Particular solutions of q-Painlevé equations and q-hypergeometric equations

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Painlevé Equations and Related Topics

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-1. Asymptotics on the Painlevé equations

Bruno's work

⋆ Power/Exponential/Elliptic (complete) asymptotics

Some are known, some are new. (including log-terms /degenerate case)

The study of **asymptotic analysis in the Painlevé equations** :

⋆ Find **asymptotic forms** (0-parameters, 1-parameters, 2-parameters)

⋆ **Resummation** of asymptotic series (**WKB, Ecall's resurgence** ,...)

⋆ **Connection formula** between the parameter (**Nonlinear Stokes**)

For P2 P3, see the **Novokshenov's book** with Fokas, Kapaev, Its

⋆ in Ch.11 $y'' = 2y^3 + ty + \alpha$ (Recent Kapaev's work).

For **Elliptic asymptotics** for P1 and P2,

Kitaev, A. V. Elliptic asymptotics of the first and second Painleve transcendents. Uspekhi Mat. Nauk/Russ. Math. Surveys **49** (1994), 77–140; 81–150.

Joshi-Kitaev, Boutroux's tritronquee for P1 (SAM2001)

Kitaev-Vartanian, degenerate third Painleve equation. (IP2004/2010)

A. A. Kapaev, Quasi-linear Stokes phenomenon for P1 (JPA 2004)

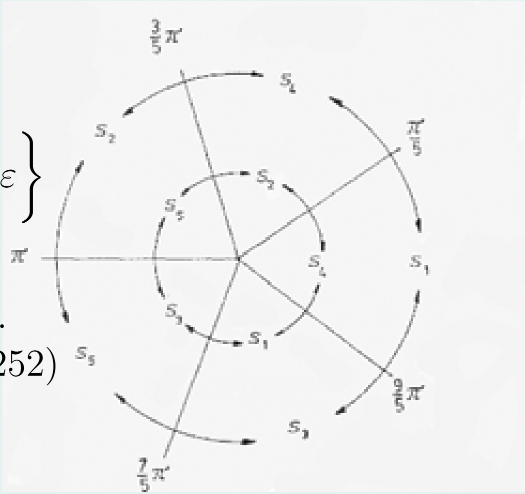
S. Shimomura, K. Takano, See **From Gauss to Painlevé** Ch.4 (1980's)

Example1: P1 case : $y'' = 6y^2 + x$

$$y(x) \sim |x|^{1/2}\wp(\frac{4}{5}e^{i\varphi}|x|^{5/4} - t(\varphi, s); g_2(\varphi), g_3(\varphi)) + O(|x|^{3/4}), \quad x \in D_k(\varphi, \varepsilon, s)$$

$$t(\phi, s) = \frac{1}{2\pi i}\left(\omega_a(\varphi)\log(is_{2-2k}) + \omega_b(\varphi)\log\frac{s_{5-2k}}{s_{2-2k}}\right)$$

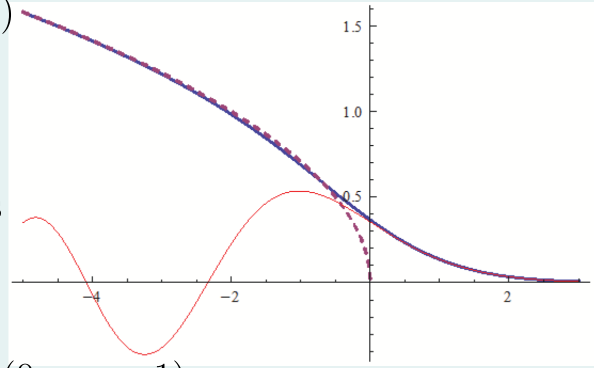
$$D_k(\varphi, \varepsilon, s) = \left\{x \in \mathbb{C}; \frac{(3+2k)\pi}{5} + \varepsilon \leqq \varphi \leqq \frac{(5+2k)\pi}{5} - \varepsilon\right\}$$



Kapaev-Kitaev Connection formulae for P1.
(Lett. Math. Phys. **27** (1993), 243–252)

Example2: Special P2 case: $y'' = 2y^3 + xy$

The Hastings-McLeod solution (1980)



$$y \sim \begin{cases} \mathrm{Ai}(x) & (x \to +\infty) \\ \sqrt{-x/2} & (x \to -\infty) \end{cases}$$

$x \to +\infty$: **Exponential** asymptotics

$x \to -\infty$: **Power** asymptotics

The Ablowitz-Segur solution (1977): $(0 < r < 1)$

$$y \sim \begin{cases} r\mathrm{Ai}(x) & (x \to +\infty) \\ d|x|^{-1/4} \sin\left(\frac{2}{3}|x|^{3/2} - \frac{3}{4}d^2 \log|x| - \theta\right) + o(|x|^{-1/4}) & (x \to -\infty) \end{cases}$$

Connection formula: $r \to (d, \theta)$

$$\begin{aligned} d^2(r) &= -\pi^{-1} \log(1 - r^2) \\ \theta(r) &= \frac{3}{2} d(r)^2 \log 2 + \arg[\Gamma(1 - id(r)^2/2)] - \frac{\pi}{4} \end{aligned}$$

0. Motivation

· There exist **different q-Painlevé equations** whose limit $q \to 1$ give the same continuous Painlevé equation.

⋆ **Two types of q-P_{III}** are known.

· There exist **different q-special functions** whose limit $q \to 1$ give the same classical special functions. They appear as **particular solutions** of q-Painlevé equations.

⋆ **Three q-Bessel**, **two q-Airy** are known.

For q-Painlevé equations, **Sakai's classification** of the initial value spaces explains why there exist different q-Painlevé equations.

But we **do not know** classification of q-special functions.

> **Unified theory of q-difference linear equations** matches to Sakai's classification.

1. Unified approach to q-difference linear equations

1.1. Basic notations

0) **q-Pochhammer symbol**:

$$(a_1,\ldots,a_r;q)_n=\prod_{i=1}^{n}(a_i;q)_n,\quad (a;q)_n=(1-a)(1-qa)\cdots(1-q^{n-1}a).$$

1) **generalized q-hypergeometric series**:

$$_r\varphi_s\,(a_1,\ldots,a_r;b_1,\ldots,b_s;q,z)=\sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_n}{(b_1,\ldots,b_s;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}z^n.$$

2) **Theta function**:

$$\theta_q(x):=\sum_{n=-\infty}^{\infty}q^{n(n-1)/2}x^n=(q,-x,-q/x;q)_\infty.$$

$$x\theta_q(xq)=\theta_q(x).$$

1.2. Different q-special functions

1) Three types of the **q-Bessel functions**:

${}_2\varphi_1\left(0,0;c;x\right), {}_0\varphi_1\left(-;c;x\right), {}_1\varphi_1\left(0,c;x\right)$.

$$(3.1): \quad J_\nu^{(1)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu {}_2\varphi_1\left(0,0;q^{\nu+1};q;-\frac{x^2}{4}\right),$$

$$J_\nu^{(2)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu {}_0\varphi_1\left(-;q^{\nu+1};q;-\frac{q^{\nu+1}x^2}{4}\right),$$

$$(3.2): \quad J_\nu^{(3)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}x^\nu {}_1\varphi_1\left(0;q^{\nu+1};q;qx^2\right).$$

(Hahn's formula): $\quad J_\nu^{(2)}(x;q) = (-\frac{x^2}{4};q)_\infty \cdot J_\nu^{(1)}(x;q).$

$\star$ In our classification, $J_\nu^{(1)}$ and $J_\nu^{(2)}$ are equivalent.

2) Two types of the q-**Airy functions**:

[HKW] Solutions of (4-1): $\underline{u(xq^2) + xu(xq) - u(x) = 0}$

$$u = A\, {}_1\varphi_1\left(0; -q; q; -x\right) + B\, \mathrm{e}^{\pi i \frac{\log x}{\log q}}\, {}_1\varphi_1\left(0; -q; q; x\right),$$

$$\mathrm{Ai}_q(x) := {}_1\varphi_1\left(0; -q; q; -x\right) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q^2; q^2)_k} x^k.$$

When $q^\nu = -1$, this q-Airy is related to $J_\nu^{(3)}(x; q)$:

$$J_\nu^{(3)}(x; q) = \frac{(-q; q)_\infty}{(q; q)_\infty} x^\nu\, {}_1\varphi_1\left(0; -q; q; qx^2\right).$$

This is **a q-analogue of relation between Airy and modified Bessel**:

$$\mathrm{Ai}(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}\, K_{1/3}(\frac{2}{3}x^{3/2}).$$

[Ismail] Solutions of (4-2): $\underline{qxu(xq^2) - u(xq) + u(x) = 0}$

$$u = A\, {}_0\varphi_1\left(-; 0; q; -tq\right) + B\, \theta_q(x)\, {}_2\varphi_0\left(0, 0; -; q; -t/q\right)$$

The first solution is called q-**Ramanujan function**:

$$A_q(x) := {}_0\varphi_1\left(-; 0; q; -tq\right) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k} (-z)^k.$$

There are **three different q-Bessel functions**, and there are **two different q-Airy functions**.

Question

How can we distinguish different q-special functions?

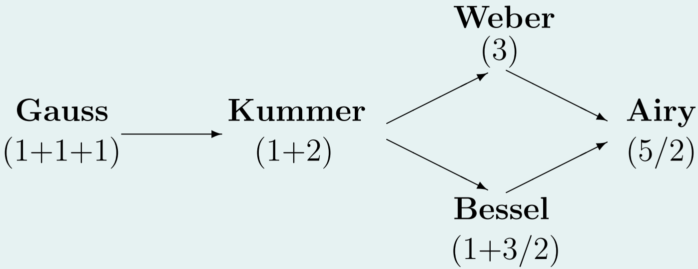
How can we show equivalence of q-special functions?

What is **equivalence**?

We go back to unified approach to **differential** special functions.

1.3. Unified approach to special functions

1) Confluence of **classical special functions** (Klein, Bôcher: 1894) [SL2000]



(r) = Poincaré rank +1

2) **Separation variables** of the Laplacian (**Meixner-Schäfke** 1954)

Bessel: Cylindrical Coordinates

Weber: Parabolic Cylindrical Coordinates

Mathieu, Spheroidal Wave: Elliptical Coordinates

3) **Laplace type** equation (**Tricomi** 1954)

$$(a_0 + b_0 x)\frac{d^2 y}{dx^2} + (a_1 + b_1 x)\frac{dy}{dx} + (a_2 + b_2 x)y = 0$$

change variables: $x \to px + q, \quad y \to g(x)y$

$\to$ **Kummer**, **Bessel**, **Weber**, **Airy**.

1.4. Unified approach to q-difference linear equations

$$(a_1 + b_1 x)u(xq^2) + (a_2 + b_2 x)u(xq) + (a_3 + b_3 x)u(x) = 0 \tag{1}$$

We admit the following transforms [Hahn] (1949):

(A) Change $\mathbf{x} \to \mathbf{cx}$:

$$\Phi \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}; \; cx \Bigg] = \Phi \begin{bmatrix} a_1 & a_2 & a_3 \\ cb_1 & cb_2 & cb_3 \end{bmatrix}; \; x \Bigg]$$

(B) Change $\mathbf{u} \to \mathbf{x}^\gamma \mathbf{u}$ ($c = q^\gamma$)

$$x^\gamma \, \Phi \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}; \; x \Bigg] = \Phi \begin{bmatrix} c^2 a_1 & c a_2 & a_3 \\ c^2 b_1 & c b_2 & b_3 \end{bmatrix}; \; x \Bigg]$$

(C) Change $\mathbf{x} \to \mathbf{1}/\mathbf{x}$

$$\Phi \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}; \; \frac{1}{x} \Bigg] = \Phi \begin{bmatrix} b_3 & b_2 & b_1 \\ q^2 a_3 & q^2 a_2 & a_1 \end{bmatrix}; \; x \Bigg]$$

(D) Change $\mathbf{u} \to (\mathbf{ax}; \mathbf{q})_\infty / (\mathbf{bx}; \mathbf{q})_\infty \mathbf{u}$ ($s = \mathrm{lq}(a_3/a_1)$)

$$\Phi \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}; \; x \Bigg] = x^s \frac{(-b_3 x/a_3; q)_\infty}{(-b_1 x/a_1 q; q)_\infty} \, \Phi \begin{bmatrix} a_3 & a_2 & a_1 \\ q b_3 & b_2 & q^{-1} b_1 \end{bmatrix}; \; x \Bigg]$$

Theorem (1) reduces to one of the following equation by transforms (A)(B)(C)(D). ($p = \sqrt{q}$)

1) When $a_1 a_3 b_1 b_3 \neq 0$, **Heine's hypergeometric** ${}_2\varphi_1(a, b; c; q; x)$:

$$(c - abqx)u(xq^2) - [c + q - (a + b)qx]u(qx) + q(1 - x)u(x) = 0.$$

2) When $b_3 = 0$, $a_1 a_3 b_1 b_2 \neq 0$, ${}_1\varphi_1(a; c; q; x)$:

$$(c - aqx)u(xq^2) - (c + q - qx)u(qx) + qu(x) = 0.$$

3.1) When $b_1 = b_2 = 0$, $a_3 \cdot a_2 a_1 b_3 \neq 0$, **Jackson's Bessel functions** $J_\nu^{(1)}(x; q)$:

$$u(xp^2) - (p^\nu + p^{-\nu})u(xp) + (1 + x^2/4)u(x) = 0.$$

3.2) When $b_1 = b_3 = 0$, $a_2 \cdot a_3 a_1 b_2 \neq 0$, **Hahn-Exton's Bessel functions** $J_\nu^{(3)}(x; q)$:

$$u(xp^2) + [-(p^\nu + p^{-\nu}) + p^{2-\nu}x^2]u(xp) + u(x) = 0.$$

3.3) When $b_3 = a_1 = 0$, $a_2 b_2 \cdot a_3 b_1 \neq 0$, **$q$-Hermite-Weber** ${}_1\varphi_1(a; 0; q; x)$

$$axu(xq^2) + (1 - x)u(xq) - u(x) = 0.$$

4.1) When $b_1 = a_2 = b_3 = 0$, **q-Airy** ${}_1\varphi_1(0; -q; q; -x)$:

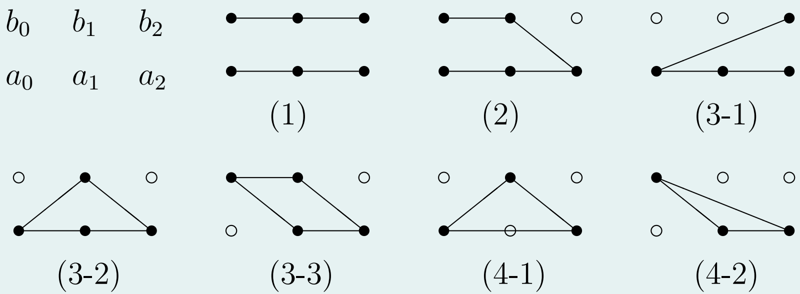
$$u(xq^2) + xu(xq) - u(x) = 0.$$

4.2) When $a_1 = b_2 = b_3 = 0$, **the Ramanujan function** ${}_0\varphi_1(-; 0; q; -tq)$:

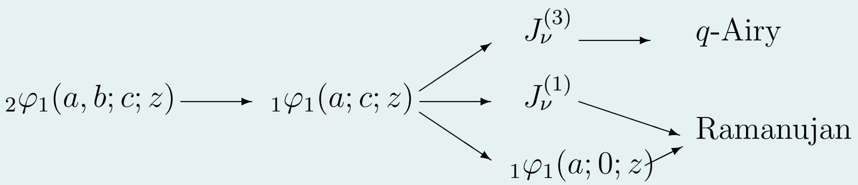
$$qxu(xq^2) - u(xq) + u(x) = 0.$$

Newton diagram

The black circle means a coefficient which is not zero.



Coalescent diagram



1.5 Covering transformation: non-integer slope

$$a(x)u(xq^2) + b(x)u(xq) + c(x)u(x) = 0,$$

We set $x = t^2$, $v(t) = u(x)$, $p = \sqrt{q}$. Then

$$(S): \qquad a(t^2)v(tp^2) + b(t^2)v(tp) + c(t^2)v(t) = 0.$$

⋆ **Covering transformation** is useful to study irregular singular points when the Poicanré rank is **non-integer** .
In our case, (3-1), (4-2) has **non-integer slopes** .

(4-2): $qxu(xq^2) - u(xq) + u(x) = 0$ has a solution **around the infinity**

$$u = \theta_p((-p)^{1/2}t)\mathrm{Ai}_q(\frac{1}{(-p)^{3/2}t}).$$

Morita's connection formula (**T. Morita** `arXiv:1104.0755`)

$$A_{q^2}\left(-\frac{q^3}{x^2}\right) = -\frac{1}{(q,-1;q)_\infty}\left\{\theta\left(\frac{x}{q}\right)\mathrm{Ai}_q(-x) + \theta\left(-\frac{x}{q}\right)\mathrm{Ai}_q(x)\right\}.$$

2. Hypergeometric solutions of q-Painlevé [**KMNOY**]

$$\begin{array}{ccccccccc} q\text{-}P_{\rm VI} & \to & q\text{-}P_{\rm V} & \to & \begin{array}{c} q\text{-}P_{\rm IV} \\ q\text{-}P_{\rm III} \end{array} & \to & q\text{-}P_{\rm II} & \to & q\text{-}P_{\rm I} \\ A_3^{(1)} & \to & A_4^{(1)} & \to & A_5^{(1)} & \to & A_6^{(1)} & \to & A_7^{(1)} \\ D_5^{(1)} & \to & A_4^{(1)} & \to & (A_2+A_1)^{(1)} & \to & (A_1+A_1)^{(1)} & \to & A_1^{(1)} \end{array}$$

Hypergeometric solutions [KMNOY]

$$\begin{array}{ccccccccc} {}_2\varphi_1 & \to & {}_1\varphi_1 & \to & \begin{array}{c} {}_1\varphi_1\,(a;0;q;z) \\ {}_1\varphi_1\,(0;b;q;z) \end{array} & \to & {}_1\varphi_1\,(0;-q;q;z) & \to & \text{none} \end{array}$$

$$\begin{array}{ccccccccc} (1) & \to & (2) & \to & \begin{array}{c} (3.3) \\ (3.2) \end{array} & \to & (4.1) & \to & \text{none} \end{array}$$

⋆ **(3.1)** is a special solution of q-$P_{\rm III}(A_3)$ (Kajiwara-Ohta-Satsuma 1995)

⋆ **(4.2)** is equivalent to (4.1) by **covering transformation** .

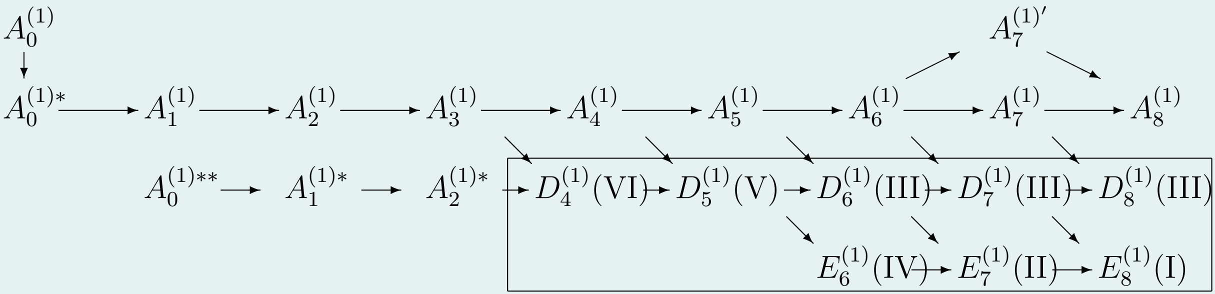
2.1 The Painlevé diagram

Okamoto-Sakai's Initial Spaces (IVS)

Okamoto: IVS of the Painlevé eq. = **Open rational surface** $X \setminus D$

Sakai: IVS = 9 points blow-up of $\mathbb{CP}^2$

The list of D



0) $A_0^{(1)}$ is a **boss** of Painlevé equations (**elliptic Painlevé**)

1) The A-series give **q-difference Painlevé equations**

2) $A_0^{(1)**}$, $A_1^{(1)*}$, $A_2^{(1)*}$: **difference Painlevé** (**Boalch**) corresponding to

$$[(111111, 222, 33)], \quad [(1111), (1111), (22)], \quad [(111), (111), (111)].$$

3) Eight in the box are **Painlevé differential equations** .

Misunderstanding of discrete Painlevé equations

Q. What are (q-)difference Painlevé equations?

A. There are many **difference** Painlevé equations, whose limit $q \to 1$ goes to the same **differential** Painlevé equations.

Example: Two different q-$P_{\rm III}(A_3^{(1)})$:

1. **Ramani-Grammaticos-Hietarinta's** q-$P_{\rm III}(A_5^{(1)})$

q-$P_{\rm VI}(A_3^{(1)})$ [JS96]

$$\frac{y\bar{y}}{a_3a_4} = \frac{(\bar{z}-b_1t)(\bar{z}-b_2t)}{(\bar{z}-b_3)(\bar{z}-b_4)}, \quad \frac{z\bar{z}}{b_3b_4} = \frac{(y-a_1t)(y-a_2t)}{(y-a_3)(y-a_4)}, \quad \left(\frac{b_1b_2}{b_3b_4} = q\frac{a_1a_2}{a_3a_4}\right).$$

We set

$$t = s^2,\ q = p^2,\ b_i = pa_i\ (i=1,2),\ b_i = a_i\ (i=3,4),\ y(t) = \overline{w}(s),\ z(t) = w(s).$$

q-$P_{\rm III}(A_3^{(1)})$ is a **symmetric specialization** of q-$P_{\rm VI}(A_3^{(1)})$: [RGH91]

$$\frac{\overline{w}\underline{w}}{a_3a_4} = \frac{(w-a_1s)(w-sa_2)}{(w-a_1)(w-a_4)}.$$

2. **Sakai's** q-$P_{\rm III}$

q-$P_{\mathbf{III}}(A_5^{(1)})$: [Sakai00, Murata]

$$\frac{y\bar{y}}{a_3a_4} = -\frac{\bar{z}(\bar{z}-b_2t)}{\bar{z}-b_3}, \qquad \frac{z\bar{z}}{b_3} = -\frac{y(y-a_1t)}{a_4(y-a_3)}$$

Distinguish two different q-$P_{\mathbf{III}}$, we should check out IVS .

⋆ This is an analogue of classical special functions.

$${}_1F_1(\nu+1/2, 2\nu+1; 2iz) = \Gamma(1+\nu)e^{iz}(z/2)^{-\nu}J_\nu(z),$$

$$U(-\nu/2, -1/2; z^2/2) = 2^{-\nu/2}e^{z^2/4}D_\nu(z).$$

Higher special functions coincide with lower special functions for special parameters.

Example. q-$P_{\mathbf{VI}}(A_5^{(1)})$ **and** q-$P_{\mathbf{III}}(A_5^{(1)})$ **for the same IVS** $(A_5^{(1)})$

q-$P_{\mathbf{VI}}(A_5^{(1)})$:

$$\frac{y\bar{y}}{a_3a_4} = \frac{b_2t(\bar{z}-b_1t)}{\bar{z}-b_3}, \quad \frac{z\bar{z}}{b_3} = -\frac{(y-a_1t)(y-a_2t)}{a_4y}, \qquad \left(\frac{b_1b_2}{b_3} = q\frac{a_1a_2}{a_4}\right).$$

q-$P_{\mathbf{III}}(A_5^{(1)})$:

$$\frac{y\bar{y}}{a_3a_4} = -\frac{\bar{z}(\bar{z}-b_2t)}{\bar{z}-b_3}, \quad \frac{z\bar{z}}{b_3} = -\frac{y(y-a_1t)}{a_4(y-a_3)}$$

These two equations have the same **IVS** . When $q \to 1$,

$$q\text{-}P_{\mathrm{VI}}(A_5^{(1)}) \to P_{\mathrm{IV}} \qquad q\text{-}P_{\mathrm{III}}(A_5^{(1)}) \to P_{\mathrm{III}}$$

We can take **different translations** of the affine Weyl group, and we may restrict parameters. Then we get **different types** of q-analogue Painlevé equations.

3. How to solve q-difference equations

3.1. Irregular singularity

Differential case

$$\frac{du^2}{dx^2} + p(x)\frac{du}{dx} + q(x)u = 0.$$

$p =$ (pole order of $p(x)$ at $x = a$), $q =$ (pole order of $p(x)$ at $x = a$),

$$r := \max(p, q/2) - 1$$

If $r = 0$, $x = a$ is a **regular singular point** .
If $r > 0$, $x = a$ is a **irregular singular point** with the Poincaré rank r.

Difference case

$$a(x)u(xq^2) + b(x)u(xq) + c(x)u(x) = 0.$$

We may assume $a(x), b(x), c(x)$ are holomorphic at $x = 0$,
and one of $a(0), b(0), c(0)$ is not zero.
The **characteristic polynomial** around $x = 0$

$$F_0(\mu) = a(0)\mu^2 + b(0)\mu + c(0)$$

Definition

If $F_0(\mu) = 0$ has two finite solutions $\mu = q^\alpha, q^\beta$, $x = 0$ is called **regular singular point**. If $F_0(\mu) = 0$ has a solution $\mu = 0$ or $\mu = \infty$, $x = 0$ is a **irregular singular point**.

When $x = 0$ is regular singular, there exist solutions of convergent series

$$u = x^\alpha \sum_{n=0}^{\infty} a_n x^n, \quad x^\beta \sum_{n=0}^{\infty} b_n x^n.$$

When $x = 0$ is irregular singular, it depends on the slope of Newton polygon. When the slope is ± 1, there exist a solution of the form

$$u = \frac{x^\alpha}{\theta(rx)} \sum_{n=0}^{\infty} a_n x^n, \quad \text{or} \quad x^\alpha \theta(rx) \sum_{n=0}^{\infty} a_n x^n.$$

Remark. In difference case, **the series might be convergent** even for irregular case.

3.2 Mastumoto's q-Riemann scheme

Start from the Laplace type:

$$(a_0 + b_0 x)u(xq^2) + (a_1 + b_1 x)u(xq) + (a_2 + b_2 x)u(x) = 0.$$

Characteristic polynomials :

$$\begin{array}{llll} x = 0: & a_0\mu^2 + a_1\mu + a_2 = 0. & \text{roots:} & \mu_1, \mu_2 \\ x = \infty: & b_0 + b_1\lambda + b_2\lambda^2 = 0. & \text{roots:} & \lambda_1, \lambda_2 \end{array}$$

Extra exponents : We set $\rho_1 = -a_0/b_0, \quad \rho_2 = -b_2/a_2,$

a) q-analogue of Fuchs' relation

$$\rho_1\rho_2\lambda_1\lambda_2\mu_1\mu_2 = 1$$

b) q-analogue of Papperitz's equation

$$\lambda_1\lambda_2(x - \rho_1)u(xq^2) - \{(\lambda_1 + \lambda_2)x - \lambda_1\lambda_2\rho_1(\mu_1 + \mu_2)\}u(xq) + (x - \lambda_1\lambda_2\mu_1\mu_2\rho_1)u(x) = 0,$$

c) q-analogue of the Riemann scheme

$$\Phi \left\{ \begin{matrix} 0 & \infty & * & & \\ \mu_1 & \lambda_1 & \rho_1 & ; & x \\ \mu_2 & \lambda_2 & \rho_2 & & \end{matrix} \right\}.$$

Application: q-analogue of Kummer's twenty-four solutions

$$(c - abqx)u(xq^2) - [c + q - (a+b)qx]u(qx) + q(1-x)u(x) = 0.$$

has the following eight solutions:

$$\begin{aligned} &{}_2\varphi_1(a, b; c; q, x), && \frac{(abx/c;q)_\infty}{(x;q)_\infty} {}_2\varphi_1\left(\frac{c}{a}, \frac{c}{b}; c; q, \frac{ab}{c}x\right), \\ &x^{1-\gamma}{}_2\varphi_1\left(\frac{aq}{c}, \frac{bq}{c}; \frac{q^2}{c}; q, x\right), && x^{1-\gamma}\frac{(abx/c;q)_\infty}{(x;q)_\infty} {}_2\varphi_1\left(\frac{q}{a}, \frac{q}{b}; \frac{q^2}{c}; q, \frac{ab}{c}x\right), \\ &x^{-\alpha}{}_2\varphi_1\left(a, \frac{aq}{c}; \frac{aq}{b}; q, \frac{cq}{abx}\right), && x^{-\alpha}\frac{(q/x;q)_\infty}{(cq/abx)_\infty} {}_2\varphi_1\left(\frac{q}{b}, \frac{c}{b}; \frac{aq}{b}; q, \frac{q}{x}\right), \\ &x^{-\beta}{}_2\varphi_1\left(b, \frac{bq}{c}; \frac{bq}{a}; q, \frac{cq}{abx}\right), && x^{-\beta}\frac{(q/x;q)_\infty}{(cq/abx;q)_\infty} {}_2\varphi_1\left(\frac{q}{a}, \frac{c}{a}; \frac{bq}{a}; q, \frac{q}{x}\right). \end{aligned}$$

Here $a = q^\alpha, b = q^\beta$ and $c = q^\gamma$. Other 16 solutions are complicated.

(1)

$$\Phi \left\{ \begin{matrix} 0 & \infty & * & \\ 1 & a & c/aq & ; \quad x \\ q/c & b & 1 & \end{matrix} \right\} = A\, {}_2\varphi_1(a,b;c;q,x) + B\, x^{1-\gamma}\, {}_2\varphi_1\left(\frac{aq}{c}, \frac{bq}{c}; \frac{q^2}{c}; q, x\right)$$

(2)

$$\Phi \left\{ \begin{matrix} 0 & \infty & * & \\ 1 & a & c/aq & ; \quad x \\ q/c & \infty & 0 & \end{matrix} \right\} = A\, {}_1\varphi_1(a;c;q,x) + B\, x^{1-\gamma}\, {}_1\varphi_1\left(\frac{aq}{c}; \frac{q^2}{c}; q, x\right)$$

(3-1)

$$\Phi \left\{ \begin{matrix} 0 & \infty & * & \\ q^{\nu} & 0 & \infty & ; \quad x \\ q^{-\nu} & 0 & -1/4 & \end{matrix} \right\} = A\, x^{\nu}\, {}_2\varphi_1(0,0;q^{1+\nu};q,-x/4) + B\, x^{-\nu}\, {}_2\varphi_1(0,0;q^{1-\nu};p,-x/4)$$

(3-2) The Riemann scheme

$$\Phi \left\{ \begin{matrix} 0 & \infty & * & \\ p^{\nu} & 0 & \infty & ; \quad x \\ p^{-\nu} & \infty & 0 & \end{matrix} \right\} = A\, x^{\nu}\, {}_1\varphi_1\left(0;q^{1+2\nu};q,q^2x\right) + B\, x^{-\nu}\, {}_1\varphi_1\left(0;q^{1-2\nu};q,q^{2-2\nu}x\right).$$

(3-3) The Riemann scheme is given by

$$\Phi\left\{\begin{matrix} 0 & \infty & * & & \\ 1 & a & 0 & ; & x \\ \infty & \infty & 0 & & \end{matrix}\right\} = A\,{}_1\varphi_0(a;-;q,x) + B\,\theta(-ax/q)_2\varphi_0(q/a,0;-;q,ax/q^2).$$

(4-1)

$$\Phi\left\{\begin{matrix} 0 & \infty & * & & \\ 1 & 0 & \infty & ; & x \\ -1 & \infty & 0 & & \end{matrix}\right\} = A\,\mathrm{Ai}_q(x) + B\,e^{\pi i \mathrm{lq}\,x}\mathrm{Ai}_q(-x).$$

where $\mathrm{lq}\,x = \log_e x/\log_e q$.

(4-2)

$$\Phi\left\{\begin{matrix} 0 & \infty & * & & \\ 1 & \infty & 0 & ; & x \\ \infty & \infty & 0 & & \end{matrix}\right\} = A\,{}_0\varphi_1(-;0;q,-qx) + B\,\theta_q(x)\,{}_2\varphi_0(0,0;-;q,-x/q).$$

4. Summary

1) There exist **seven type of q-hypergeometric equations** .

$\boldsymbol{{}_2\varphi_1}$, $\boldsymbol{{}_1\varphi_1}$, ${}_2\varphi_1(0,0;c)$, $\boldsymbol{{}_1\varphi_1(0;c)}$, $\boldsymbol{{}_1\varphi_1(a;0)}$, $\boldsymbol{{}_1\varphi_1(0;-q)}$, ${}_0\varphi_1(-;0)$

2) **Five** of seven q-hypergeometric correspond to

particular solutions of q-Painlevé equations .

3) **Jackson's first q-Bessel** ${}_2\varphi_1(0,0;c)$ corresponds to particular solutions of q-$P_{\mathbf{III}}(A_3^{(1)})$, which is a symmetric specialization of q-$P_{VI}(A_3^{(1)})$.

4) **The Ramanujan function** ${}_0\varphi_1(-;0)$ are **equivalent to the q-Airy** by **covering transformation** .

Ramanujan and **q-Airy** live on different sides: $x=0$ and $x=\infty$.

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