QUADRATIC EQUATIONS AND MONODROMY EVOLVING DEFORMATIONS

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1. Introduction

In this paper we study a special class of monodromy evolving deformations (MED). Chakravarty and Ablowitz [4] showed that a fifth-order equation which arises in complex Bianchi IX cosmological models can be represented by MED. We show that Halphen-type quadratic equations are obtained as MED.

In [4], Chakravarty and Ablowitz show that a fifth order equation

$$\begin{aligned} \omega_1' &= \omega_2\omega_3 - \omega_1(\omega_2+\omega_3) + \phi^2, \\ \omega_2' &= \omega_3\omega_1 - \omega_2(\omega_3+\omega_1) + \theta^2, \\ \omega_2' &= \omega_3\omega_1 - \omega_2(\omega_3+\omega_1) - \phi\theta, \\ \phi' &= \omega_1(\theta-\phi) - \omega_3(\theta+\phi), \\ \theta' &= -\omega_2(\theta-\phi) - \omega_3(\theta+\phi). \end{aligned} \tag{1}$$

can be represented by monodromy evolving deformations. The system (1) comes from the self-dual Bianchi IX system and is solved by the Schwarzian functions $S(z;0,0,a)$ (three angles of the Schwarzian triangle are $0,0$ and $a\pi$). Since generic Schwarzian functions have natural boundary or moving branch points, (1) cannot be obtained as monodromy preserving deformations, because monodromy preserving deformations has the Painlevé property.

The system (1) can be represented as the compatibility conditions for

$$\frac{\partial Y}{\partial x} = \frac{\mu I - (C_+x^2 + 2Dx + C_-)}{P} Y, \tag{2}$$

$$2\frac{\partial Y}{\partial t} = [\nu - (C_+x + D)]Y - Q(x)\frac{\partial Y}{\partial x}. \tag{3}$$

Here

$$P = \alpha_+x^4 + (\beta_+ + \beta_-)x^2 + \alpha_-, \quad Q = \alpha_+x^3 + \beta_+x,$$
$$C_\pm = (i\omega_1 \pm \phi)\sigma_1 \pm (\omega_2 \pm i\theta)\sigma_2, \quad D = -\omega_3\sigma_3,$$

for $\alpha_\pm = (\omega_1 - \omega_2) \mp (\theta + \phi), \beta_\pm = (\omega_1 + \omega_2 - 2\omega_3) \pm i(\theta - \phi)$. μ is a constant parameter and

$$\frac{\partial \nu}{\partial x} = \frac{(\beta_- + 4\omega_3) - \alpha_+x^2}{P}\mu. \tag{4}$$

Here the standard Pauli spin matrices σ_j's are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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Since ν is not a rational function on xm, (3) does not give a monodromy preserving deformation of (2).

Halphen studied two types of quadratic equations. His first equation in [7]

$$\begin{aligned} X'+Y' &= 2XY \\ Y'+Z' &= 2YZ, \\ Z'+X' &= 2ZX, \end{aligned} \tag{5}$$

is very famous and is appeared in many mathematical fields. It is a reduction from the Bianchi IX cosmological models or the self-dual Yang-Mills equation [5] [6] and gives a special self-dual Einstein metric [2]. If we set $y = 2(X+Y+Z)$, y satisfies Chazy's equation

$$y''' = 2yy'' - 3(y')^2. \tag{6}$$

Chazy's equation appeared in his classification of the third order Painlevé type equation [3], but (6) does not have the Painlevé property because generic solutions has natural boundary.

Halphen's second equation [8]

$$\begin{aligned} x_1' &= x_1^2 + a(x_1-x_2)^2 + b(x_2-x_3)^2 + c(x_3-x_1)^2, \\ x_2' &= x_2^2 + a(x_1-x_2)^2 + b(x_2-x_3)^2 + c(x_3-x_1)^2, \\ x_3' &= x_3^2 + a(x_1-x_2)^2 + b(x_2-x_3)^2 + c(x_3-x_1)^2, \end{aligned} \tag{7}$$

is less familiar but it is a more general system than (5). Here we use a different form the original Halphen's equation (See [12]). In case $a = b = c = -\frac{1}{8}$, (7) is equivalent to (5) by the transform $2X = x_2 + x_3, 2Y = x_3 + x_1, 2Z = x_1 + x_2$.

Halphen's first equation (5) can be solved by theta constants [7] [11]. His second equation (7) can be solved by the Gauss hypergeometric function [12]. When $a = b = c = -\frac{1}{8}$, (7) is solved by $F(1/2, 1/2, 1; z)$ which is related to $\theta_3(0, \tau)$.

Halphen's second equation is also appeared as a reduction of the self-dual Yang-Mills equation or the Einstein self-dual equation [1]. Since Halphen's equation or Chazy's equation do not have the Painlevé property, they do not obtained as monodromy preserving deformations. In [14], we showed that any Halphen's second equation are obtained as MED.

More generally, a quadratic equation

$$\frac{dx_i}{dt} = \sum_{j,k=1}^{n} c^i_{jk} x_j x_k \qquad (i = 1, 2, ..., n) \tag{8}$$

is studied in many fields. One of the most famous example is the Lotka-Volterra equation. Here c_{jk} is a complex constant and $c_{jk} = c_{kj}$. In 1960's Markus studies stability around the origin of a quadratic equation [10]. He use a non-associative algebra $V = \oplus_{j=1}^{n} \mathbb{C} a_j$ associated with (8):

$$a_j \cdot a_k = \sum_{i=1}^{n} c^i_{jk} a_i.$$

Markus gave a classification of quadratic systems when $n = 2$ by means of the non-associative algebra. The author was taught Markus' work by John McKay.

For Halphen's first and second equation, the associated non-associative algebra has a unit. Conversely, a quadratic equation with rank three can be solved by (confluent) hypergeometric functions, if and only if the associated non-associative algebra has a unit. Such quadratic systems are completely classified in [13].

The aim of this paper is to represent the Halphen-type system as monodromy evolving deformations. Since Halphen-type equation do not have the Painlevé property, it is never represented by monodromy preserving deformations.

In [5] [6], they obtained the Lax pair of Halphen's first equation and Chazy's equation, which are special cases of Halphen's second equation. And our Lax pair is different from their results even when Halphen's first equation since ours are monodromy evolving deformations but [5] [6] gave the Lax pair as a reduction of the self-dual Yang-Mills equation.

Both [4] and the author treat special cases of monodromy evolving deformations. We do not have general theory of monodromy evolving deformation. But it is important to study monodromy evolving deformations to consider nonlinear equations which does have the Painlevé property, such as Chazy's equation and Halphen's equations.

2. Reduction of the self-dual Yang-Mills equation

We take a $\mathfrak{g}$-valued 1-form

$$A = \sum_{j=1}^{4} A_j(x)dx_j,$$

where $\mathfrak{g}$ is a Lie algebra and $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. The curvature 2-form $F = \sum_{j<k} F_{jk} dx_j \wedge dx_k$ is given by

$$F_{jk} = \partial_j A_k - \partial_k A_j - [A_j, A_k],$$

where $\partial_j j = \partial/\partial x_j$. The SDYM equation is

$$F_{12} = F_{34}, \quad F_{13} = F_{41}, \quad F_{14} = F_{23}. \tag{9}$$

If the A_j's are functions only of $t = x_1$ and we take a special gauge such that A_1=0, then (9) is the Nahm equations

$$\begin{array}{rcl} \partial_t A_2 & = & [A_3, A_4] \\ \partial_t A_3 & = & [A_4, A_2], \\ \partial_t A_4 & = & [A_2, A_3]. \end{array} \tag{10}$$

We take $Diff(S^3)$, the infinite-dimensional Lie algebra of vector fields on S^3 as the Lie algebra g above. Let X_1, X_2 and X_3 are divergence-free vector fields on S^3 and satisfy commutation relations

$$[X_j, X_k] = \sum_l \varepsilon_{jkl} X_l,$$

where ε_{jkl} is the standard anti-symmetric form with $\varepsilon_{123} = 1$. Let $O_{jk} \in SO(3)$ be a matrix such that

$$\sum_{j,k,l} \varepsilon_{jkl} O_{jp} O_{kq} O_{lr} = O_{pqr},$$

$$X_j(O_{lk}) = \sum_p \varepsilon_{jkp} O_{lp}.$$

Then we choose the connection of the form

$$A_l = \sum_{j,k=1}^{3} O_{lj} M_{jk}(t) X_k.$$

Then the 3×3 matrix valued function $M = M(t)$ satisfies the ninth-order Darboux-Halphen (DH-IX) system [6]

$$\frac{dM}{dt} = (\operatorname{adj} M)T + M^T M - (\mathrm{Tr} M)M. \tag{11}$$

Here we set $\operatorname{adj} M := \det M \cdot M^{-1}$, and M^T is the transpose of M. The DH-IX system (11) was also derived by Hitchin [9] where it represents an $SU(2)$-invariant hypercomplex four-manifold. Since the Weyl curvature of a hypercomplex four-manifold is self-dual, (11) gives a class of self-dual Weyl Bianchi IX space-times.

In [6], Chakravarty, Ablowitz, and Takhtajan show that the Halphen's first equation (5) is a special case of the DH-IX system when $M_{jk}(t) = \omega_k(t)$.

In [4], Chakravarty and Ablowitz show that (1) is a special case of the DH-IX system when

$$M = \begin{pmatrix} \omega_1 & \theta & 0 \\ \phi & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}.$$

And they show that (1) is described by monodromy evolving deformations as (2-3).

We decompose the matrix $M = M_s + M_a$, where M_s is the symmetric part of M and M_a is the anti-symmetric part of M. We assume that the eigenvalues of the symmetric part M_s of M are distinct. Then M_s can be diagonalized using a complex orthogonal matrix P and we can write

$$M_s = P d P^{-1}, \quad M_a = P a P^{-1},$$

where $d = \operatorname{diag}(\omega_1, \omega_2, \omega_3)$. The matrix element of a skew-symmetric matrix a are denoted as $a_{12} = -a_{21} = \tau_3, a_{23} = -a_{32} = \tau_1, a_{31} = -a_{13} = \tau_2$. In [1], Ablowitz, Chakravarty and Halburd show that equation (11) can be reduced to the third-order system

$$\begin{aligned} \omega_1' &= \omega_2\omega_3 - \omega_1(\omega_2 + \omega_3) + \tau^2, \\ \omega_2' &= \omega_3\omega_1 - \omega_2(\omega_3 + \omega_1) + \tau^2, \\ \omega_2' &= \omega_3\omega_1 - \omega_2(\omega_3 + \omega_1) + \tau^2. \end{aligned} \tag{12}$$

where

$$\tau^2 = \alpha_1^2(\omega_1 - \omega_2)(\omega_3 - \omega_1) + \alpha_2^2(\omega_2 - \omega_3)(\omega_1 - \omega_2) + \alpha_3^2(\omega_3 - \omega_1)(\omega_2 - \omega_3).$$

The system (12) is equivalent to Halphen's second equation (7) by

$$2\omega_1 = -x_2 - x_3, \quad 2\omega_2 = -x_3 - x_1, \quad 2\omega_3 = -x_1 - x_2,$$

and

$$8a = \alpha_1^2 + \alpha_2^2 - \alpha_3^2 - 1, \quad 8b = -\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - 1, \quad 8c = \alpha_1^2 - \alpha_2^2 + \alpha_3^2 - 1.$$

The system (12) can be solved by the Schwarzian function $S(x; \alpha_1, \alpha_2, \alpha_3)$.

3. Halphen's equation

In this section we review Halphen's equation. Halphen's first equation (5) can be solved by theta constants. See [11].

$$X = 2\frac{\partial}{\partial t}\log\left[\theta_2\left(0, \frac{at+b}{ct+d}\right)(ct+d)^{-1/2}\right]$$
$$Y = 2\frac{\partial}{\partial t}\log\left[\theta_3\left(0, \frac{at+b}{ct+d}\right)(ct+d)^{-1/2}\right],$$
$$Z = 2\frac{\partial}{\partial t}\log\left[\theta_4\left(0, \frac{at+b}{ct+d}\right)(ct+d)^{-1/2}\right].$$

Here $\theta_j(z,\tau)$ is Jacobi's theta function and $ad-bc=1$. When $j=2,3,4$, $\theta_j(z,\tau)$ is an even function as z. Since generic solutions of (5) have natural boundary, (5) does not have the Painlevé property.

If we set $y = 2(X+Y+Z)$, y satisfies Chazy's equation (6), which is solved by

$$y(t) = 4\frac{\partial}{\partial t}\log\left[\vartheta_1'\left(0, \frac{at+b}{ct+d}\right)(ct+d)^{-3/2}\right],$$

where $ad-bc=1$.

Halphen's first equation and Chazy's equation are special cases of Halphen's second equation (7), which is solved by hypergeometric functions. For details, see [12]. We take a Fuchsian equation

$$\frac{d^2y}{dz^2} = \left(\frac{a+b}{z^2} + \frac{c+b}{(z-1)^2} - \frac{2b}{z(z-1)}\right)y.$$

Let t be a ratio of two solutions of the above equation. We set

$$x_1 = \frac{d}{dt}\log y, \quad x_2 = \frac{d}{dt}\log\frac{y}{z}, \quad x_3 = \frac{d}{dt}\log\frac{y}{z-1}.$$

Then x_1, x_2 and x_3 satisfy (7).

Chazy's equation (6) can be solved by the hypergeometric equation

$$x(1-x)\frac{d^2y}{dx^2} + \left(\frac{1}{2} - \frac{7}{6}x\right)\frac{dy}{dx} - \frac{1}{144}y = 0,$$

and is a special case of Halphen's second equation (7) when

$$a = -\frac{31}{288}, \qquad b = -\frac{23}{288}, \qquad c = -\frac{41}{288}.$$

For (7), the associated algebra has the following multiplication table:

$$a_1\cdot a_1 = a_1 + (c+a)(a_1+a_2+a_3), a_1\cdot a_2 = -a(a_1+a_2+a_3),$$
$$a_2\cdot a_2 = a_2 + (a+b)(a_1+a_2+a_3), a_2\cdot a_3 = -b(a_1+a_2+a_3),$$
$$a_3\cdot a_3 = a_3 + (b+c)(a_1+a_2+a_3), a_3\cdot a_1 = -c(a_1+a_2+a_3).$$

And this algebra has a unit $a_1+a_2+a_3$.

In [13], we showed that if and only if a non-associative and commutative algebra of rank three, the corresponding quadratic nonlinear equation can be solved by (confluent) hypergeometric functions (there exists some exceptional cases, which can be solved by elementary functions). And if the quadratic equation can be solved by hypergeometric functions, it is (7). A relation between Halphen-type equations and non-associative algebras was suggested by John McKay.

4. MONODROMY EVOLVING EQUATIONS

In general the solution of

$$\frac{dY}{dx} = \sum_{k=1}^{n} \frac{A_k}{x - x_j} Y,$$

can be developed as

$$\begin{aligned} Y(x) \sim & Y_j(x)(x - x_j)^{L_j}, \qquad Y_j(x) = O((x - x_j)^0), \\ Y(x) \sim & Y_\infty(x) x^{L_\infty}, \qquad Y_\infty(x) = I + O(1/x), \end{aligned}$$

around the singular points $x = x_1, x_2, ..., x_n$ and ∞. Here I is the unit matrix.

When the monodromy is preserved, Y satisfies the deformation equation

$$\frac{\partial}{\partial x_j} Y(x) = -\frac{A_j}{x - x_j} Y.$$

And the compatibility condition gives the Schlesinger equation

$$\frac{\partial A_k}{\partial x_j} Y(x) = \frac{[A_k, A_j]}{x_k - x_j}, \qquad j \neq k.$$

We will study monodromy evolving deformations when L_j will evolve according to

$$\frac{\partial L_j}{\partial x_k} = f_{jk} I \tag{13}$$

for any $j, k = 1, 2, ..., n$. Here $f_{jk} = f_{jk}(t)$ is a scalar function. The local exponent L_∞ evolve as

$$\frac{\partial L_\infty}{\partial x_k} = -\sum_{j=1}^{n} f_{jk} I,$$

because the sum of eigenvalues of all local exponents is invariant.

Theorem 1. *If the local exponents L_j evolve as* (13)*, Y satisfies the deformation equation*

$$\frac{\partial Y}{\partial x_k} = \left(-\frac{A_k}{x - x_k} + \sum_{j=1}^{n} f_{jk} \log(x - x_j) \right) Y.$$

Proof. Since

$$\begin{aligned} \frac{d}{dx} Y(x) Y(x)^{-1} & \sim \frac{d}{dx} Y_j(x) Y_j(x)^{-1} + \frac{L_j}{x - x_j} Y_j(x) Y_j(x)^{-1} \\ & \sim Y_j(a_j) L_j Y_j(x_j)^{-1} \frac{1}{x - a_j} + O((x - x_j)^0) \end{aligned}$$

for $j = 1, 2, ..., n$, we have

$$A_j = Y_j(x_j) L_j Y_j(x_j)^{-1}. \tag{14}$$

Therefore near $x = x_j$, we have the following expansions:

$$\begin{aligned} \frac{\partial Y}{\partial x_k} Y^{-1} & \sim \frac{\partial Y_\infty}{\partial x_k}(x) Y_\infty(x)^{-1} - \sum_{j=1}^{n} f_{jk} \log x, \\ & \sim -\sum_{j=1}^{n} f_{jk} \log x + O(\frac{1}{x}), \end{aligned}$$

since $\frac{\partial Y_\infty}{\partial x_k} \sim O(\frac{1}{x})$ by $Y_\infty(0) = I$.

$$\begin{aligned}\frac{\partial Y}{\partial x_k}Y^{-1} &\sim \frac{\partial Y_j}{\partial x_k}(x)Y_j(x)^{-1} + f_{jk}\log(x-x_j), \quad (j \neq k),\\ \frac{\partial Y}{\partial x_k}Y^{-1} &\sim \frac{\partial Y_k}{\partial x_k}(x)Y_k(x)^{-1} - Y_k(x)\frac{L_k}{x-x_k}Y_k(x)^{-1} + f_{kk}\log(x-x_k)\\ &\sim -\frac{A_k}{x-x_k} + f_{kk}\log(x-x_k) + (O(x-x_k)^0).\end{aligned}$$

We use (14) to show the last line. Therefore

$$\frac{\partial Y}{\partial x_k} = \left(-\frac{A_k}{x-x_k} + \sum_{j=1}^{n} f_{jk}\log(x-x_j)\right)Y.$$

□

Since the deformation equation contains a logarithmic term

$$\nu_k = \sum_{j=1}^{n} f_{jk}\log(x-x_j),$$

the monodromy data is not preserved. ν_k satisfies

$$\frac{d\nu_k}{dx} = \sum_{j=1}^{n}\frac{f_{jk}}{x-x_j},$$

which is essentially equivalent to (4) in the work of Chakravarty and Ablowitz. It seems difficult to study monodromy evolving deformations when f_{jk}'s is not scalar functions.

5. Halphen's second equation and MED

We show that Halphen's second equation is represented by monodromy evolving deformations. We set

$$Q(x) = x^2 + a(x_1-x_2)^2 + b(x_2-x_3)^2 + c(x_3-x_1)^2.$$

Halphen's second equation is

$$x_j' = Q(x_j) \qquad j = 1, 2, 3.$$

We set

$$P(x) = (x-x_1)(x-x_2)(x-x_3)$$

and consider the following 2 × 2 linear system.

$$\frac{\partial Y}{\partial x} = \left(\frac{\mu}{P} + \sum_{j=1}^{3}\frac{c_j S}{x-x_j}\right)Y, \tag{15}$$

$$\frac{\partial Y}{\partial t} = \left(\nu + \sum_{j=1}^{3} c_j x_j S\right)Y - Q(x)\frac{\partial Y}{\partial x}. \tag{16}$$

Here μ and c_j's are constants with $c_1 + c_2 + c_3 = 0$, and S is any traceless constant matrix. We assume

$$\frac{\partial \nu}{\partial x} = -\frac{x + x_1 + x_2 + x_3}{P}\mu.$$

Theorem 2. *The compatibility condition of* (15) *and* (16) *gives the Halphen's second equation.*

We can prove the theorem above directly. Therefore (15) and (16) are a Lax pair of Halphen's second equation.

The local monodromy of $Y_j(x)$ around $x = x_j$ is adjoint to $e^{2\pi i L_j}$. This deformation does not preserve monodromy data. The local exponent L_j at $x = x_j$ evolve as

$$\frac{dL_j}{dt} = \frac{2x_j + x_k + x_l}{\prod_{m \neq j}(x_j - x_m)}\mu, \tag{17}$$

where $\{j, k, l\} = \{1, 2, 3\}$ as a set. The singular points x_j also deform as

$$\frac{dx_j}{dt} = Q(x_j), \tag{18}$$

which is nothing but Halphen's second equation.

We can eliminate the variables μ and ν in (15) and (16) by the rescaling $Y = fZ$ for a scalar function $f = f(x, t)$ [4]. f satisfies the linear equations

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\mu}{P} f, \\ \frac{\partial f}{\partial t} &= \nu f - Q(x)\frac{\partial f}{\partial x}. \end{aligned}$$

The integrability condition for f is

$$\frac{\partial P}{\partial t} + Q\frac{\partial P}{\partial x} - P\frac{\partial Q}{\partial x} - (x + x_1 + x_2 + x_3)P = 0.$$

And Z satisfies

$$\begin{aligned} \frac{\partial Z}{\partial x} &= \sum_{j=1}^{3} \frac{c_j S}{x - x_j} Z, \\ \frac{\partial Z}{\partial t} &= \sum_{j=1}^{3} c_j x_j S Z - Q(x)\frac{\partial Z}{\partial x}. \end{aligned}$$

The integrability condition for Z is the sixth Painlevé equation. In our case, we take the Riccati solution of the sixth Painlevé equation, which reduce to the hypergeometric equation, since the residue matrix of $(Cx + D)/P$ at the infinity is zero. But this hypergeometric equation is different from the hypergeometric functions which solve (7).

The Halphen's equation is described as MED, but it stands on a similar position as the Riccati solution of the Painlevé equations. Studies of generic solutions of MED or other special solutions of MED, such as algebraic solutions or elliptic solutions are future problems.

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