Discrete Hamiltonian Structure of Schlesinger Transformations

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1 Introduction

Schlesinger transformation

We consider the linear differential equation of Fuchsian type:

\[
\frac{dY(x)}{dx} = A(x)Y(x), \quad A(x) = \sum_{i=1}^{n} \frac{A_i}{x - u_i}
\]

where \(u_i \in \mathbb{C}\) are distinct with each other, \(A_i\) and \(Y(x)\) are \(m \times m\) matrices, and \(A_i (i = 1, \cdots, n)\) and \(A_\infty := -A_1 - \cdots - A_n\) are semisimple (i.e. diagonalizable).

Without loss of generality, we can assume one of the eigenvalues of \(A_i\) to be zero.
Proposition 1 (Schlesinger 1912). There exists a unique $m \times m$ matrix $\mathbf{R}(x)$ whose entries are rational function of $x$ such that $\mathbf{Y}(x) = \mathbf{R}(x)\mathbf{Y}(x)$ satisfies

$$\frac{d\mathbf{Y}(x)}{dx} = \tilde{\mathbf{A}}(x)\mathbf{Y}(x), \quad \tilde{\mathbf{A}}(x) = \sum_{i=1}^{n} \frac{\tilde{\mathbf{A}}_i}{x - u_i}, \quad \tilde{\mathbf{A}}_{\infty} = \mathbf{A}_{\infty}$$

where the eigenvalues $\tilde{\theta}_i^j$ of $\tilde{\mathbf{A}}_i$ are the same with $\theta_i^j$ except $\tilde{\theta}_1^1 = \theta_1^1 - 1$ and $\tilde{\theta}_2^1 = \theta_2^1 + 1$, and the transformation $\mathbf{A}(x) \rightarrow \tilde{\mathbf{A}}(x)$ commutes with the isomonodromic deformation.

Moreover, $\mathbf{R}(x)$ has the form $\mathbf{R}(x) = 1 + \frac{\mathbf{R}_1}{x - u_1}$ with rank $\mathbf{R}_1 = 1$. 
Lax form and deforming matrix

From the compatibility, we have

\[
\frac{d \mathbf{R}(x)}{dx} = \mathbf{\bar{A}}(x) \mathbf{R}(x) - \mathbf{R}(x) \mathbf{A}(x),
\]

(1)

and \( \mathbf{R}(x) \) can be written as

\[
\mathbf{R}(x) = \mathbf{I} + \frac{u_1 - u_2}{x - u_1} \frac{\mathbf{b}_2 \mathbf{c}_2^{\dagger}}{\mathbf{c}_2^{\dagger} \mathbf{b}_2},
\]

(2)

where \( \mathbf{b}_2 \) and \( \mathbf{c}_2^{\dagger} \) are eigenvectors of \( \mathbf{A}_2 \) and \( \mathbf{A}_2^{\dagger} \) with eigenvalue \( \theta_2 \) and \( \theta_2 + 1 \) respectively.
2 Discrete Hamiltonian of Elementary Schlesinger transformation

Canonical variables

Let $r_i$ denote the rank of $A_i$. $A_i$ can be written as

$$A_i = B_i C_i^\dagger,$$  \hspace{1cm} (3)

where

$$B_i = \begin{bmatrix} b^1_i & \cdots & b^{r_i}_i \end{bmatrix}, \quad C_i = \begin{bmatrix} c^1_i & \cdots & c^{r_i}_i \end{bmatrix}^\dagger;$$

$$C_i^\dagger B_i = \Theta_i := \text{diag}[\theta^1_i, \ldots, \theta^{r_i}_i].$$

We define the symplectic form as

$$\sum_{i=1}^n \sum_{j=1}^{r_i} \sum_{k=1}^{r_i} d(b^j_i)_k \wedge d(c^j_i)_k.$$  \hspace{1cm} (4)
Discrete Hamiltonian of Elementary Schlesinger transformation

Let

\[ J = \{(i, j) | 1 \leq i \leq n, 1 \leq j \leq r_i \} \quad \text{and} \quad J' = J \backslash \{(1, 2), (2, 1)\}, \] (5)

and for a multi-index \( I = (i, j) \in J' \), we put

\[ b_I = b_{i,j}, \quad c^\dagger_I = c^\dagger_{j,i}, \quad \theta_I = \theta^j_i, \]

\[ R_I = R(u_i) \quad \text{when} \quad i \neq 1 \]

and \( R_1 = I - \frac{b_1^1 c_2^1}{c_2^1 b_1^1} - \frac{b_2^1 c_1^1}{c_1^1 b_2^1}. \)
Theorem 1. Let

$$S(B, \bar{C}^\dagger; \Theta) = \sum_{I \in \mathcal{I}'} \theta_I \log(\bar{c}_I^\dagger R_I b_I), \quad \text{and} \quad (6)$$

$$\mathcal{H}^+(B, \bar{C}^\dagger; \Theta) = (\theta_2^1 - \theta_1^1 + 1) \log(\bar{c}_2^1 b_2^1) + \theta_1^1 \log(\bar{c}_2^1 b_1^1)$$

$$+ (\theta_1^1 - 1) \log(\bar{c}_1^1 b_2^1) + S(B, \bar{C}^\dagger; \Theta). \quad (7)$$

Then, the Sclesinger transformation is written as the discrete Hamiltonian equation as

$$\bar{c}_j^i = \frac{\partial \mathcal{H}^+(B, \bar{C}^\dagger; \Theta)}{\partial \bar{c}_i^j} (B, \bar{C}^\dagger; \Theta) \quad \text{and} \quad c_j^i = \frac{\partial \mathcal{H}^+(B, \bar{C}^\dagger; \Theta)}{\partial b_i^j} (B, \bar{C}^\dagger; \Theta). \quad (9)$$
3 Example: Discrete Painlevé V

We consider the same Fuchsian equation with the case of $P_{VI}$, i.e. of matrix size $m = 2$ with $n = 3$ finite poles and rank-one condition $r_1 = r_2 = r_3 = 1$ for the residue matrices at those poles. The elementary Schlesinger transformation changes the Riemann scheme as

$$
\begin{cases}
  x = 0 & x = 1 & x = t & x = \infty \\
  \theta_0 & \theta_1 & \theta_t & \kappa_1 \\
  0 & 0 & 0 & \kappa_2
\end{cases}
\quad \mapsto \quad
\begin{cases}
  x = 0 & x = 1 & x = t & x = \infty \\
  \theta_0 & \theta_1 - 1 & \theta_t + 1 & \kappa_1 \\
  0 & 0 & 0 & \kappa_2
\end{cases}
$$

where we put $u_1 = 1$, $u_2 = t$, $u_3 = 0$. 

Although the Hamiltonian system is implicit evolution equation, we can solve it explicitly as

$$\tilde{c}_1^\dagger = \gamma_1 \left( (\theta_1 - \theta_t - 1) \frac{c_1^\dagger}{c_1^\dagger b_t} + c_t^\dagger + \frac{(1-t)c_1^\dagger b_0}{c_1^\dagger b_t} \left( c_0^\dagger - \frac{c_0^\dagger b_t}{c_1^\dagger b_t} c_1^\dagger \right) \right)$$

$$\tilde{c}_t^\dagger = \gamma_t c_1^\dagger$$

$$\tilde{c}_0^\dagger = \gamma_0 \left( c_0^\dagger + \frac{(1-t)c_0^\dagger b_t}{tc_1^\dagger b_t} c_1^\dagger \right)$$

$$\tilde{b}_1 = (\theta_1 - 1) \frac{b_t}{c_1^\dagger b_t}$$

$$\tilde{b}_t = \theta_1 \left( \frac{b_t}{\tilde{c}_t^\dagger b_t} - \frac{b_t}{c_1^\dagger b_t} \right) + (\theta_t + 1) \frac{b_t}{c_t^\dagger b_t} - \theta_0 \frac{b_t}{\tilde{c}_t^\dagger b_t}$$

$$+ \theta_0 \left( \frac{(c_0^\dagger b_0)b_t - (1-t)(c_0^\dagger b_t)b_0}{(c_0^\dagger b_0)(\tilde{c}_t^\dagger b_t) - (1-t)(c_t^\dagger b_0)(c_0^\dagger b_t)} \right)$$

$$\tilde{b}_0 = \theta_0 \left( \frac{(c_t^\dagger b_t)b_0 - (1-t)(c_t^\dagger b_0)b_t}{(c_0^\dagger b_0)(\tilde{c}_t^\dagger b_t) - (1-t)(c_t^\dagger b_0)(c_0^\dagger b_t)} \right)$$

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Since the Schlesinger transformation commutes with the gauge transformation \((b_i, c_i^\dagger)_{i=1,2,3} \mapsto (Pb_i, c_i^\dagger P^{-1})_{i=1,2,3}\) for a regular matrix \(P\), the Schlesinger transformations for different choices of representatives are conjugate with each other.

Hence, we can take the same representative with the case of \(P_{VI}\). By a gauge transformation, we can choose the coordinates of \(A_i\) as

\[
A_0 = \begin{bmatrix} 1 \\ -w \end{bmatrix} \begin{bmatrix} \theta_0 + aw & a \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 \\ b-w \end{bmatrix} \begin{bmatrix} \theta_1 - c(b-w) & c \end{bmatrix},
\]

\[
A_t = \begin{bmatrix} 1 \\ d-w \end{bmatrix} \begin{bmatrix} \theta_t + (a+c)(d-w) & -a-c \end{bmatrix}, \quad A_\infty = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}
\]

From the last constraint, we have two equations. So three variables remain essentially, while the dimension of accessory parameters is two.
Let $p$ and $q$ be the canonical variables as

$$p = cd, \quad q = \frac{(a + c)}{c},$$

then $\tilde{p}$ and $\tilde{q}$ can be written by $p$ and $q$ and the canonical form

$$dp \wedge dq$$

is preserved. This fact implies that the mapping can be written in discrete Hamiltonian form, though the Hamiltonian is changed from the original one, since the Hamiltonian is not preserved by canonical transformations in the discrete case.
Further changing variables as

\[ f = pq, \quad g = t \left( q - \frac{\theta_1 + \kappa_2}{p} \right), \]

we have the discrete Painlevé V equation in the usual form as

\[ \bar{f} + f = \kappa_2 + \frac{\theta_1 - 1 + \kappa_1}{1 - g} - \frac{\theta_t + \kappa_1}{1 - (g/t)} \]

\[ \bar{g}g = \frac{t(\bar{f} + \theta_t + 1)(\bar{f} - \theta_1 + 1 - \kappa_2)}{\bar{f}(\bar{f} - \kappa_2)}. \]
4 Example: Discrete Painlevé $A_2^{(1)*}$

Let $m = 3$, $n = 2$, $r_1 = r_2 = 2$, $r_\infty = 3$. Taking suitable gauge and the trivial transformation $(B_i, C_i^\dagger)_{i=1,2} \mapsto (B_i Q, Q^{-1} C_i^\dagger)_{i=1,2}$ for a diagonal matrix $Q = \text{diag}(\lambda_i^1, \lambda_i^2)$, we can set $B_i$, $C_i$ ($i = 1, 2$) as

$$A_i = B_i C_i,$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} \theta_1^1 & 0 & a \\ 0 & \theta_1^2 & b \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -\theta_2^1 + c & -c & \theta_2^1 \\ d & \theta_2^2 - d & 0 \end{bmatrix}.$$
and the eigenvalues of

\[ \mathbf{A}_\infty = -\mathbf{A}_1 - \mathbf{A}_2. \]

are \( \kappa_1, \kappa_2 \) and \( \kappa_3 \). From the last constraint we have

\[
-\kappa_1 \kappa_2 \kappa_3 = ((\theta_1^2 + \theta_2^2)(\theta_2^1 - c) + (\theta_1^2 + \theta_2^1)d)a \\
+ (\theta_2^2c - \theta_2^1(\theta_2^2 + d) - \theta_1^1(\theta_2^2 - c + d))b \\
+ \theta_2^1(-\theta_1^2d + \theta_1^1(\theta_1^2 + \theta_2^2 + d))
\]

\[
\kappa_1 \kappa_2 + \kappa_2 \kappa_3 + \kappa_3 \kappa_1 = (\theta_2^1 - c + d)a + (-\theta_2^2 + c - d)b \\
+ \theta_2^1(\theta_1^2 + \theta_2^2) - \theta_2^2d + \theta_1^1(\theta_1^2 + \theta_2^1 + \theta_2^2 + d),
\]

thus \( a \) and \( b \) are determined by \( c \) and \( d \).
The elementary Schlesinger transformation gives a two dimensional dynamical system which is turned out to be one the discrete Painlevé equations of type $A_2^{(1)*}$, i.e. a particular translation of the affine Weyl group of type $E_6^{(1)}$. However, this equation becomes very complicated, so we consider another Schlesinger transformation of different direction which gives a rather simple discrete Painlevé equation of $A_2^{(1)*}$ type.

Now we consider the following Schlesinger transformation

\[
\begin{pmatrix}
x = u_1 & x = u_2 & x = \infty \\
1 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 1 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
x = u_1 & x = u_2 & x = \infty \\
1 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 1 \\
\end{pmatrix}
\]
Then we have the resulting evolution equation

\[ \bar{c} = \text{(some rational function of } c, d) \]
\[ \bar{d} = \text{(some rational function of } c, d). \]

By changing variables as

\[ u = \frac{c - d - \theta_2^1}{d}, \quad v = \frac{1}{c - d - \theta_2^1} \]

and further

\[ u = \frac{\theta_2^1 - \theta_2^2}{f + \theta_2^2}, \quad v = \frac{\theta_1^1 - \theta_1^2}{(\theta_2^1 - \theta_2^2)(g - (\theta_1^1 - \theta_1^2))}, \]

we have

\[ dP(A_2^{(1)*}) : \begin{cases} \bar{f} + g = \frac{(g + b_1)(g + b_2)(g + b_3)(g + b_4)}{(g - b_5)(g - b_6)} \quad \bar{f} - b_1 \end{cases} \]
\[ \bar{f} + \bar{g} = \frac{(\bar{f} - b_1)(\bar{f} - b_2)(\bar{f} - b_3)(\bar{f} - b_4)}{(\bar{f} + b_7 - \lambda)(\bar{f} + b_8 - \lambda)} \]
where

\[
\begin{align*}
  b_1 &= \theta_1^2 + \kappa_1, \quad b_2 = \theta_1^2 + \kappa_2, \quad b_3 = \theta_1^2 + \kappa_3, \quad b_4 = 0 \\
  b_5 &= \theta_1^1 - \theta_1^2, \quad b_6 = -\theta_1^2 - 1, \quad b_7 = \theta_2^1, \quad b_8 = \theta_2^2, \quad \lambda = -1.
\end{align*}
\]
A  Types of discrete Painlevé equation (Sakai’s Label) and what was the problem?

Both continuous and discrete Painlevé equations can be classified by the types of the anti-canonical divisor of their space of initial values, which can be obtained by successive 9 blow-ups on the (singular) cubic curve on the projective plane.

\[ D_4^{(1)}, A_2^{(1)*} \text{ and } A_2^{(1)} \] cubic curve on \( \mathbb{P}^2(\mathbb{C}) \)

Figure 1: \( D_4^{(1)}, A_2^{(1)*} \text{ and } A_2^{(1)} \) cubic curve on \( \mathbb{P}^2(\mathbb{C}) \)
Here $D_{4}^{(1)}$ and $A_{2}^{(1)*}$ are additive (or difference) type, while $A_{2}^{(1)}$ is multiplicative ($q$ difference).

[Sakai2007] has proposed a problem to write discrete Painlevé equations of type $A_{0}^{(1)**}$, $A_{1}^{(1)*}$ and $A_{2}^{(1)*}$ as a Schlesinger transformation. This problem was solved theoretically by [Boalch2009] where he showed that the moduli spaces of monodromy groups of some Fuchsian equations are analytically isomorphic to the spaces of initial conditions for these discrete Painlevé equations. Here we demonstrate these correspondences explicitly on the level of equations for the above two cases\textsuperscript{a}.

\textsuperscript{a}[Arinkin-Borodin2006] clarified the correspondence between $dP(A_{2}^{(1)*})$ and difference Fuchsian equation.