The Painlevé transcendents with Solvable monodromy

September 26th 2007

Yousuke Ohyama
Osaka University, Japan

at Mathematical Institute, Oxford

0. List of Painlevé equations

$$\begin{aligned}
\text{P1)} \qquad y'' &= 6y^2 + t, \\
\text{P2)} \qquad y'' &= 2y^3 + ty + \alpha, \\
\text{P4)} \qquad y'' &= \frac{1}{2y}y'^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \\
\text{P3)} \qquad y'' &= \frac{1}{y}y'^2 - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y}, \\
\text{P5)} \qquad y'' &= \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) \\
&\qquad + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1},
\end{aligned}$$

$$\begin{aligned}
\text{P6)} \quad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)y'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\
+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right].
\end{aligned}$$

1. Why will we study the Painlevé?

Two motivations of the Painlevé equation:

- The Painlevé property (without moving singularities)
- Monodromy preserving deformations

Miwa-Malgrange Theorem

MPD $\Longrightarrow$ PP

- (For non-PP, we need Monodromy Evolving deformations)

- What is a meaning of the Painlevé property?

  PP $\Longrightarrow$ MPD ??
- Are the Painlevé equations are integrable?
- What is a meaning of integrability?
- (How about discrete case?)

What is a meaning of differential equations?

What is a meaning of solving differential equations?

It is impossible to answer at least now....

Classical Integrable system

solve by Rational, Exponential, Abelian functions

$2n$-dim, symplectic, genus n Jacobian fibration

Study of Integrable system

Find a procedure to reduce known functions

Find a criterion of Integrable system (**P-test**,...)

What is a meaning of known functions?

known functions = Umemura's classical functions
= diff-Galois group is algebraic group
= Abelian + Linear equation

But generic Painlevé functions are transcendental.

What is a meaning of SOLVING the Painlevé equations?

transcendental = unknown, new

solve = reduce to known functions

· **Classification of all classical sol. of the Painleve eq.**

algebraic sol. classified except PVI

Riccati sol's are all classified. ($\longrightarrow$hypergeometric)

· **How about other solutions ?**

$\Longrightarrow$ **The Painlevé Property**

$\Longrightarrow$ **Monodromy Problem**

2. Solve the Painlevé eq. by PP

P2 has a simple pole:

$$y = \frac{1}{t-c} + \sum_{n=0}^{\infty} a_n (t-c)^n$$

$$a_0 = 0,\ a_1 = -\frac{c}{6},\ a_2 = -\frac{1+\alpha}{4},\ a_3 = h,$$

$$a_4 = \frac{c(1+3\alpha)}{72},\ a_5 = \frac{27 - 2c^3 + 108\alpha + 81\alpha^2 - 216ch)}{3024}$$

y is tangent at $t = \infty$.

We need blow-up to study the pole solutions.

$\Longrightarrow$ By finite times of blow-up, we get Initial Values Space

Initial Values space = Complete Painlevé test

Classify all rational surfaces with a good condition

$\Longrightarrow$ **Sakai's classification** (9 points blow-up of $\mathbb{CP}^2$)

All continuous Painlevé + q-Painlevé + elliptic Painlevé

Fuchs-Poincaré Theorem

The 1st order eq. $f(t, y, y') = 0$ has PP

$\Longrightarrow f(t, y, z) = 0$: a genus 0,1 curve for generic t as a f'n of y, z

$\Downarrow$

Problem

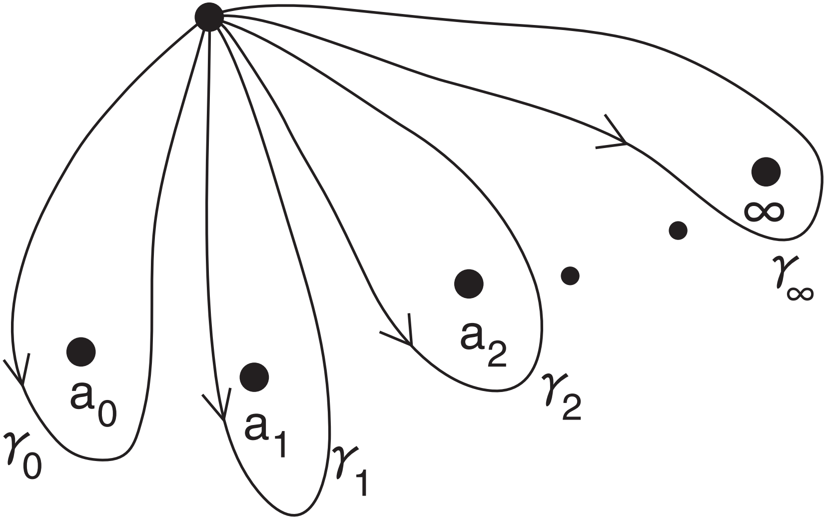
If 2nd order eq. $f(t, y, y', y'') = 0$ has PP,
what is the surface $f(t, y, z, w) = 0$?

Partial Answer by Sakai's next talk "**ODE on RES**"

3. Solve the Painlevé eq. by Monodromy

$$\frac{dY}{dz} = \left(\frac{A_0}{z-a_0} + \frac{A_1}{z-a_1} + \cdots + \frac{A_n}{z-a_n} \right) Y$$

A_j: $m \times m$ matrices, $M_\infty M_n \cdots M_1 M_0 = 1$



The Riemann-Hilbert correspondence

$$\mathcal{RH} : \text{connection } \{A_j\} \longrightarrow \text{monodromy } \{M_j\}$$

The map $\mathcal{RH}$ is highly transcendental, it is difficult to calculate the map exactly. We will study a ‘special value’ of $\mathcal{RH}$.

Riemann-Hilbert Problem and Isomonodromy Problem

Schlesinger and **Garnier** tried

to solve **RH Problem** by **MPD**.

[Example]

· When singularities $x = a_j$ are symmetric

· By confluence of a singularity $x = a_j \to a_k$, LEq becomes simple.

Principal.

If we research isomonodromic family of connections, we can understand the connection or monodromy data more.

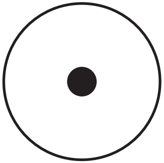
3.1 Symmetric solution of P4

$$\frac{\partial Y}{\partial x} = A(x,t)Y, \quad \frac{\partial Y}{\partial t} = B(x,t)Y, \quad (\mathrm{MJ},\ 1981)$$

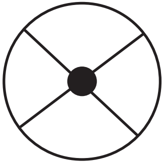
$$A(x,t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} t & u \\ \frac{2(z-\theta_0-\theta_\infty)}{u} & -t \end{pmatrix} + \begin{pmatrix} -z+\theta_0 & -\frac{uy}{2} \\ \frac{2z(z-2\theta_0)}{uy} & z-\theta_0 \end{pmatrix} \frac{1}{x},$$

$$B(x,t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & u \\ \frac{2(z-\theta_0-\theta_\infty)}{u} & 0 \end{pmatrix}.$$

$$\alpha = 2\theta_\infty - 1, \quad \beta = -8\theta_0^2.$$



x=0



x=∞

The Stokes matrices around ∞:

$$G_1 = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

Relation between the parameters $p = ab, q = bc, r = cd$.

$$(*) \qquad q(1+q)e^{2i\pi\theta_\infty} + [pr + (1+p)(1+r)q]e^{-2i\pi\theta_\infty} = 2q\cos 2\pi\theta_0.$$

Take a new variables $y, w = z/y$. a polynomial connection:

$$A(x,t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} t & u \\ \frac{2(yw-\theta_0-\theta_\infty)}{u} & -t \end{pmatrix} + \begin{pmatrix} -yw+\theta_0 & -uy/2 \\ \frac{2w(yw-2\theta_0)}{u} & yw-\theta_0 \end{pmatrix} \frac{1}{x}$$

$$\begin{aligned} \frac{dy}{dt} &= -4yw + y^2 + 2ty + 4\theta_0 \\ \frac{dw}{dt} &= 2w^2 - 2yw - 2tw + (\theta_0 + \theta_\infty) \end{aligned}$$

[Theorem] (Kaneko) Set $t = 0, y = w = 0$

$$A(x,t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & u \\ 2(-\theta_0 - \theta_\infty)/u & 0 \end{pmatrix} + \begin{pmatrix} \theta_0 & 0 \\ 0 & -\theta_0 \end{pmatrix} \frac{1}{x}$$

$$Y(x) = \begin{pmatrix} L_{k,m}(x) & L_{k,-m}(x) \\ \frac{2k-2m-1}{u(2m+1)} L_{k+\frac{1}{2},m+\frac{1}{2}}(x) & \frac{2(2m+1)}{u} L_{k+\frac{1}{2},-m-\frac{1}{2}}(x) \end{pmatrix},$$

$$L_{k,m}(x) = x^{\theta_0} e^{-\frac{x^2}{2}} {}_1F_1(m - k + \frac{1}{2}, 2m + 1; x^2)$$

$$k = \frac{2\theta_\infty - 1}{4}, \qquad m = \frac{2\theta_0 - 1}{4}$$

$\Longrightarrow$ We can determine monodromy, Stokes.

Remark. Similar symmetric solutions exist for P1, P2 (Kitaev).

定年後に数学博士号

大阪大大学院・71歳の金子さん

難解方程式で新発見も

企業を定年退職後に大阪大で数学を学び、研究してきた七十一歳の男性がこの春、博士号を授与されることになった。研究内容は世界の最先端として高い評価を受けている。若手の活躍が目立つ数学では六十歳を過ぎて学び始め、論文を書く人は極めてまれだ。

大阪府池田市の金子和雄さん（七一）は大阪大大学院で機械工学の修士課程を終え、企業の技術者として蒸気タービンの設計などに携わった。九年前に定年退職。「まだ何かできそうだ」ともう一度勉強する決心をした。

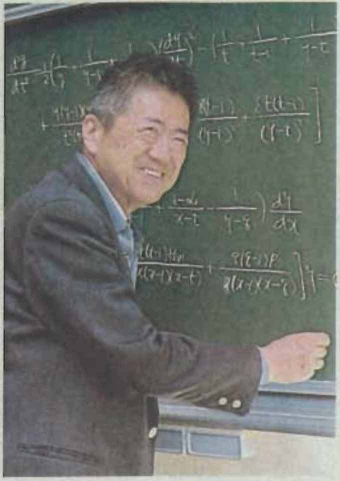
数学を選んだのは〝消去法〟だったという。「学生時代は製図や実験に時間を取られ、じっくり勉強した記憶がない。物理は体力的に実験についていけそうもなく、数学へ。行きたいとは思っていたんですが」と話す。

学部の聴講生から始めて、順調に大学院に進学。数学者を悩ませてきた「パンルベ方程式」という難解な微分方程式の研究に打ち込み、この方程式を満たす解となる新しい関数を発見した。成果は国内の学術誌や海外の研究会で発表され、高い評価を受けている。

「苦しんで苦しんで計算して、やっと一つ、答えを見つける。小さな前進ですけれど、そのうれしさは、何ものにも替え難いものです」

指導教官の大山陽介同大助教授（四四）は「博士課程で研究が止まる人も少なくないが、金子さんは本当に順調に続けてこられた。若い学生にも良い影響を与えたと思う。細かい計算に丁寧に、執拗（しつよう）に取り組む姿勢は私も見習わなければ」と語る。

学位授与式は今月二十三日。四月以降も大学で「気力の続く限り」（金子さん）研究を続ける。



定年退職後に大阪大で数学を学び博士号を授与される金子和雄さん

残留孤児支援策―代表が十日、大阪市内で―原告団からは、京都訴

71歳理学博士誕生

姫路の金子さん

定年後、大阪大で研究

学位授与式に臨む金子和雄さん（中央）＝23日正午、大阪府吹田市の大阪大学で ▲

元エンジニアで、定年退職後に大阪大で数学を研究してきた兵庫県姫路市の金子和雄さん(71)が23日、難解な微分方程式に挑んだ成果が認められ、理学博士号を授与された。午後から同大学で行われた授与式では、情報科学研究科の代表を務めた。金子さんは「まるで問題の魅力に吸い込まれるようだった」と感慨深げに話している。

金子さんは大阪大工学部卒。機械工学の修士号を取得後、三菱重工業などで37年間、蒸気タービンの設計・製造技術者として活躍した。63歳で定年退職した後、学生時代から興味があった数学を「もう一度、じっくり勉強したい」と決意して理学部へ聴講に通い、2000年に学士編入した。

卒業前、フランスの数学者が見つけてから1世紀近く解けないままという「パンルベ方程式」に出会い、本格的な研究のために大学院情報科学研究科に進学。六つの式のうち、三つの式に解のヒントとなる新しい関数式を発見し、高い評価を受けた。学位論文は「モノドロミ可解なパンルベ超越函数（かんすう）」。

休日も惜しんで研究に励み、国内外の研究会にも積極的に参加。孫ほど年の離れた院生らともうち解けた。指導教官の大山陽介助教授(44)は「山積みの難題も、一つずつ結び目をほぐすような地道な努力を続け、大いに刺激を受けた」と感服する。

「必死でやっている間、ほかに何もいらないほど楽しい」と金子さん。今後も研究生などとして研究を続けるという。

Theorem 1. *Set* $b_1 = \cos 2\pi\theta_0, \quad b_2 = \cos 2\pi\theta_\infty$.

1. *If* $(b_1^2 - 1)(b_0 - b_1) = 0$, *(*) has a singularity* (A_1 *or* A_2)*.*
2. $\mathcal{RH}$ *is a simultaneous resolution.*
3. $(b_1^2 - 1)(b_0 - b_1)$ *is an analytic invariant of* $W(A_2^{(1)})$.

Theorem 2.

Parameter	Stokes data	Solution
$\theta_0 = 1/6, \theta_\infty = 1/2$	$a = b = c = d = -\sqrt{-1}$	$y = -2t/3$
$\theta_0 = \theta_\infty$	$b = d = 0$ (node)	Weber
$\theta_0 = \theta_\infty = 0$	$a = c = b = d = 0$ (A_2)	Hermite
any	$\mathbb{Z}_2$ symmetry	$y(0) = w(0) = 0$ Kaneko

3.2 Analytic solutions around $t = 0$ of P6

For a generic solution of P6, $t = 0$ is

an essential singularity and a branch point.

Theorem 3. (Kaneko, Bruno)

For generic parameters, P6 has four meromorphic solutions at $t = 0$.

$$\begin{aligned}
\text{I)}: \quad y(t) &= \frac{\alpha_4}{\alpha_4-\alpha_0}t + \frac{\alpha_0\alpha_4\left[-1-\alpha_1^2+\alpha_3^2+(\alpha_4-\alpha_0)^2\right]}{2\left[1-(\alpha_4-\alpha_0)^2\right](\alpha_4-\alpha_0)^2}t^2 + O(t^3),\\
\text{II)}: \quad y(t) &= \frac{\alpha_4}{\alpha_4+\alpha_0}t + \frac{-\alpha_0\alpha_4\left[1+\alpha_1^2-\alpha_3^2-(\alpha_4+\alpha_0)^2\right]}{2\left[1-(\alpha_4+\alpha_0)^2\right](\alpha_4+\alpha_0)^2}t^2 + O(t^3),\\
\text{III)}: \quad y(t) &= \frac{\alpha_1+\alpha_3}{\alpha_1} + \frac{-\alpha_3\left[1+\alpha_4^2-\alpha_0^2-(\alpha_1+\alpha_3)^2\right]}{2\alpha_1\left[1-(\alpha_1+\alpha_3)^2\right]}t + O(t^2),\\
\text{IV)}: \quad y(t) &= \frac{\alpha_1-\alpha_3}{\alpha_1} + \frac{\alpha_3\left[1+\alpha_4^2-\alpha_0^2-(\alpha_1-\alpha_3)^2\right]}{2\alpha_1\left[1-(\alpha_1-\alpha_3)^2\right]}t + O(t^2).
\end{aligned}$$

$$\alpha = \frac{{\alpha_1}^2}{2}, \quad \beta = -\frac{{\alpha_4}^2}{2}, \quad \gamma = \frac{{\alpha_3}^2}{2}, \quad \delta = -\frac{{\alpha_0}^2-1}{2}.$$

Remark. $t = 0$ is the Briot-Bouquet type sing. of P6.

For meromorphic solutions we take two limits of

$$\frac{dY}{dx} = \left(\frac{A_1}{x} + \frac{A_2}{x-t} + \frac{A_3}{x-1} \right) Y,$$

1) $t \longrightarrow 0$:

$$\frac{A_1}{x} + \frac{A_2}{x-t} = \frac{A_1 + A_2}{x} + \frac{tA_2}{x(x-t)}$$

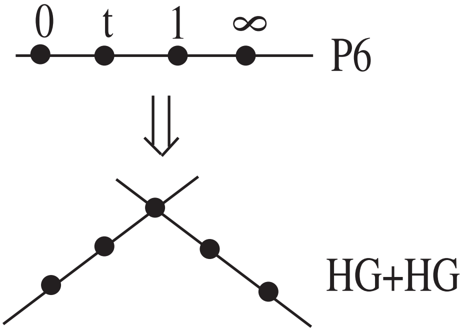
If $\lim_{t \to 0} tA_2 = 0$, no irregular singularity.

2) We set $x = zt$ and $t \longrightarrow 0$.

$$\frac{dY}{dz} = \left(\frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-1/t} \right) Y,$$

If $\lim_{t \to 0} tA_3 = \text{finite}$, no irregular singularity.

We get two limits, which reduce to Hypergeometric (**Jimbo** 1982).



For generic solution, both limits are Euler-Gauss hypergeometric.

Theorem 4. (Kaneko) *For four meromorphic solutions of P6, one of two limits above becomes a reduced equation (monodromy data is diagonal). Converse is true.*
For the solution (I) (II), the first limit is HG and $[M_0, M_t] = 0$.
For (III) (IV), the second limit is HG and $[M_1, M_\infty] = 0$.

Remark.

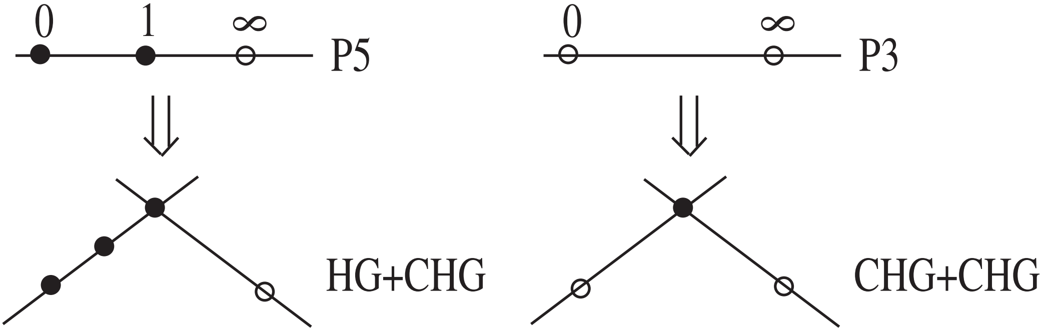
Similar results hold for P5, P3.

- P5 has three meromorphic solutions at $t = 0$.
- P3($D_6^{(1)}$) has two meromorphic solutions at $t = 0$.
- P3($D_7^{(1)}$) has one meromorphic solution at $t = 0$.
- P3($D_8^{(1)}$) has no meromorphic solutions at $t = 0$.

P3($D_6^{(1)}$): $\delta \neq 0, \gamma \neq 0$

P3($D_7^{(1)}$): $\delta = 0, \gamma \neq 0$ (or $\delta \neq 0, \gamma = 0$)

P3($D_8^{(1)}$): $\delta = 0, \gamma = 0$



Observation

1) #{meromorphic sol's at the origin}=#{parameters α, β...}

2) All meromorphic solutions are equivalent to the Bäcklund transf.

3) For P5, one is HG+diagonal, two are diagonal+CHG.

4) For P3($D_6^{(1)}$), one is CHG+diagonal, another is diagonal+CHG.

5) For P3($D_7^{(1)}$), the limit is CHGL+diagonal.

The confluent HG limit equation (CHGL) ${}_0F_1(c; x)$:

$$xu'' + \frac{c}{x}u' - u = 0$$

3.3 Umemura's classical solutions

Umemura's classical functions = solutions of LE + Abelian

LE = Picard-Vessiot Theory

· As solutions of Painlevé, LE becomes the Riccati solutions (HG).

· As solutions of Painlevé, abelian functions (elliptic) does not appear except asymptotics.

· Picard's solution does **not classical** in Umemura's sense.

$$y = \wp(a\omega_1(t) + b\omega_2(t); \omega_1(t), \omega_2(t))$$

depends both on z and τ.

The Riccati solution of P6

$$A(x,t)=\frac{A_1}{x}+\frac{A_2}{x-1}+\frac{A_3}{x-t},\qquad A_j=\begin{pmatrix} a_j & b_j\\ 0 & d_j\end{pmatrix}.$$

$A_\infty=-(A_0+A_1+A_2)$: diagonal $\Leftrightarrow b_1+b_2+b_3=0.$

Schlesinger equation:

$$\frac{dA_1}{dt}=\frac{[A_3,A_1]}{t},\qquad \frac{dA_2}{dt}=\frac{[A_3,A_2]}{t-1}.$$

becomes

$$\frac{db_1}{dt}=\frac{1}{t}\left(e_3b_1-e_1b_3\right),\ \frac{db_2}{dt}=\frac{1}{t-1}\left(e_3b_2-e_2b_3\right),$$

$$e_j=a_j-d_j$$

can be solved directly!

b_1 satisfies the Euler-Gauss Hypergeometric

$$t(t-1)\frac{d^2}{dt^2}b_1-\left[(e_1+e_2+2e_3)t-e_0-e_2+1\right]\frac{d}{dt}+e_3(e_1+e_2+e_3)b_1=0$$

3.4 R. Fuchs Problem

R. Fuchs' three Math. Ann. papers

1) **[1906]** P6 is isomonodromic deformations

2) **[1911]** a Forgotten paper, the same title as [1906]
determined Linear Monodromy of Picard's solution
proposed **When LEq is a pull-back of HG?**

3) **[1914]** Crashed by Garnier paper
studied asymptotic behavior at $t = 0$ of P6

The third paper is wrong, but still contains a good idea. (Kaneko's recent paper)

R. Fuchs' problem:

When can we reduce the linearized equation of a Painleve y

$$\frac{d^2u}{dx^2} = P(x;t,y)u$$

to an equation without the deformation parameter t

$$\frac{d^2v}{d\xi^2} = q(\xi)v$$

by an algebraic transformation

$$\xi = R(x;t), \qquad u = \sqrt{dx/d\xi}\ v?$$

The answer = **algebraic solutions**?

For P1-P5, yes (O-Okumura).

rational sol's of P2-P5 = **pull-back of CHG(L)**

For P6, Kitaev(-Vidunas), Boalch,... (except Picard)

4 Monodromy solvable solutions

For some Painlevé solutions,
we can determine the monodromy data (Stokes).

1) **Umemura's classical solutions**
   Riccati: Monodromy is upper triangular.
   Algebraic: pull-back of Hypergeometric (R. Fuchs 1911).
2) **Symmetric solutions**: P1, P2, P4
3) **Picard's solution** of P6 (R. Fuchs 1911)
4) **Analytic around $t = 0$**: P3, P5, P6.
5) **Boutroux solutions**: P1, P2. or Tronquée solution...

> One meaning of Solving the Painlevé equations is to determine the linear monodromy of the Painlevé function.

Appendix. Picard's solution of P6

Picard's solution of **P6(0,0,0,1/2)**

$$y = \wp(a\omega_1 + b\omega_2; \omega_1, \omega_2) + \frac{t+1}{3}$$

$$\omega_j = \int_{\gamma_j} \frac{dx}{\sqrt{x(x-1)(x-t)}}.$$

For Picard's solution, the monodromy matrices:

$$M_j = \begin{pmatrix} -1 + a_j b_j & -b_j^2 \\ a_j^2 & -1 - a_j b_j \end{pmatrix} \qquad j = 0, 1, t, \infty$$

$$a_0 b_1 - a_1 b_0 = 2\cos(a-b)\pi, \quad a_0 b_\infty - a_\infty b_0 = 2\sin a\pi$$

$$a_0 b_t - a_t b_0 = 2\cos b\pi, \quad a_1 b_\infty - a_\infty b_1 = -2\sin(2a-b)\pi$$

$$a_1 b_t - a_t b_1 = -2\cos a\pi, \quad a_t b_\infty - a_\infty b_t = -2\sin b\pi$$

R. Fuchs (1911) Math. Ann. **70**, 525–549.
[Kitaev-Korotkin (1998), Mazzocco (2001)]

References

[1] R. Fuchs, Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im endlichen gelegenen wesentlich singulären Stellen. Math. Ann. **70** (1911), 525–549.

[2] R. Fuchs, Über die analytische Natur der Lösungen von Differential gleichungen zweiter ordnung mit festen kritischen Punkten, Math. Ann. **75** (1914), 469–496.

[3] M. Jimbo, Monodromy problem and the boundary condition for some Painlevé equations, *Publ. Res. Inst. Math. Sci.* **18** (1982), 1137–1161.

[4] K. Kaneko, "Painleve VI transcendents which are meromorphic at a fixed singularity", Proc. Japan Acad. Ser. A Math. Sci. 82 (2006), 71–76.

[5] K. Kaneko, "The Painleve transcendents with solvable monodromy", math.CA/0701241, **Doctor these in Osaka University.**

[6] K. Kaneko, and Y. Ohyama, "Fifth Painleve transcendents which are analytic at the origin", Funkcial. Ekvac. (2007),

[7] K. Kaneko, "Painlevé VI transcendents on the Fricke's cubic surface"