kummer 1-cocycles ρ_a ($a \in \mathbb{N}$) and their generalizations for $\widehat{GT}$ were developed recently by Ihara.

Using G.Anderson's hyperadelic Gamma functions and other arithemetic tools, Ihara introduces between $G_{\mathbb{Q}}$ and $\widehat{GT}$ more intermediate subgroups $\widehat{GTA}$, $\widehat{GTK}$ etc.

Yet, nobody knows whether there are possible equalities among the inclusions:

$$G_{\mathbb{Q}} \subset \{\mathbb{I}\Gamma, \widehat{GTA}, \widehat{GTK}\} \subset \widehat{GT}.$$

Recent inclination is to suspect at least $G_{\mathbb{Q}} \neq \widehat{GT}$.

4. More geometry of $M_{0,4} = \mathbf{P}^1 - \{0, 1, \infty\}$

The quotient line of $\mathbf{P}^1_t$ modulo the S_3-symmetry is essentially the j-line of the elliptic modulus. We denote this line by $\mathbf{P}^1_s$ coordinatized by

$$s = \phi(t) = \frac{27}{4} \frac{t^2(t-1)^2}{(t^2 - t + 1)^3}.$$

The cover $\phi : \mathbf{P}^1_t \to \mathbf{P}^1_s$ is ramified over $s = 0, 1, \infty$, ramification indices over $s = 1$, $s = \infty$ must divide 2,3 respectively.

Write x_s, y_s, z_s for the standard loops running around $s = 0, 1, \infty$, then one gets an open immersion:

$$\pi_1(\mathbf{P}^1_t - \{0,1,\infty\}) \\ \hookrightarrow \pi_1(\mathbf{P}^1_s - \{0,1,\infty\})/\langle\langle y^2, z^3\rangle\rangle$$

Recall we had a path δ from $\overrightarrow{01}$ to $\frac{1}{2}$ on $\mathbf{P}^1_t$. On the other hand, we have $\bar{\gamma}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ for the s-coordinate on $\mathbf{P}^1_s$. Calculating how ϕ maps tangential base points, we get:

$$\begin{array}{ccccl} \overrightarrow{01} & \stackrel{\delta}{\longrightarrow} & \frac{1}{2} & & \text{over } /\mathbf{P}^1_t \\ \phi\Big\downarrow & & \phi\Big\downarrow & & \\ \frac{4}{27}\overrightarrow{01} & \underset{\text{``}\bar{\gamma}\text{''}}{\longrightarrow} & \frac{1}{12}\overrightarrow{10} & & \text{over } /\mathbf{P}^1_s \end{array}$$

Since the $G_{\mathbb{Q}}$-actions on δ, $\bar{\gamma}$ produce the profinite words $g_\sigma(x,y)$, $f_\sigma(x_s,y_s)$, we get a relation in $\pi_1(\mathbf{P}^1_s - \{0,1,\infty\})/\langle\langle y^2, z^3\rangle\rangle$, roughly in the following form:

$$g_\sigma(x,y) \doteqdot f_\sigma(x_s,y_s) \quad (\sigma \in G_{\mathbb{Q}}).$$

Similar considerations on the intermediate covers and a certain lifting through the surjection

$$\hat{B}_3 \to \pi_1(\mathbf{P}^1_s - \{0,1,\infty\}_{/\overline{\mathbb{Q}}})/\langle\langle y^2, z^3\rangle\rangle$$

yields equations representing g_σ, h_σ directly by f_σ :

Theorem (N. -Tsunogai)

For $\sigma \in G_{\mathbb{Q}}$, we have the following equations in $\hat{B}_3 = \langle a_1 a_2 \mid a_1 a_2 a_1 = a_2 a_1 a_2 \rangle$:

$$g_\sigma(a_1^2, a_2^2) = \eta^{2\rho_2 - \rho_3} f_\sigma(a_1, \eta) a_1^{-2\rho_2 + 3\rho_3},$$
$$= f_\sigma(a_1^2, \eta) a_1^{4\rho_2}.$$

Here, η designates $a_1 a_2 a_1$.

We also have similar equations for h_σ:

Theorem (N. -Tsunogai)

$$h_\sigma(a_1^2, a_2^2) = (\xi_\pm)^{\rho_2 + \frac{\lambda \mp 1 - 6\rho_3}{4}} f_\sigma(a_1, \xi_\pm) a_1^{3\rho_3 - 2\rho_2 - \frac{\lambda \mp 1}{2}},$$
$$= (\xi_\pm)^{\frac{\lambda \mp 1 - 6\rho_3}{4}} f_\sigma(a_1^2, \xi_\pm) a_1^{3\rho_3 - \frac{\lambda \mp 1}{2}}.$$

Here, ξ_+, ξ_- denote $a_1 a_2$, $a_2 a_1$ respectively, and the sign $\mp$ is taken according as $\lambda \equiv \pm 1 \bmod 6$ respectively.

Equating RHSs of the above formulae, we obtain equations involving only f_σ:

(IV'') $f(a_1, a_1a_2) = (a_1a_2)^{-\rho_2} f(a_1^2, a_1a_2)a_1^{2\rho_2}$,

(V) $f(a_1, \eta) = \eta^{\rho_3 - 2\rho_2} f(a_1^2, \eta)a_1^{6\rho_2 - 3\rho_3}$.

Theorem (N. -Schneps)

$$(IV')\ f_\sigma(a_1, a_2^2) = a_2^{4\rho_2(\sigma)} f_\sigma(a_1^2, a_2^2) a_1^{2\rho_2(\sigma)} (a_1a_2^2)^{-2\rho_2(\sigma)}.$$

Historically, (IV) was found from $M_{0,5} \doteqdot M_{1,2}$, and then a stronger (IV$'$) was obtained. Finally, (IV$'$) was rephrased as (IV$''$) fitting into the framework of the S_3-cover of j-line: $(IV) \Leftarrow (IV') \Leftrightarrow (IV'')$.

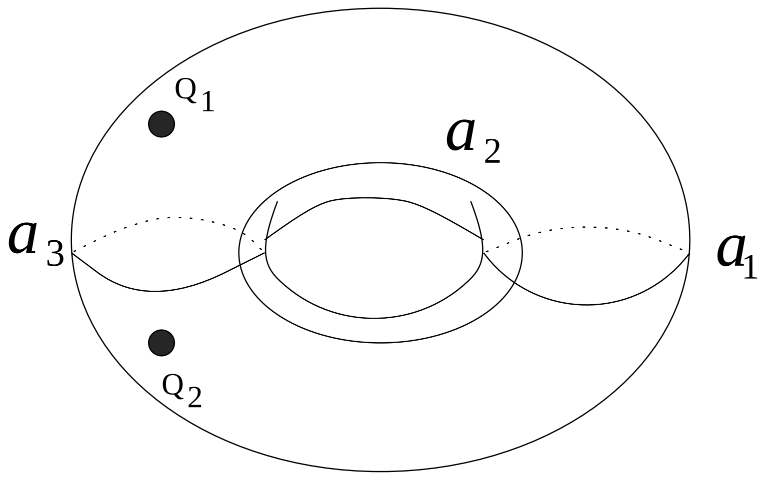
5. Family of elliptic curves

The moduli stack $M_{1,2}$ parametrizes (E, O, Q), the elliptic curves with one distiguished point other than the origin.

The geometric fundamental group

$$\mathfrak{M}_{1,2} = \pi_1(M_{1,2}/_{\overline{\mathbb{Q}}})$$

is the profinite completion of the mapping class group of torus with two marked points, generated by certain Dehn twists a_1, a_2, a_3.



These Dehn twist generators satisfy the braid relations:

$a_1 a_3 = a_3 a_1,$

$a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$ $(i = 1, 2).$

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In fact, it is known that $\mathfrak{M}_{1,2} \cong \hat{B}_4^*$, where B_4^* means the braid group modulo center.

The natural forgetful morphism $M_{1,2} \to M_{1,1}$ gives the universal family of elliptic curves over the "fine j-line" $M_{1,1}$. The geometric fundamental group $\mathfrak{M}_{1,1}$ of the latter stack is isormophic to the profinite completion of $\mathrm{SL}_2(\mathbb{Z})$ which is isomorphic to $\hat{B}_3^\star = \hat{B}_3/\langle \eta^4 \rangle$.

The above forgetful morphism induces a surjection

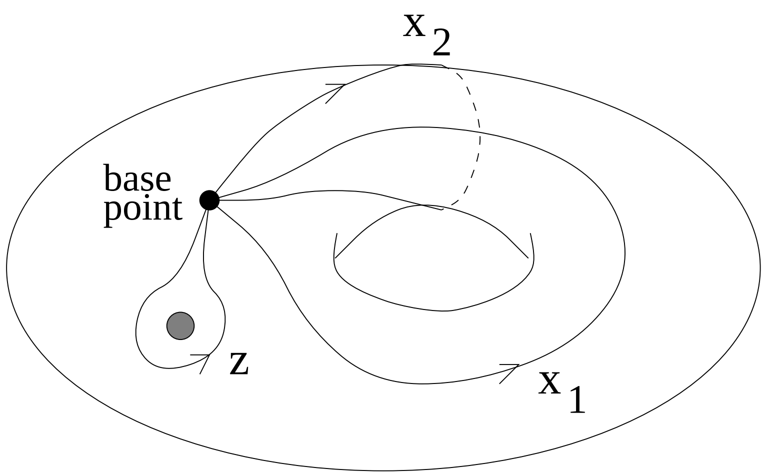
$$\mathfrak{M}_{1,2} = \hat{B}_4^* \twoheadrightarrow \widehat{\mathrm{SL}_2(\mathbb{Z})} = \hat{B}_3^\star$$

The kernel $\mathrm{Ker}(\mathrm{B}_4^* \to \mathrm{B}_3^\star)$ is the free profinite group generated by

$x_1 = a_1^{-1}a_3a_2a_3^{-1}a_1a_2^{-1}, \quad x_2 = a_1a_3^{-1}.$

$$1 \to \langle x_1, x_2\rangle \to B_4^* \to B_3^\star \to 1$$

If we put $z = (a_1a_2)^6$ in $\hat{B}_4^*$, then it holds that $x_1x_2x_1^{-1}x_2^{-1}z = 1$; giving the relation for the fundamental group of elliptic curve minus origin, where z gives monodromy around the origin.



Given an elliptic curve (E, O) over $\mathbb{Q}$, we have a representing homomorphism $G_{\mathbb{Q}} \to \pi_1(M_{1,1})$ such that the pull back of the image in $\pi_1(M_{1,2})$ is isomorphic to $\pi_1(E \setminus \{O\})$.

In this way, $\pi_1(M_{1,2})$ encodes all information of the punctured elliptic curves.

According to Grothendieck's anabelian conjecture (settled by Tamagawa, Mochizuki), the isomorphism classes of hyperbolic curves over number fields are determined by the arithmetic fundamental groups.

We want to know various aspects of

$$1 \longrightarrow \mathfrak{M}_{1,2} \longrightarrow \pi_1(M_{1,2}) \longrightarrow G_{\mathbb{Q}} \longrightarrow 1.$$

Note that the middle group may also be divided as

$$1 \to \langle x_1, x_2 \rangle \to \pi_1(M_{1,2}) \to \pi_1(M_{1,1}) \to 1$$

with $\pi_1(M_{1,1}) = G_{\mathbb{Q}} \ltimes \widehat{\mathrm{SL}_2(\mathbb{Z})}$.

In other words, we want to know how three profinite groups

$$G_{\mathbb{Q}},\ \widehat{\mathrm{SL}_2(\mathbb{Z})} \text{ and } \hat{F}_2$$

are interacting together in $\pi_1(M_{1,2}/_{\mathbb{Q}})$.

6. Tate elliptic curve

Let q be a variable.

The Tate elliptic curve over the field $\mathbb{Q}((q))$ in Weierstrass form is defiend by :

$$Y^2 = 4X^3 - g_2(q)X - g_3(q),$$

where

$$g_2(q) = 20(-\frac{B_4}{8} + \sum_{n\geq 1} \sigma_3(n)q^n),$$
$$g_3(q) = \frac{7}{3}(\frac{B_6}{12} - \sum_{n\geq 1} \sigma_5(n)q^n).$$

($B_4 = -1/30$, $B_6 = 1/42$ are the Bernoulli numbers and $\sigma_k(n)$ is the sum of the k-th powers of the divisors of n.)

Using the $\overline{\mathbb{Q}}$-Puiseux power series in $q^{1/N}$ and $t^{1/N}$ ($t = -2X/Y$, $N \geq 1$), one can define a standard tangential base point near the origin (= infinity point). The picture of the affine Tate elliptic curve is tagent to the degenerate fibre of $M_{1,2} \to M_{1,1}$ over the point $j = \infty$, and so is the fundamental groups. The above tangential base point gives a section $G_{\mathbb{Q}} \to \pi_1(M_{1,2})$.

Theorem. The Galois action on $x_1 = a_1^{-1}a_3a_2a_3^{-1}a_1a_2^{-1}$ and $x_2 = a_1a_3^{-1}$ is given by the formula:

$$x_1 \mapsto z^{\frac{1-\lambda}{2}} f(x_1x_2x_1^{-1}, z)x_1 f(x_2^{-1}, z)^{-1},$$
$$x_2 \mapsto f(x_2^{-1}, z)x_2^{\lambda} f(x_2^{-1}, z)^{-1}$$
$$z \mapsto z^{\lambda}$$

where $(\lambda, f) \in \widehat{GT}$ is the corresponding pair to $\sigma \in G_{\mathbb{Q}}$.

The above "limit $G_{\mathbb{Q}}$-action" on x_1, x_2, z can be generalized to arbitrary type of maximal degeneration of marked algebraic curves (N., Amer. J. Math. 1999).

This is a description of vertical tube (near the degenerate curve) in the moduli space $M_{1,2}$. For the horizontal tube (near the origins locus), it is useful to lift things to level 2 elliptic curves, as the level 2 modular curve $\cong \mathbf{P}^1 - \{0, 1, \infty\}$! This lies in $M_{1,2}[2]$ tangent to the origins locus.

Combining these vertical and horizontal description, we get a $G_{\mathbb{Q}}$-action on the whole $\mathfrak{M}_{1,2}$ as follows:

$$\sigma(a_1) = a_1^{\lambda},$$

$$\sigma(a_2) = f(a_1^2, a_2^2)^{-1} a_2^{\lambda} f(a_1^2, a_2^2),$$

$$\sigma(a_3) = f(\eta^4, a_3)^{-1} a_3^{\lambda} f(\eta^4, a_3).$$

$$(\eta = a_1 a_2 a_1)$$

Here, the tangential base point differs from the above one by the factor $(a_1^2)^{4\rho_2}$ according to the fact that the principal coefficient of the level 2 modular function $\lambda(q)$ is 16.

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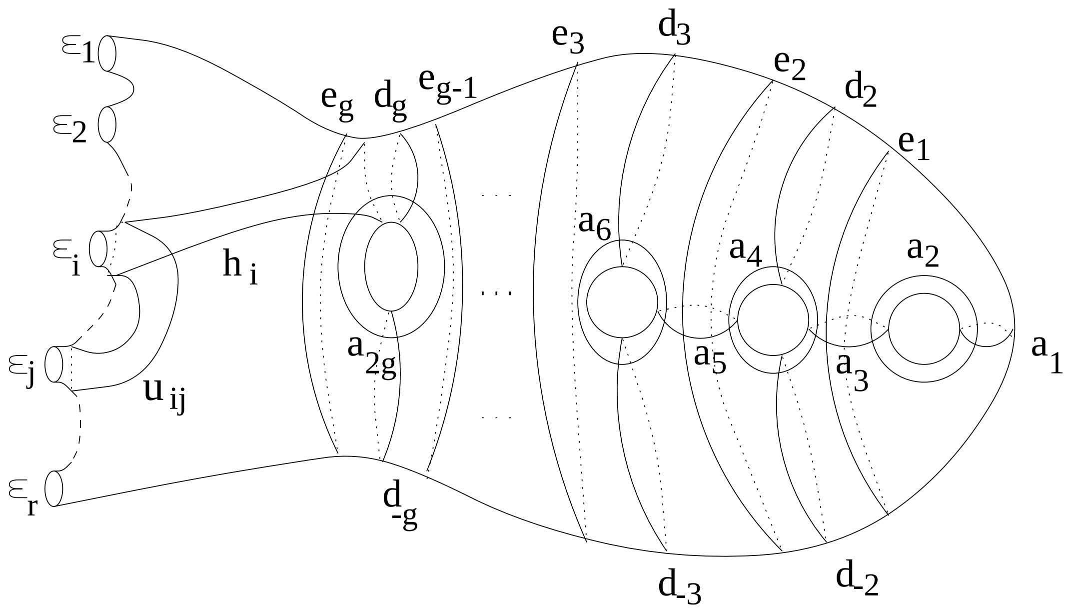
The original guess of the equation (IV) was obtained by (carefully) comparing the above action with the standard $G_{\mathbb{Q}}$ action on $\hat{B}_4$:

$$\sigma(a_1) = a_1^{\lambda},$$
$$\sigma(a_2) = f(a_1^2, a_2^2)^{-1} a_2^{\lambda} f(a_1^2, a_2^2),$$
$$\sigma(a_3) = f(\eta^2, a_3^2)^{-1} a_3^{\lambda} f(\eta^2, a_3^2).$$

Remark (N. -Schneps) The above $G_{\mathbb{Q}}$-action on the standard Dehn twist generators a_1, a_2, a_3 can also be "systematically" generalized to the $\mathbb{\Gamma}$-action on the general mapping class groups $\mathfrak{M}_{g,n}$.



The story of Galois-Teichmüller modular groups of small type also should continue more, as Grothendieck called them as “real jewel” in his Esquisse d’un Programme.