Quadratic equations and monodromy evolving deformations

September 24th 2007

Yousuke OHYAMA
Osaka University, Japan

at Kent University

1 Monodromy Evolving Deformations

Chakravarty and Ablowitz's DH-V equation

$$\begin{aligned}
\omega_1' &= \omega_2\omega_3 - \omega_1(\omega_2+\omega_3) + \phi^2,\\
\omega_2' &= \omega_3\omega_1 - \omega_2(\omega_3+\omega_1) + \theta^2,\\
\omega_2' &= \omega_3\omega_1 - \omega_2(\omega_3+\omega_1) - \phi\theta,\\
\phi' &= \omega_1(\theta-\phi) - \omega_3(\theta+\phi),\\
\theta' &= -\omega_2(\theta-\phi) - \omega_3(\theta+\phi).
\end{aligned}$$

Monodromy Evolving Deformations

$$\begin{aligned}
\frac{\partial Y}{\partial x} &= \frac{\mu I - (C_+x^2 + 2Dx + C_-)}{P}Y,\\
2\frac{\partial Y}{\partial t} &= [\nu - (C_+x + D)]Y - Q(x)\frac{\partial Y}{\partial x}.
\end{aligned}$$

$$
\begin{aligned}
P &= \alpha_+ x^4 + (\beta_+ + \beta_-)x^2 + \alpha_-, \\
Q &= \alpha_+ x^3 + \beta_+ x, \\
C_\pm &= (i\omega_1 \pm \phi)\sigma_1 \pm (\omega_2 \pm i\theta)\sigma_2, \\
D &= -\omega_3 \sigma_3,
\end{aligned}
$$

$\alpha_\pm = (\omega_1 - \omega_2) \mp (\theta + \phi), \beta_\pm = (\omega_1 + \omega_2 - 2\omega_3) \pm i(\theta - \phi).$

μ : constant parameter

$$\frac{\partial \nu}{\partial x} = \frac{(\beta_- + 4\omega_3) - \alpha_+ x^2}{P}\mu.$$

σ_j: Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

DH-V: solved by the Schwarzian function $S(x; 0, 0, a)$.

2 Halphen's system

Halphen's first equation in [CR **92** (1881), 1101–1103]

$$\begin{aligned} X' + Y' &= 2XY, \\ Y' + Z' &= 2YZ, \\ Z' + X' &= 2ZX. \end{aligned}$$

Halphen's second equation in [CR **92** (1881), 1404–1406]

$$\begin{aligned} x_1' &= x_1^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2, \\ x_2' &= x_2^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2, \\ x_3' &= x_3^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2. \end{aligned}$$

The 1st is a special case of the 2nd:

$$a = b = c = -\frac{1}{8},$$

$$2X = x_2 + x_3, 2Y = x_3 + x_1, 2Z = x_1 + x_2.$$

Chazy's equation: For Halphen's first, we set $y = 2(X + Y + Z)$,

$$y''' = 2yy'' - 3(y')^2.$$

Solution of Halphen's 1st:

$$X = 2\frac{\partial}{\partial t}\log\left[\theta_2\left(0, \frac{at+b}{ct+d}\right)(ct+d)^{-1/2}\right]$$

$$Y = 2\frac{\partial}{\partial t}\log\left[\theta_3\left(0, \frac{at+b}{ct+d}\right)(ct+d)^{-1/2}\right],$$

$$Z = 2\frac{\partial}{\partial t}\log\left[\theta_4\left(0, \frac{at+b}{ct+d}\right)(ct+d)^{-1/2}\right].$$

$\theta_j(z, \tau)$: Jacobi's theta function, $ad - bc = 1$.

Solution of Chazy:

$$y(t) = 4\frac{\partial}{\partial t}\log\left[\vartheta_1'\left(0, \frac{at+b}{ct+d}\right)(ct+d)^{-3/2}\right].$$

Solution of Halphen's 2nd: Start from

$$\frac{d^2y}{dz^2} = \left(\frac{a+b}{z^2} + \frac{c+b}{(z-1)^2} - \frac{2b}{z(z-1)}\right) y.$$

t: a ratio of two solutions

$$x_1 = \frac{d}{dt}\log y, \quad x_2 = \frac{d}{dt}\log\frac{y}{z}, \quad x_3 = \frac{d}{dt}\log\frac{y}{z-1}.$$

$\Longrightarrow x_1, x_2$ and x_3 satisfy Halphen's 2nd.

In case of $a = b = c = -\frac{1}{8}$ (Halphen's 1st),
the starting linear equation is solved by $F(1/2, 1/2, 1; z)$.

$$K = \frac{\pi}{2}F(1/2, 1/2, 1; k^2) = \frac{\pi}{2}\theta_3^2(0, \tau)$$

3 Reduction of self-dual Yang-Mills equation

$\mathfrak{g}$-valued 1-form in $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

$$A = \sum_{j=1}^{4} A_j(x) dx_j$$

The curvature 2-form $F = \sum_{j<k} F_{jk} dx_j \wedge dx_k$:

$$F_{jk} = \partial_j A_k - \partial_k A_j - [A_j, A_k],$$

where $\partial_j = \partial/\partial x_j$.

The self-dual Yang-Mills equation:

$$F_{12} = F_{34}, \quad F_{13} = F_{42}, \quad F_{14} = F_{23}.$$

$A_j = A_j(x_1)$, $A_1 = 0 \Longrightarrow$ SDYM is **the Nahm equation**

$$\begin{aligned} \partial_t A_2 &= [A_3, A_4], \\ \partial_t A_3 &= [A_4, A_2], \\ \partial_t A_4 &= [A_2, A_3]. \end{aligned}$$

$\mathfrak{g} = \mathfrak{diff}(S^3)$: the ∞-dim Lie algebra of vector fields on S^3

X_1, X_2, X_3: divergence-free vector fields on S^3

$$[X_j, X_k] = \sum_l \varepsilon_{jkl} X_l,$$

ε_{jkl}: the standard anti-symmetric form $\varepsilon_{123} = 1$.

$O_{jk} \in SO(3)$:

$$\sum_{j,k,l} \varepsilon_{jkl} O_{jp} O_{kq} O_{lr} = \varepsilon_{pqr},$$

$$X_j(O_{lk}) = \sum_p \varepsilon_{jkp} O_{lp}.$$

By the Euler angles (θ, ϕ, ψ),

$$\begin{aligned}
X_1 &= \cos\psi \frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \cot\theta \sin\psi \frac{\partial}{\partial\psi}, \\
X_2 &= -\sin\psi \frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \cot\theta \cos\psi \frac{\partial}{\partial\psi}, \\
X_3 &= \frac{\partial}{\partial\psi}.
\end{aligned}$$

$$O = \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta & -\cos\phi\sin\psi - \sin\phi\cos\psi\cos\theta & \sin\phi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta & -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta & -\cos\phi\sin\theta \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{pmatrix}$$

Choose the connection of the form

$$A_l = \sum_{j,k=1}^{3} O_{lj} M_{jk}(t) X_k.$$

The 3×3 matrix valued function $M = M(t)$ satisfies
the ninth-order Darboux- Halphen (DH-IX) system

$$\frac{dM}{dt} = (\mathrm{adj}\, M)T + M^T M - (\mathrm{Tr} M)M. \tag{1}$$

$\mathrm{adj}\, M := \det M \cdot M^{-1}$,
M^T: the transpose of M

DH-IX = $SU(2)$-invariant **hypercomplex four-manifold**

- the Weyl curvature of a hypercomplex four-manifold is self-dual
- DH-IX = self-dual Weyl **Bianchi IX space-times**.

[Hitchin98]

Chakravarty-Ablowitz +Clarkson+Takhtajan+Halburd+Herbst

[CAC90, CAT92]: $M_{jk}(t) = \omega_k(t)\delta_{jk}$: diagonal $\Longrightarrow$ Halphen's 1st

· Bianchi IX, hyperKähler (AH metric):

$$g = \omega_1\omega_2\omega_3 dt^2 + \frac{\omega_2\omega_3}{\omega_1}\sigma_1^2 + \frac{\omega_3\omega_1}{\omega_2}\sigma_2^2 + \frac{\omega_1\omega_2}{\omega_3}\sigma_3^2$$

[CA96]: special **off-diagonal** case

$$M = \begin{pmatrix} \omega_1 & \theta & 0 \\ \phi & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}.$$

$\Longrightarrow$ DH-V (monodromy evolving)

· special self-dual Bianchi IX,

$$g \sim dt^2 + \frac{(\omega_2\sigma_1 + \theta)^2}{\Delta^2} + \frac{(\phi\sigma_1 + \omega_1)^2}{\Delta^2} + \frac{\sigma_3^2}{\omega_3^2}$$

$$\Delta = \omega_1\omega_2 - \phi\theta$$

[ACH99]:

generic case= self-dual Bianchi IX= Halphen 2nd

$M = M_s + M_a$ =(symmetric part)+(anti-symmetric part)

Assumption: eigenvalues of M_s are distinct

$$M_s = PdP^{-1}, \quad M_a = PaP^{-1},$$

$$d = \mathrm{diag}\,(\omega_1, \omega_2, \omega_3).$$

$$a_{12} = -a_{21} = \tau_3, a_{23} = -a_{32} = \tau_1, a_{31} = -a_{13} = \tau_2.$$

DH-IX $\Longrightarrow$

$$\begin{aligned} \omega_1' &= \omega_2\omega_3 - \omega_1(\omega_2 + \omega_3) + \tau^2, \\ \omega_2' &= \omega_3\omega_1 - \omega_2(\omega_3 + \omega_1) + \tau^2, \\ \omega_2' &= \omega_3\omega_1 - \omega_2(\omega_3 + \omega_1) + \tau^2. \end{aligned}$$

$$\tau^2 = \alpha_1^2(\omega_1 - \omega_2)(\omega_3 - \omega_1) + \alpha_2^2(\omega_2 - \omega_3)(\omega_1 - \omega_2) + \alpha_3^2(\omega_3 - \omega_1)(\omega_2 - \omega_3).$$

The ACH system is equivalent to Halphen's 2nd

$$2\omega_1 = -x_2 - x_3, \quad 2\omega_2 = -x_3 - x_1, \quad 2\omega_3 = -x_1 - x_2,$$

$$8a = \alpha_1^2 + \alpha_2^2 - \alpha_3^2 - 1, \quad 8b = -\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - 1, \quad 8c = \alpha_1^2 - \alpha_2^2 + \alpha_3^2 - 1.$$

The ACH system:

solved by the Schwarzian function $S(x; \alpha_1, \alpha_2, \alpha_3)$.

4 Monodromy evolving equation

$$\frac{dY}{dx} = \sum_{k=1}^{n} \frac{A_k}{x - x_j} Y,$$

Local expansions

$$Y(x) \sim Y_j(x)(x - x_j)^{L_j}, \qquad Y_j(x) = O((x - x_j)^0),$$
$$Y(x) \sim Y_\infty(x) x^{-L_\infty}, \qquad Y_\infty(x) = I + O(1/x),$$

Monodromy preserving deformations:

$$\frac{\partial}{\partial x_j} Y(x) = -\frac{A_j}{x - x_j} Y.$$

Compatibility condition = **the Schlesinger equation**

$$\frac{\partial A_k}{\partial x_j} = \frac{[A_k, A_j]}{x_k - x_j}, \qquad j \neq k.$$

Monodromy Evolving Deformations L_j evolve as

$$\frac{\partial L_j}{\partial x_k} = f_{jk} I, \qquad (j, k = 1, 2, \dots, n).$$

$f_{jk} = f_{jk}(t)$ is a scalar function.

Then L_∞ evolve as

$$\frac{\partial L_\infty}{\partial x_k} = -\sum_{j=1}^{n} f_{jk} I,$$

(the sum of eigenvalues of all local exponents should be zero.)

Theorem

If the local exponents L_j evolve as above, Y satisfies the deformation equation

$$\frac{\partial Y}{\partial x_k} = \left(-\frac{A_k}{x - x_k} + \sum_{j=1}^{n} f_{jk} \log(x - x_j) \right) Y.$$

Proof. Since

$$\begin{aligned}\frac{d}{dx}Y(x)Y(x)^{-1} &\sim \frac{d}{dx}Y_j(x)Y_j(x)^{-1} + \frac{L_j}{x-x_j}Y_j(x)Y_j(x)^{-1} \\ &\sim Y_j(a_j)L_jY_j(x_j)^{-1}\frac{1}{x-a_j} + O((x-x_j)^0)\end{aligned}$$

for $j = 1, 2, ..., n$, we have

$$A_j = Y_j(x_j)L_jY_j(x_j)^{-1}.$$

Around $x = \infty$,

$$\begin{aligned}\frac{\partial Y}{\partial x_k}Y^{-1} &\sim \frac{\partial Y_\infty}{\partial x_k}(x)Y_\infty(x)^{-1} - \sum_{j=1}^{n} f_{jk}\log x, \\ &\sim -\sum_{j=1}^{n} f_{jk}\log x + O(\frac{1}{x}),\end{aligned}$$

since $\dfrac{\partial Y_\infty}{\partial x_k} \sim O(\frac{1}{x})$ by $Y_\infty(0) = I$.

Around $x = x_j$,

$$\begin{aligned}
\frac{\partial Y}{\partial x_k} Y^{-1} &\sim \frac{\partial Y_j}{\partial x_k}(x) Y_j(x)^{-1} + f_{jk} \log(x - x_j), \quad (j \neq k), \\
\frac{\partial Y}{\partial x_k} Y^{-1} &\sim \frac{\partial Y_k}{\partial x_k}(x) Y_k(x)^{-1} - Y_k(x) \frac{L_k}{x - x_k} Y_k(x)^{-1} + f_{kk} \log(x - x_k) \\
&\sim -\frac{A_k}{x - x_k} + f_{kk} \log(x - x_k) + (O(x - x_k)^0).
\end{aligned}$$

By the residue theorem,

$$\frac{\partial Y}{\partial x_k} = \left(-\frac{A_k}{x - x_k} + \sum_{j=1}^{n} f_{jk} \log(x - x_j) \right) Y.$$

$\square$

The deformation equation contains a logarithmic term

$$\nu_k = \sum_{j=1}^{n} f_{jk} \log(x - x_j),$$

the monodromy data is not preserved.
(Chakravarty-Ablowitz's term ν is also a log term)

5 Halphen's 2nd as MED

Let x_1, x_2, x_3 be functions of t.

$$P(x) := (x - x_1)(x - x_2)(x - x_3)$$

$$Q(x) := x^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2.$$

Halphen's 2nd:

$$x_j' = Q(x_j) \qquad j = 1, 2, 3.$$

Halphen's 2nd as MED

The compatibility condition gives the Halphen's 2nd:

$$\begin{aligned}\frac{\partial Y}{\partial x} &= \left(\frac{\mu}{P} + \sum_{j=1}^{3} \frac{c_j S}{x - x_j}\right) Y, \\ \frac{\partial Y}{\partial t} &= \left(\nu + \sum_{j=1}^{3} c_j x_j S\right) Y - Q(x)\frac{\partial Y}{\partial x}.\end{aligned}$$

μ, c_j's: constants $(c_1 + c_2 + c_3 = 0)$

S: a traceless constant matrix

$$\frac{\partial \nu}{\partial x} = -\frac{x + x_1 + x_2 + x_3}{P}\mu.$$

- The local exponent matrix L_j at $x = x_j$ evolve as

$$\frac{dL_j}{dt} = \frac{2x_j + x_k + x_l}{\prod_{m \neq j}(x_j - x_m)}\mu, \tag{2}$$

where $\{j, k, l\} = \{1, 2, 3\}$ as a set.

- The singular points x_j deform as

$$\frac{dx_j}{dt} = Q(x_j). \tag{3}$$

which is nothing but Halphen's second equation.

- Put $Y = fZ$ for a scalar $f = f(x,t)$.

$$\frac{\partial f}{\partial x} = \frac{\mu}{P} f,$$
$$\frac{\partial f}{\partial t} = \nu f - Q(x)\frac{\partial f}{\partial x}.$$

- The integrability condition for f:

$$\frac{\partial P}{\partial t} + Q\frac{\partial P}{\partial x} - P\frac{\partial Q}{\partial x} - (x + x_1 + x_2 + x_3)P = 0.$$

- Z gives the sixth Painlevé equation (**Riccati**)

$$\frac{\partial Z}{\partial x} = \sum_{j=1}^{3} \frac{c_j S}{x - x_j} Z,$$

$$\frac{\partial Z}{\partial t} = \sum_{j=1}^{3} c_j x_j S Z - Q(x)\frac{\partial Z}{\partial x}.$$

6 Review and future problems

Halphen is a special solution of MED, like the Riccati solutions in the Painlevé equation.

MPD	MED
Painlevé	Our New system
Riccati	Halphen

- How is other special solutions of MED?
- How can we characterize MPD in MED?

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