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FUNDAMENTAL GROUPS AND GEOMETRY IN POSITIVE CHARACTERISTIC

"Galois Actions and Geometry" Workshop, MSRI

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October 11, 1999

§. **Anabelian Geometry.**

k: field, $\overline{k}$: algebraic closure

U: (geometrically connected) variety $/k$, $\overline{U} = U \underset{k}{\otimes} \overline{k}$

Then we have

$$1 \to \pi_1(\overline{U}) \to \pi_1(U) \stackrel{\mathrm{pr}_U}{\to} G_k \to 1$$

or

$$G_k \stackrel{\rho_U}{\to} \mathrm{Out}(\pi_1(\overline{U}))$$

<u>Anabelian Geometry</u>:

Recover **Geometry** of U from (outer) **Galois Action** ρ_U on geometric fundamental group $\pi_1(\overline{U})$ for 'anabelian' U/k

(Birational version → talks by Pop and Efrat)

However, this talk:

– No Galois Actions (by considering char. > 0)

– Very Little Geometry (by considering curves)

§. Geometric π_1 of Varieties.

$f : \mathcal{U} \to S$: "good" family over connected S/k

e.g. proper smooth minus relatively normal crossing divisors

Then:

<u>$\mathrm{char}(k) = 0$</u>

$\pi_1(\forall$geom. fiber of $f)$ are isomorphic

$\approx \pi_1(\overline{U})$ depends only on the connected component
of moduli space to which U belong

<u>$\mathrm{char}(k) = p > 0$</u>

$\pi_1($geom. fiber of $f)$ may depend on geom. point of S

(though $\pi_1(\forall$geom. fiber of $f)^{p'}$ are isomorphic)

$\Longrightarrow$ possibility to recover geometry only from $\pi_1(\overline{U})$

§. Geometric π_1 of Curves.

From now on,

$k = \overline{k}$

U: smooth curve $/k$

X: smooth compactification of U, $g =$ genus of X

$S = X - U$, $n = \sharp(S)$

(g, n): "type" of U

<u>$\mathrm{char}(k) = 0$</u>

We have

$$\pi_1(U) \overset{\sim}{\leftarrow} \widehat{\Pi}_{g,n}$$

where

$$\begin{aligned}\Pi_{g,n} &= \pi_1^{\mathrm{top}}(\text{Riemann surface of type } (g,n)) \\ &\simeq \langle a_1, b_1, ..., a_g, b_g, c_1, ..., c_n \mid \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{n} c_j = 1 \rangle\end{aligned}$$

In particular, $\pi_1(U)$ depends only on (g, n).

$\underline{\mathrm{char}(k) = p > 0}$

$\pi_1(U)$ is not finitely generated (for $n > 0$), but we have

$$\pi_1(U) \twoheadrightarrow \pi_1^{\mathrm{t}}(U) \twoheadleftarrow \widehat{\Pi}_{g,n}$$

(π_1^{t} = tame π_1, which allows only tame ramification over S.)
This induces

$$\pi_1(U)^{p'} \overset{\sim}{\to} \pi_1^{\mathrm{t}}(U)^{p'} \overset{\sim}{\leftarrow} (\widehat{\Pi}_{g,n})^{p'}$$

For pro-p quotients, we have, if $n > 0$,

$$\pi_1(U)^p = \text{free pro-}p\text{ group of rank } \sharp(k)$$

and, in general,

$$\pi_1^{\mathrm{t}}(U)^p = \pi_1(X)^p = \text{free pro-}p\text{ group of rank } \gamma$$

where

$$\begin{aligned} \gamma &= p\text{-rank (Hasse-Witt invariant) of } X \\ &= \dim_{\mathbb{F}_p} J[p](k) \quad (J = \mathrm{Jac}_X) \end{aligned}$$

($0 \leq \gamma \leq g$, and $\gamma = g \iff J$: ordinary)

§. Isomorphism Classes.

From now on, $\mathrm{char}(k) = p > 0$ unless otherwise stated.

$$\begin{array}{rl} \mathcal{C}_k & = \{\text{smooth connected curves } /k\}/ \underset{\mathrm{Sch}}{\simeq} \\ \cup & \\ \mathcal{C}(g,n)_k & = \{\text{smooth connected curves } /k \text{ of type } (g,n)\}/ \underset{\mathrm{Sch}}{\simeq} \end{array}$$

Then we have

$$\pi_1 \ (\text{or } \pi_1^{\mathrm{t}}) : \mathcal{C}_k \to \{\text{profinite groups}\}/ \simeq$$

Theorem 1 (T).

(i) $\pi_1|_{\mathcal{C}(0,n)_{\overline{\mathbb{F}}_p}}$ is injective.

(ii) $\pi_1^{\mathrm{t}}|_{\mathcal{C}(0,n)_{\overline{\mathbb{F}}_p}}$ is injective.

Theorem 2 (Pop-Saïdi). (→ Saïdi's talk)

For $g \geq 2$, $\pi_1|_{\mathcal{C}(g,0,+\text{ some conditions})_{\overline{\mathbb{F}}_p}}$ is almost injective, i. e. its fibers are finite sets.

§. Type (g, n).

Theorem 3 (T).

(i) (g, n) can be recovered from $\pi_1(U)$.

(ii) (g, n) can be recovered from $\pi_1^{\mathrm{t}}(U)$, except for $(g, n) = (0, 0), (0, 1)$.

Example. (Harbater, Bouw) E: elliptic curve $/k$

If $\mathrm{char}(k) = 0$, we have

$$\pi_1(E - \{O\}) \simeq \pi_1(\mathbf{P}^1 - \{0, 1, \infty\})(\simeq \widehat{F}_2)$$

If $\mathrm{char}(k) = p > 0$, we have

(i) $$\pi_1(E - \{O\}) \not\simeq \pi_1(\mathbf{P}^1 - \{0, 1, \infty\})$$

(ii) $$\pi_1^{\mathrm{t}}(E - \{O\}) \not\simeq \pi_1^{\mathrm{t}}(\mathbf{P}^1 - \{0, 1, \infty\})$$

(i) can be reduced to (ii) by using Hurwitz formula. (ii) is easy (resp. not so easy) for E ordinary (resp. supersingular).

Key to Th3(ii): Generalization of Raynaud's theory of Θ-divisors.

§. Inertia Subgroups.

Points of $S \rightsquigarrow$ inertia subgroups in $\pi_1(U)$ (or $\pi_1^{\mathrm{t}}(U)$)

Proposition 4 (T).

(i) The set of inertia subgroups (and their higher ramification subgroups) in $\pi_1(U)$ can be recovered from $\pi_1(U)$.

More precisely, given U_i/k_i for $i = 1, 2$, any isomorphism

$$\pi_1(U_1) \simeq \pi_1(U_2)$$

preserves the set of inertia (and higher ramification) subgroups.

(ii) The set of inertia subgroups in $\pi_1^{\mathrm{t}}(U)$ can be recovered from $\pi_1^{\mathrm{t}}(U)$. More precisely, given U_i/k_i for $i = 1, 2$, any isomorphism

$$\pi_1^{\mathrm{t}}(U_1) \simeq \pi_1^{\mathrm{t}}(U_2)$$

preserves the set of inertia subgroups.

Key: Apply Th3 for all covers of U. For higher ramification subgroups, use the Lefschetz trace formula, which connects Artin characters with global information obtained from π_1.

§. Digression: "Container" of Galois Group.

$\underline{\mathrm{char}(k) = 0}$

Fix $\widehat{\Pi}_{g,n} \xrightarrow{\sim} \pi_1(U)$ so that $\langle \overline{c}_j \rangle$'s are inertia subgroups ($\overline{c}_j = \mathrm{Im}(c_j)$).

$$\mathrm{Aut}^*(\pi_1(U)) \stackrel{\mathrm{def}}{=}$$
$$\{\sigma \in \mathrm{Aut}(\pi_1(U)) \mid \exists \lambda \in \widehat{\mathbb{Z}}^\times, \exists s \in S_n,\ \sigma(\overline{c}_j) \sim \overline{c}_{s(j)}{}^\lambda\}$$

$$\mathrm{Out}^*(\pi_1(U)) \stackrel{\mathrm{def}}{=} \mathrm{Aut}^*(\pi_1(U)) / \mathrm{Inn}(\pi_1(U))$$

(independent of the choice of $\widehat{\Pi}_{g,n} \xrightarrow{\sim} \pi_1(U)$)

Known: $\mathrm{Im}(\mathrm{Aut}_{\mathrm{Sch}}(U)) \subset \mathrm{Out}^*(\pi_1(U))$

(Note: $U = U_0 \underset{k_0}{\otimes} k \implies \rho_{U_0}(G_{k_0}) \subset \mathrm{Im}(\mathrm{Aut}_{\mathrm{Sch}}(U))$)

Important: Define a good (smaller) container and capture the Galois group

→ talks by Schneps, Zapponi and Nakamura;

→ talks by Hain and Matsumoto in pro-algebraic group context

$\underline{\mathrm{char}(k) = p > 0}$

Fix $\widehat{\Pi}_{g,n} \twoheadrightarrow \pi_1^{\mathrm{t}}(U)$ so that $\langle \overline{c}_j \rangle$'s are inertia subgroups.

$$\mathrm{Aut}^*(\pi_1^{\mathrm{t}}(U)) \stackrel{\mathrm{def}}{=}$$

$$\{\sigma \in \mathrm{Aut}(\pi_1^{\mathrm{t}}(U)) \mid \exists \lambda \in p^{\widehat{\mathbb{Z}}} \subset (\widehat{\mathbb{Z}}^{p'})^\times, \exists s \in S_n,\ \sigma(\overline{c}_j) \sim \overline{c}_{s(j)}{}^\lambda\}$$

$$\mathrm{Out}^*(\pi_1^{\mathrm{t}}(U)) \stackrel{\mathrm{def}}{=} \mathrm{Aut}^*(\pi_1^{\mathrm{t}}(U)) / \mathrm{Inn}(\pi_1^{\mathrm{t}}(U))$$

(independent of the choice of $\widehat{\Pi}_{g,n} \twoheadrightarrow \pi_1^{\mathrm{t}}(U)$)

Known: $\mathrm{Im}(\mathrm{Aut}_{\mathrm{Sch}}(U)) \subset \mathrm{Out}^*(\pi_1^{\mathrm{t}}(U))$

(Note: $U = U_0 \underset{k_0}{\otimes} k$ with $\overline{k}_0 = k \implies \rho_{U_0}(G_{k_0}) \subset \mathrm{Im}(\mathrm{Aut}_{\mathrm{Sch}}(U))$)

Proposition 5 (T).

$$\mathrm{Aut}(\pi_1^{\mathrm{t}}(U)) = \mathrm{Aut}^*(\pi_1^{\mathrm{t}}(U))$$

$$\mathrm{Out}(\pi_1^{\mathrm{t}}(U)) = \mathrm{Out}^*(\pi_1^{\mathrm{t}}(U))$$

Question. ($n = 0$.) For $g \geq 2$, is the image of

$$\mathrm{Out}(\pi_1(X)) \to \mathrm{Aut}(H^2(\pi_1(X), \widehat{\mathbb{Z}}^{p'})) = (\widehat{\mathbb{Z}}^{p'})^\times$$

contained in $p^{\widehat{\mathbb{Z}}}$? (OK for $\mathrm{Im}(\mathrm{Aut}_{\mathrm{Sch}}(X))$.)

§. Base Fields and Fields of Definition.

U/k as above ($\mathrm{char}(k) = p > 0$)

Definition.

$\mathrm{td}(k)$ = transcendence degree of k over $\overline{\mathbb{F}}_p$

$\mathrm{ed}(U) = \min\{\mathrm{td}(k_0) \mid k_0 \subset k,\ U \simeq \exists U_0 \underset{k_0}{\otimes} k\}$ ($\leq \mathrm{td}(k)$)

<u>*Remark*</u>.

(i) $\mathrm{td}(k)$ determines the isom. class of k, since $k = \overline{k}$.

(ii) $\mathrm{ed}(U) \leq \dim(M_{g,n})$ ($= 3g - 3 + n$, if $\chi < 0$)

For algebraically closed fields $k_0 \subset k$,

$$\pi_1(U_0 \underset{k_0}{\otimes} k) \twoheadrightarrow \pi_1(U_0)$$

$$\pi_1^{\mathrm{t}}(U_0 \underset{k_0}{\otimes} k) \xrightarrow{\sim} \pi_1^{\mathrm{t}}(U_0)$$

Question.

(i) For $n > 0$, can $\mathrm{td}(k)$ be recovered from $\pi_1(U)$?

(ii) For $(g, n) \neq (1, 0)$, can $\mathrm{ed}(U)$ be recovered from $\pi_1^{\mathrm{t}}(U)$?

Remark.

(i) We have

$$\pi_A(U_0 \underset{k_0}{\otimes} k) = \pi_A(U_0)$$

where π_A denotes the set of finite quotients of π_1.

Moreover, Abhyankar's conjecture (proved by Raynaud and Harbater) gives explicit description of $\pi_A(U)$ for $n > 0$, which depends only on (g, n) and p.

(ii) $\pi_A^{\mathrm{t}}(U)$ has as much information as $\pi_1^{\mathrm{t}}(U)$, but has not been understood well.

Lemma 6. For $n > 0$,

$$\dim_{\mathbb{F}_p}(H^1(\pi_1(U), \mathbb{F}_p)) = \begin{cases} \omega & \mathrm{td}(k) \leq \omega \\ \mathrm{td}(k) & \mathrm{td}(k) > \omega \end{cases}$$

In particular, $\mathrm{td}(k)$ can be recovered from $\pi_1(U)$ for $\mathrm{td}(k) > \omega$.

Proposition 7 (T). Assume $g = 0$.
(i) For $n > 0$, whether $\mathrm{td}(k) = 0$ or not can be detected from $\pi_1(U)$.
(ii) Whether $\mathrm{ed}(U) = 0$ or not can be detected from $\pi_1^{\mathrm{t}}(U)$.
In particular, $\mathrm{ed}(U)$ can be recovered from $\pi_1^{\mathrm{t}}(U)$ for $n \leq 4$.

Proof. Corollary of (proof of) Theorem 1. For (i), use also an old lemma of Abhyankar.

§. Hopfian Properties.

Definition.

$$\pi_1(U)\text{: Hopfian} \iff \forall \pi_1(U) \twoheadrightarrow \pi_1(U) \text{ is bijective}$$

Since every finitely generated profinite group is Hopfian, assume $n > 0$ in this §.

Lemma 8.

$$\pi_1(U)\text{: Hopfian} \implies \mathrm{td}(k) < \omega \;\&\; (g, n) \neq (0, 1)$$

Question H. $\Longleftarrow$?

Definition. $\pi_1(U)$: weakly Hopfian $\iff$
$\pi_1(U) \twoheadrightarrow \pi_1(U)$ is bijective, if it induces surjections of inertia subgroups and higher ramification subgroups

Lemma 9.

$$\pi_1(U)\text{: weakly Hopfian} \implies \mathrm{td}(k) < \omega$$

Question WH. $\Longleftarrow$?

Yes to Question H $\implies$ Yes to Question WH $\implies$
$\mathrm{td}(k)$ can be recovered from $\pi_1(U)$ for $g = 0$.

§. Free Quotients.

Definition.

$$\rho(U) = \max\{r \mid \exists \pi_1(U) \twoheadrightarrow \widehat{F}_r\}$$

$$\rho^{\mathrm{t}}(U) = \max\{r \mid \exists \pi_1^{\mathrm{t}}(U) \twoheadrightarrow \widehat{F}_r\}$$

<u>*Remark*</u>.

(i) $\exists \pi_1^{\mathrm{t}}(U) \twoheadrightarrow \widehat{F}_r \iff \pi_A^{\mathrm{t}}(U) \supset \pi_A(\widehat{F}_r)$

(ii) For $n > 0$, $\pi_A(U) \supset \pi_A(\widehat{F}_r) \iff r \leq 2g + n - 1$

Lemma 10.

(i) $\rho(U) \geq \rho^{\mathrm{t}}(U) \geq \rho(X)$

(ii) $\rho(U) \leq 2g + n - 1$ for $(g, n) \neq (0, 0)$

(iii) $\rho^{\mathrm{t}}(U)(\leq \gamma) \leq g$

<u>*Remark*</u>.

(i) For $n > 0$, no upper bound better than (ii) is known, even for a single U.

(ii) Degeneration method (or "patching") sometimes gives a lower bound. ($\rightarrow$ talks by Stevenson and Harbater)

For example, if X is generic, then we have

$$\rho^{\mathrm{t}}(U) = \rho(X) = g$$

<u>*Question*</u>. Assume $k = \overline{\mathbb{F}}_p$.

(i) $\rho^{\mathrm{t}}(U) \leq 1$?

(ii) $\rho(U) \leq 1$?

(Possibly absurd. Yes $\implies \nexists \pi_1(\mathbf{A}^1_{\overline{\mathbb{F}}_p}) \twoheadrightarrow \mathbb{Z}_p * \mathbb{Z}_p$)