Some Illustrative Examples for Anabelian Geometry in High Dimensions

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§1. Introduction

In [G1-2], Grothendieck conjectured that smooth (possibly non-complete) irreducible *hyperbolic* curves X over a finitely generated field (say, over a fixed algebraic number field k) are determined uniquely by their algebraic fundamental groups $\pi_1(X)$ (which are naturally extensions of $G_k = \mathrm{Gal}(\bar{k}/k)$ by $\pi_1(\overline{X})$, where $\bar{k}$ is an algebraic closure of k and $\overline{X} = X \times \bar{k}$). This means, for example, that the cross ratio of four k-rational points $a_1, \ldots, a_4$ on $\mathbb{P}^1$ should be determined by $\pi_1(\mathbb{P}^1 - \{a_1, \ldots, a_4\})$. This conjecture of Grothendieck was first proved in the case of genus 0 ([N1]), then by Tamagawa in the case of arbitrary non-complete curves ([T1]), and finally, by Mochizuki [M1][M2] in all cases (even over local fields). On the other hand, Pop has proved that finitely generated fields K over a prime field (say, over $\mathbb{Q}$) are determined uniquely by their absolute Galois groups $G_K = \mathrm{Gal}(\bar{K}/K)(= \pi_1(\mathrm{Spec}\, K))$ [P]. So, if we call, after Grothendieck, a class $\mathfrak{X}$ of algebraic varieties *anabelian*, when the functor

$$\mathfrak{X} \ni X \longmapsto \pi_1(X) \in \{\text{Profinite groups}\}$$

(where, by definition, the only morphisms are isomorphisms (and those modulo inner automorphisms)) is fully faithful, then all hyperbolic curves over k, as well as spectra of local rings at the generic points of any irreducible algebraic varieties over k, form anabelian classes. By using the term "anabelian", Grothendieck seems to suggest that one could expect such phenomena to occur even in higher dimensional situations, and that whether this holds or not would be tightly related to whether the geometric part $\pi_1(\overline{X})$ of $\pi_1(X)$ is "far from" being an abelian group.

Thus, the main problems are (i) to find wider classes of anabelian varieties ("only" the higher dimensional case is left open), and (ii) to see how the geometry of X is reflected in the group theory of $\pi_1(X)$. This note is to give two types of examples in the higher dimensional case, one being an "alarming example", and the other, "supporting".

Among the first things to note is that Pop's theorem suggests a possibility that every irreducible variety over k has a non-empty open subvariety which is "anabelian" (including Artin type neighborhoods of each nonsingular closed point [AGV] Exp.XI, Prop.3.3, cf. also [N4])[1]. Another is that by Lefschetz' theorem, general hyperplane cuts of quasi-projective varieties of dimension > 2 leave fundamental groups invariant, warning us that one should choose a good model from each equivalence class of varieties having the same "anabelian" fundamental groups. (Should they always be $K(\pi,1)$?) About hyperbolicity, since varieties of general type and hyperbolic manifolds (in the sense of [Ii][Ko]) are natural higher dimensional generalizations of hyperbolic curves, one asks what additional conditions would be necessary in order that these varieties be anabelian. As a test, we examine whether the "automorphism group" of $\pi_1(X)$ (one can put several different senses in this) is canonically isomorphic to $\mathrm{Aut}_k X$. We shall show that some Shimura varieties of the most classical split type, namely the Hilbert modular varieties (e.g. surfaces) and the Siegel modular varieties, *cannot* be anabelian[2]. Finally, we shall also show, as a "favorable anabelian example", that if X is a braid configuration of a hyperbolic curve, then 'Aut'$\pi_1(X)$ corresponds bijectively with $\mathrm{Aut}_k X$, by combining a previous result of the second named author (partly with Takao) and the (above mentioned) result of Tamagawa and Mochizuki. Of course, these are only two (types of) examples, but it might be worthwhile to keep them in mind (somewhere in the corner) in studying "higher anabelian varieties".

§2. Group of self-equivalence classes

In this article, we only consider smooth algebraic varieties X over a number field k which is embedded in the field $\mathbb{C}$ of complex numbers. Given such an X/k, identify $\pi_1(X)$ with the total Galois group of the tower of all finite etale coverings of X. Then, since the tower consists of the constant field extension and the geometric covering extension, we have a canonical exact sequence of Galois groups:

$$(2.1) \qquad 1 \to \pi_1(\overline{X}) \to \pi_1(X) \overset{p_{X/k}}{\to} G_k \to 1.$$

Here, the kernel group $\pi_1(\overline{X})$ classifies the geometric covering extensions over $\overline{X} = X \times \bar{k}$ and is isomorphic to the profinite completion of the usual

[1] In Grothendieck [G2] p.3, one reads "Andernteils sehe ich eine Mannigfaltigkeit jedenfalls dann als "anabelsch" (ich könnte sagen "elementar anabelsch") an, wenn sie sich durch successive (glatte) Faserungen aus anabelschen Kurven aufbauen lässt. Demnach (einer Bemerkung von M.Artin zufolge) hat jeder Punkt einer glatten Mannigfaltigkeit X/K ein Fundamentalsystem von (affinen) anabelschen Umgebungen."

[2] Unexpectedly from a (vague) statement by Grothendieck "ich würde annehmen, dass dasselbe (i.e., anabelianity) auch für die Modulmultiplizitäten polarisierter abelscher Mannigfaltigkeiten gelten dürfte" ([G2] p.3).

discrete fundamental group of the associated manifold $X(\mathbb{C})$. Every automorphism $f \in \mathrm{Aut}_k X$ induces an element of $\mathrm{Aut}_{\bar{k}}(\overline{X})$, which in turn induces (by extension and conjugation) an automorphism of $\pi_1(X)$ determined up to inner automorphisms by elements of $\pi_1(\overline{X})$. This automorphism of $\pi_1(X)$ reduces to the identity on G_k, as f is defined over k. In other words, we have a natural homomorphism

$$\mathrm{Aut}_k X \longrightarrow E_k(X) :\stackrel{def}{=} \mathrm{Aut}_{G_k}(\pi_1(X))/\mathrm{Int}\pi_1(\overline{X}),$$

where $\mathrm{Aut}_{G_k}(\pi_1(X)) := \{\alpha \in \mathrm{Aut}(\pi_1(X)) \mid p_{X/k} = p_{X/k} \circ \alpha\}$ and $\mathrm{Int}\pi_1(\overline{X})$ is its subgroup of the inner automorphisms by the elements of $\pi_1(\overline{X})$. The first naive criterion for anabelianity of X would be to check whether the group $E_k(X)$ recovers (or, in the strongest sense, is isomorphic to) $\mathrm{Aut}_k X$.

Another candidate for approximating $\mathrm{Aut}_k X$ in terms of π_1 is what is called the *Galois centralizer*: The exact sequence (2.1) induces the exterior Galois representation

$$\varphi_{X/k} : G_k \longrightarrow \mathrm{Out}\pi_1(\overline{X})$$

which associates to each $\sigma \in G_k$ the outer class of the conjugate actions on $\pi_1(\overline{X})$ by the preimages of σ in $\pi_1(X)$. The *Galois centralizer* $\mathrm{Out}_{G_k}\pi_1(\overline{X})$ is, by definition, the centralizer of the image of $\varphi_{X/k}$ in $\mathrm{Out}\pi_1(\overline{X})$. Every element of $E_k(X)$ gives, by restriction, an element of $\mathrm{Out}_{G_k}\pi_1(\overline{X})$; thus, having obtained a diagram of group homomorphisms

$$\mathrm{Aut}_k X \to E_k(X) \stackrel{\mathrm{res}}{\to} \mathrm{Out}_{G_k}\pi_1(\overline{X}),$$

we may also ask whether $\mathrm{Out}_{G_k}\pi_1(\overline{X})$ recovers $\mathrm{Aut}_k X$.

The two groups $E_k(X)$ and $\mathrm{Out}_{G_k}\pi_1(\overline{X})$ are isomorphic if the center Z of $\pi_1(\overline{X})$ is trivial. In fact, we have the following exact sequence (a profinite version of Wells' exact sequence, see [N3] 1.5.5)

$$(2.2) \qquad 1 \to H^1_{cont}(G_k, Z) \to E_k(X) \to \mathrm{Out}_{G_k}\pi_1(\overline{X}) \to H^2_{cont}(G_k, Z),$$

where the arrows are homomorphisms except for the last one which is merely a mapping of sets preserving origins, and G_k acts on Z by conjugation. As a third candidate for approximating $\mathrm{Aut}_k X$, we define $\bar{E}_k(X)$ to be the image of the homomorphism $E_k(X) \to \mathrm{Out}_{G_k}\pi_1(\overline{X})$.

A test for anabelianity. *If X/k deserves to be called anabelian, then at least one of* $\mathrm{Out}_{G_k}\pi_1(\overline{X})$, $E_k(X)$, $\bar{E}_k(X)$ *should coincide with* $\mathrm{Aut}_k X$.

§3. Locally symmetric spaces

We shall give two "alarming examples" of varieties which are "hyperbolic" in the usual sense but not "anabelian" in the sense of the above test. First, we discuss the case of Siegel modular varieties, and then the Hilbert modular case.

Example (S). Let $A_{g,n}$ be the Siegel modular variety of degree $g \geq 2$ and level $n \geq 3$ defined over the cyclotomic field $k = \mathbb{Q}(\mu_n)$. Then none of $E_k(A_{g,n})$, $\mathrm{Out}_{G_k}\pi_1(\overline{A}_{g,n})$, $\bar{E}_k(A_{g,n})$ is isomorphic to $\mathrm{Aut}_k(A_{g,n})$.

For 'hyperbolists', $A_{g,n}$ is a very "pleasant" object; the associated manifold $A_{g,n}(\mathbb{C})$ is the quotient of the Siegel upper half space by a torsion-free discrete subgroup $\Gamma_g(n) = \{A \in \mathrm{Sp}(2g,\mathbb{Z}) \mid A \equiv 1_{2g} \bmod n\} \subset \mathrm{Sp}(2g,\mathbb{R})$; hence it is a locally symmetric space having negative curvature. In particular, it is K($\Gamma_g(n)$,1). Moreover, $A_{g,n}(\mathbb{C})$ is a hyperbolic complex manifold in the sense of Kobayashi [Ko], and is a variety of log general type in the sense of Iitaka [Ii] (cf. [Mu] §4). Therefore, $\mathrm{Aut}_{\mathbb{C}}(A_{g,n})$ is a finite group. By rigidity theorems for locally symmetric spaces, $\mathrm{Aut}_{\mathbb{C}}(A_{g,n})$ amounts to 'a half' of the finite group $\mathrm{Out}\Gamma_g(n)$ ([Mo], [Ma]; [No]). One encounters $A_{g,n}$ as a typical example of a hyperbolic variety in text books of hyperbolic geometry.

Meanwhile, for 'anabelianists', two "unpleasant" phenomena occur in $A_{g,n}$. Although $\pi_1(A_{g,n}(\mathbb{C})) = \Gamma_g(n)$ has trivial center and is residually finite, its profinite completion $\pi_1(\overline{A}_{g,n}) = \hat{\Gamma}_g(n)$ has a big center when $g > 1$. In fact, since $\mathrm{Sp}(2g,\mathbb{Z})$ $(g > 1)$ has the congruence subgroup property (Bass-Lazard-Serre[BLS], Mennicke[Me]),

$$(3.1) \qquad \hat{\Gamma}_g(n) = \prod_{p\nmid n} \mathrm{Sp}_{2g}(\mathbb{Z}_p) \times \prod_{p\mid n}\{A \in \mathrm{Sp}_{2g}(\mathbb{Z}_p) \mid A \equiv 1_{2g} \bmod n\}.$$

So, the center Z of $\pi_1(\overline{A}_{g,n})$ is an infinite group that corresponds via (3.1) to the infinite product $\prod'_p\{\pm 1\}$ of the center $\{\pm 1\}$ of $\mathrm{Sp}_{2g}(\mathbb{Z}_p)$, where p runs over all primes $p \nmid n$, with an addition of $p = 2$ when $2 \parallel n$. On the other hand, k has infinitely many quadratic extensions. Therefore, the cohomology group $H^1_{cont}(G_k, Z)$ is infinite. Therefore, by (2.2), $E_k(A_{g,n})$ is also infinite and hence *cannot* approximate the finite group $\mathrm{Aut}(A_{g,n})$.

Remark. Existence of non-trivial torsions in $\hat{\Gamma}_g(n)$ $(n \geq 3)$ implies that $\mathrm{Sp}(2g,\mathbb{Z})$ is *not* a good group in the sense of Serre [Se1][3]. This is also rele-

[3] A discrete group is called *good* if the Galois cohomology of its profinite completion coincides with the corresponding (discrete) group cohomology. The cohomological dimension of $\hat{\Gamma}_g(n)$ is infinite due to the raised torsion, while that of the $K(\Gamma_g(n),1)$-space $A_{g,n}(\mathbb{C})$ (for $n \geq 3$) is bounded ([T2]).

vant to the goodness condition appearing in the profinite Gottlieb theorem [N3] 1.3.

On the other hand, the image of the exterior Galois representation $\varphi : G_k \to \mathrm{Out}\,\pi_1(\overline{A}_{g,n})$ is an *infinite abelian* group. To see this, let $K_{g,N}$ be the function field of $A_{g,N}$ over $\mathbb{Q}(\mu_N)$ ($N \geq 1$), and let L_g denote the composite of $K_{g,N}$ for all $N \geq 1$. Then, according to Shimura ([Sh1] Th.3 (brief account), [Sh2] Th.7.2 (details)), L_g is a Galois extension of $K_{g,1}$ with exact constant field $\mathbb{Q}(\mu_\infty)$, and moreover, there is an equivalence of two short exact sequences of profinite groups

$$
\begin{array}{llllllll}
(3.2) & 1 \to \mathrm{Gal}(L_g/K_{g,1}(\mu_\infty)) & \to & \mathrm{Gal}(L_g/K_{g,1}) & \to & \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) & \to & 1 \\
 & \quad\downarrow \wr & & \downarrow \wr & & \downarrow \wr\, \chi & & \\
(3.3) & 1 \to \mathrm{Sp}(2g, \hat{\mathbb{Z}})/\{\pm 1\} & \to & \mathrm{GSp}(2g, \hat{\mathbb{Z}})/\{\pm 1\} & \xrightarrow{\nu} & \hat{\mathbb{Z}}^\times & \to & 1.
\end{array}
$$

Here, χ is the cyclotomic character, GSp is the group of (symplectic) similitudes, and ν is the multiplier. Now let $n \geq 3$ and N run over all multiples of n. Then $\overline{A}_{g,N}$ is the finite etale covering of $\overline{A}_{g,n}$ that corresponds to the open normal subgroup $\hat{\Gamma}_g(N)$ of $\pi_1(\overline{A}_{g,n}) \cong \hat{\Gamma}_g(n)$, and by the congruence subgroup property, every finite etale covering of $\overline{A}_{g,n}$ is a subcovering of some $\overline{A}_{g,N}$. Therefore, by the above result of Shimura, the exterior action of $G_k = G_{\mathbb{Q}(\mu_n)}$ on $\pi_1(\overline{A}_{g,n})$ factors through $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}(\mu_n))$ and is given by the exterior action of

$$\{a \in \hat{\mathbb{Z}}^\times; a \equiv 1 (\mathrm{mod}\ n)\} = \prod_{p \nmid n} \mathbb{Z}_p^\times \times \prod_{p \mid n} \{a \in \mathbb{Z}_p^\times; a \equiv 1 (\mathrm{mod}\ n)\}$$

on $\hat{\Gamma}_g(n)$ (see (3.1)) *via* the "1_{2g} (mod n) part" of (3.3). But the kernel of the exterior action of $\mathbb{Z}_p^\times$ on $\mathrm{Sp}(2g, \mathbb{Z}_p)/\{\pm I\}$ is exactly $(\mathbb{Z}_p^\times)^2$, because the centralizer of $\mathrm{Sp}(2g, \mathbb{Z}_p)/\{\pm I\}$ in $\mathrm{GSp}(2g, \mathbb{Z}_p)/\{\pm I\}$ consists only of scalar matrices $a_p \cdot I_{2g} (a_p \in \mathbb{Z}_p^\times)$ (whose multiplier being a_p^2). Therefore, $\varphi(G_k)$ contains an infinite abelian group

$$\prod_{p \nmid n} (\mathbb{Z}_p^\times / (\mathbb{Z}_p^\times)^2).$$

Now since $\varphi(G_k)$ is abelian, $\mathrm{Out}_{G_k}(\pi_1(\overline{A}_{g,n}))$ contains $\varphi(G_k)$ itself which is infinite. Therefore, $\mathrm{Aut}_k A_{g,n}$ cannot be isomorphic to $\mathrm{Out}_{G_k}(\pi_1(\overline{A}_{g,n}))$.

It remains to examine whether the group $\bar{E}_k(A_{g,n})$ happens to approximate $\mathrm{Aut}_k(A_{g,n})$ or not. But this is again negative: In fact, $\bar{E}_k(A_{g,n})$ still contains the exterior Galois image $\varphi(G_k)$, because of the following

Lemma. *Suppose that the projection $p_{X/k} : \pi_1(X) \to G_k$ has a splitting homomorphism $s : G_k \to \pi_1(X)$ such that the induced lift $\tilde{\varphi} : G_k \to \mathrm{Aut}\pi_1(\overline{X})$ of $\varphi_{X/k} : G_k \to \mathrm{Out}\pi_1(\overline{X})$ via s has an abelian image. Then, $\varphi_{X/k}(G_k)$ is contained in $\bar{E}_k(X)$.*

When $X = A_{g,n}$, $k = \mathbb{Q}(\mu_n)$, the assumption of the Lemma is satisfied by taking the preimages of the matrices $diag(a, \dots, a, 1 \dots, 1)$ for $s(G_k)$.

Proof of the lemma. It is not difficult to see from the assumption that, for any fixed $\sigma_0 \in G_k$, the map

$$xs(\sigma) \mapsto s(\sigma_0)xs(\sigma_0)^{-1}s(\sigma) \qquad (x \in \pi_1(\overline{X}),\ \sigma \in G_k)$$

gives a group automorphism of $\pi_1(X) = \pi_1(\overline{X}) \rtimes s(G_k)$. This provides us with a desired preimage of $\varphi(\sigma_0)$ in $E_k(X)$. ♢

Example (H). Let F be a totally real number field of finite degree g over $\mathbb{Q}$, $\mathcal{O}_F$ be the ring of integers of F, and for each positive integer n, let $\Delta_F(n)$ be the Hilbert modular group of level n;

$$\Delta_F(n) = \{A \in SL(2, \mathcal{O}_F); A \equiv 1_2 (\mathrm{mod}\, n)\}.$$

Then the group $\Delta_F(n)$ acts on the product $\mathcal{H}^g$ of g copies of the complex upper half plane $\mathcal{H}$, in the usual manner, and if $n \geq 3$, $\Delta_F(n)$ is torsion-free. In this case, the quotient $\Delta_F(n)\backslash\mathcal{H}^g$ is also known to be Kobayashi-hyperbolic and of log-general type.

The congruence subgroup property for $SL(2, \mathcal{O}_F)$ is also valid if $g > 1$ [Se2]. Moreover, by Shimura ([Sh2] Th.7.2), $\Delta_F(n)\backslash\mathcal{H}^g$ has a standard model $A_{F,n}$ over $\mathbb{Q}(\mu_n)$ (not $F(\mu_n)$), and if $K_{F,n}$ denotes its function field, and $L_F = \bigcup_n K_{F,n}$, then there is again an equivalence of two short exact sequences:

$$\begin{array}{ccccccc}
1 \to \mathrm{Gal}(L_F/K_{F,1}(\mu_\infty)) & \to & \mathrm{Gal}(L_F/K_{F,1}) & \to & \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \to 1 \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr\, \chi \\
1 \to \mathrm{SL}(2, \hat{\mathcal{O}}_F)/\{\pm 1\} & \to & \mathrm{GL}(2, \hat{\mathcal{O}}_F)|_{\det \in \hat{\mathbb{Z}}^\times}/\{\pm 1\} & \overset{\det}{\to} & \hat{\mathbb{Z}}^\times \to 1.
\end{array}$$

Using those primes p that decompose completely in F, we see easily that the image of $G_{\mathbb{Q}(\mu_n)}$ in $\mathrm{Out}\pi_1(\overline{A}_{F,n})$ is again an infinite abelian group.

Now it follows by the arguments parallel to those used in the Siegel modular case, that $A_{F,n}$ also fails the anabelianity test of §2.

§4. Braid configuration spaces

In this section, we consider braid configuration spaces of hyperbolic curves as typical candidates for higher dimensional anabelian varieties. Let $r \geq 1$ and C be a hyperbolic curve over a number field k. The r-dimensional (pure) braid configuration space $C^{(r)}$ is defined to be the product of r copies of C minus all the weak diagonals:

$$C^{(r)} := \{(x_1, \ldots, x_r) \in C^r \mid x_i \neq x_j \ (i \neq j)\}.$$

For this type of variety, $\pi_1(\overline{C}^{(r)})$ is a successive extension of free profinite groups (and a profinite surface group), and hence has trivial center (cf. e.g. [N3] 1.2). Therefore, three groups considered in the previous section coincide for braid configuration spaces: $E_k(C^{(r)}) = \bar{E}_k(C^{(r)}) = \mathrm{Out}_{G_k}\pi_1(\overline{C}^{(r)})$. We may also consider the 'pro-l version' of these groups for any fixed prime l, by replacing $\pi_1(\overline{C}^{(r)})$ by its maximal pro-l quotient $\pi_1^{pro-l}(\overline{C}^{(r)})$ and by using a naturally induced exact sequence from (2.1) for $X = C^{(r)}$:

$$1 \to \pi_1^{pro-l}(\overline{C}^{(r)}) \to \pi_1^{(l)}(C^{(r)}) \to G_k \to 1.$$

In the similar way, $E_k^{(l)}(C^{(r)})$, $\bar{E}_k^{(l)}(C^{(r)})$, $\mathrm{Out}_{G_k}\pi_1^{pro-l}(\overline{C}^{(r)})$ are defined and again these three groups are isomorphic. Thus, our anabelianity test (§2) for $C^{(r)}$ is reduced to the following

Question. Does $\mathrm{Out}_{G_k}\pi_1(\overline{C}^{(r)})$ or $\mathrm{Out}_{G_k}\pi_1^{pro-l}(\overline{C}^{(r)})$ recover $Aut_k(C^{(r)})$?

Before discussing the above question, we shall summarize recent achievements by Tamagawa and Mochizuki for Grothendieck's *fundamental conjecture of anabelian geometry* ([G1][G2]). Let C, C' be hyperbolic curves over k. Then,

Theorem A. (Tamagawa [T1]: affine case, Mochizuki [M1]: proper case) *The natural mapping*

$$\mathrm{Isom}_k(C, C') \to \mathrm{Isom}_{G_k}(\pi_1(C), \pi_1(C'))/\mathrm{Int}\pi_1(\overline{C}')$$

is a bijection.

Theorem B. (Mochizuki [M2]) *The natural mapping*

$$\mathrm{Isom}_k(C, C') \to \mathrm{Isom}_{G_k}(\pi_1^{(l)}(C), \pi_1^{(l)}(C'))/\mathrm{Int}\pi_1^{pro-l}(\overline{C}')$$

is a bijection.

Corollary AB. $\mathrm{Aut}_k(C) \cong \mathrm{Out}_{G_k}\pi_1(\overline{C}) \cong \mathrm{Out}_{G_k}\pi_1^{pro-l}(\overline{C})$.

Notes: The above works by Tamagawa and Mochizuki include much more essential ingredients beyond the number-basefield case. In effect, Tamagawa [T1] established new aspects of the finite-basefield case and Mochizuki [M2] introduced new ideas for the p-adic-basefield case. The statements of Theorems A, B can be divided into the following two forms respectively.

$$(A): \begin{cases} \text{(Equiv)}: & \text{If } \pi_1(C) \cong \pi_1(C') \text{ over } G_k, \text{ then } C \cong C' \text{ over } k. \\ \text{(Aut)}: & \mathrm{Aut}_k(C) \cong \mathrm{Out}_{G_k}\pi_1(\overline{C}). \end{cases}$$

$$(B): \begin{cases} (\text{Equiv}_l): & \text{If } \pi_1^{(l)}(C) \cong \pi_1^{(l)}(C') \text{ over } G_k, \text{ then } C \cong C' \text{ over } k. \\ (\text{Aut}_l): & \mathrm{Aut}_k(C) \cong \mathrm{Out}_{G_k}\pi_1^{pro-l}(\overline{C}). \end{cases}$$

Prior to [T1], [M1-2], some special cases had been studied by the second named author and H.Tsunogai for the number-basefield case of (Equiv) and (Aut_l). See ([AI] [N] ∼) [N1-7] [NT] (cf. [V]).

For the above Question, we have two kinds of results as follows.

Theorem C. ([T1] + [N3])

$$\mathrm{Aut}_k(C^{(r)}) \cong \mathrm{Out}_{G_k}\pi_1(\overline{C}^{(r)}) \text{ for } C = \mathbb{P}^1 - \{0, 1, \infty\},\ r \geq 1.$$

In fact, prior to [T1], the problem for general r had been reduced to the case of $r = 1$ in [N3]. This, combined with [T1], settles Theorem C. In the case of $C = \mathbb{P}^1 - \{0, 1, \infty\}$, $C^{(r)}$ can be regarded as the moduli space $M_{0,n}$ of the n-pointed projective lines with $n = r + 3$, whose automorphism group is isomorphic to S_n, the symmetric group of degree n, for $n \geq 5$. Applying Theorem C and the triviality[4] of $\mathrm{Out}G_{\mathbb{Q}}$ to a group theoretical lemma ([N3] 1.6.2), we obtain a Galois analog of Ivanov's rigidity theorem ([Iv])[5]:

Corollary C. $\mathrm{Out}\,\pi_1(M_{0,n}/\mathbb{Q}) \cong S_n$ $(n \geq 5)$, $\quad (\cong S_3\ (n = 4))$.

For general hyperbolic curves C, we also have

Theorem D. ([M2] + [NTa])

$$\mathrm{Aut}_k(C^{(r)}) \cong \mathrm{Out}_{G_k}\pi_1^{pro-l}(\overline{C}^{(r)}) \text{ for } r \geq 1.$$

In fact, in a joint article with Takao [NTa], we had shown the following sequence of injective homomorphisms:

$$\mathrm{Aut}_k C \times S_r \hookrightarrow \mathrm{Aut}_k C^{(r)} \hookrightarrow \mathrm{Out}_{G_k}\pi_1^{pro-l}(\overline{C}^{(r)}) \hookrightarrow \mathrm{Out}_{G_k}\pi_1^{pro-l}(\overline{C}) \times S_r$$

---

[4] Neukirch ∼ Komatsu, Ikeda, Iwasawa, Uchida, cf. [Ne].

[5] Ivanov's rigidity asserts that the outer automorphism group of $\pi_1(M_{0,n}(\mathbb{C}))$ is an extension of $\{\pm 1\}$ by S_n for $n \geq 5$ ([Iv]). This result is generalized to the surface mapping class groups ([Iv] [Mc]). Our Corollary C particularly indicates that Galois compatibility condition drives out 'anti-holomorphic' self-equivalences from $\mathrm{Out}\pi_1$.

for any hyperbolic curve C of non-exceptional type. (For exceptional hyperbolic curves whose geometric types are $\mathbb{P}^1 - \{0, 1, \infty\}$ or one-point punctured elliptic curves, we need some modifications to the above sequence.) This, combined with [M2], settles Theorem D. Curiously, the following purely geometric statement follows immediately from this combination.

Corollary D. $\mathrm{Aut}_k(C^{(r)}) \cong \mathrm{Aut}_k(C) \times S_r$, *unless* $\overline{C}$ *is isomorphic to* $\mathbb{P}^1 - \{0, 1, \infty\}$ *or an elliptic curve minus one point. In the latter of the exceptional cases, the statement holds if* S_r *is replaced by* S_{r+1}.

These results apparently suggest that the Galois fundamental groups of braid configuration varieties[6] differ from those of Hilbert/Siegel modular varieties in group-theoretical nature. We observe, especially, the following distinguished anabelian (=far from abelian) properties of the former fundamental groups:

(E1) Every open subgroup of the geometric profinite fundamental group has trivial center (cf. [N3] 1.3, [N4]).

(E2) The Galois image in the outer automorphism group of the geometric profinite fundamental group has only finitely many centralizing elements.

Problem. *Prove these two properties (E1-2) for the 'Galois-Teichmüller modular group'* $\pi_1(M_{g,n}/\mathbb{Q})$ ([G1][O2]).

Problem. *Find new examples of algebraic varieties possessing the properties (E1-2).*

[6] Some analogous statements also hold for products of configuration varieties of hyperbolic curves as results of Theorems A, B, C, D.

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